25/06/2020 (1)

$$\underbrace{\alpha_R(y)}_{3\text{-}D} = \underbrace{\alpha(y)}_{2\text{-}D} - \underbrace{\alpha_i(y)}_{\text{Depends on the flow and wing load.}}$$

$$\alpha_R(y) = [\alpha_G(y) - \alpha_{L0}(y)] - \alpha_i(y)$$

$$\Gamma(y) = \frac{\cancelto{2\pi}{a_0(y)}}{2} c(y) U_\infty [\alpha(y) - \alpha_i(y)]$$

→ this is the physical meaning of the Lifting Line Theory fundamental equation.

Elliptic Loading: $\Gamma(\theta) = \Gamma_0 \sin\theta$; $A_1 = \dfrac{\Gamma_0}{2bU_\infty}$

$$w_i(\theta) = U_\infty \alpha_i(\theta) = U_\infty A_1 \Rightarrow \boxed{w_i = \frac{\Gamma_0}{2b}}$$

and $\alpha_i(\theta) = \dfrac{\Gamma_0}{2bU_\infty} = A_1$ ⇒ both w_i and α_i are constant spanwise.

○ substituting the above results for their counterparts in the Fundamental equation, one gets:

$$\Gamma_0 \sqrt{1-\left(\frac{2y}{b}\right)^2} = \frac{1}{2} a_0 \, C(y) \left[U_0 \alpha(y) - \frac{\Gamma_0}{2b} \right]$$

in particular for an elliptic plan form:

$$c_y = c_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2} \Rightarrow \text{think "spitfire" (W.W.II)}$$

$c_0 \Rightarrow$ root chord.

$$\Gamma_0 = \frac{2bU_\infty}{1+\left(\frac{4b}{a_0 c_0}\right)}, \quad \text{ellipse:} \begin{cases} S_w = \frac{\pi}{4} c_0 b \\ AR = 4b/\pi c_0 \end{cases}$$

$$\Gamma_0 = \frac{2bU_\infty \alpha}{1+(\pi AR/a_0)}$$

$$C_L = \pi AR A_1 = \pi AR \frac{\Gamma_0}{2bU_\infty} = \frac{\pi AR \alpha}{1+\left(\frac{\pi AR}{a_0}\right)}$$

$$C_{L\alpha} = a = \frac{a_0}{1+\left(\frac{a_0}{\pi AR}\right)} \xRightarrow{a_0 = 2\pi} a = \frac{2\pi}{1+\left(\frac{2}{AR}\right)}$$

the 3-D finite wing $C_{L\alpha} = a = m$

defined as: $\quad m_0 \alpha_R \Big|_{3\text{-}D} = m \alpha \Big|_{2\text{-}D} = C_L$

$m_0 = a_0 = 2\pi$

$$\alpha_R = \alpha - \alpha_i \Rightarrow \alpha(y) = \alpha_R(y) + \alpha_i(y)$$

$$\frac{C_L}{m} = \frac{C_L}{m_0} + m_0 \frac{\alpha_i}{m_0} = \frac{C_L + m_0 \alpha_i}{m_0}$$

$$m = \frac{C_L m_0}{C_L + m_0 \alpha_i} \Rightarrow m = \frac{m_0}{1 + \frac{m_0 \alpha_i}{C_L}}$$

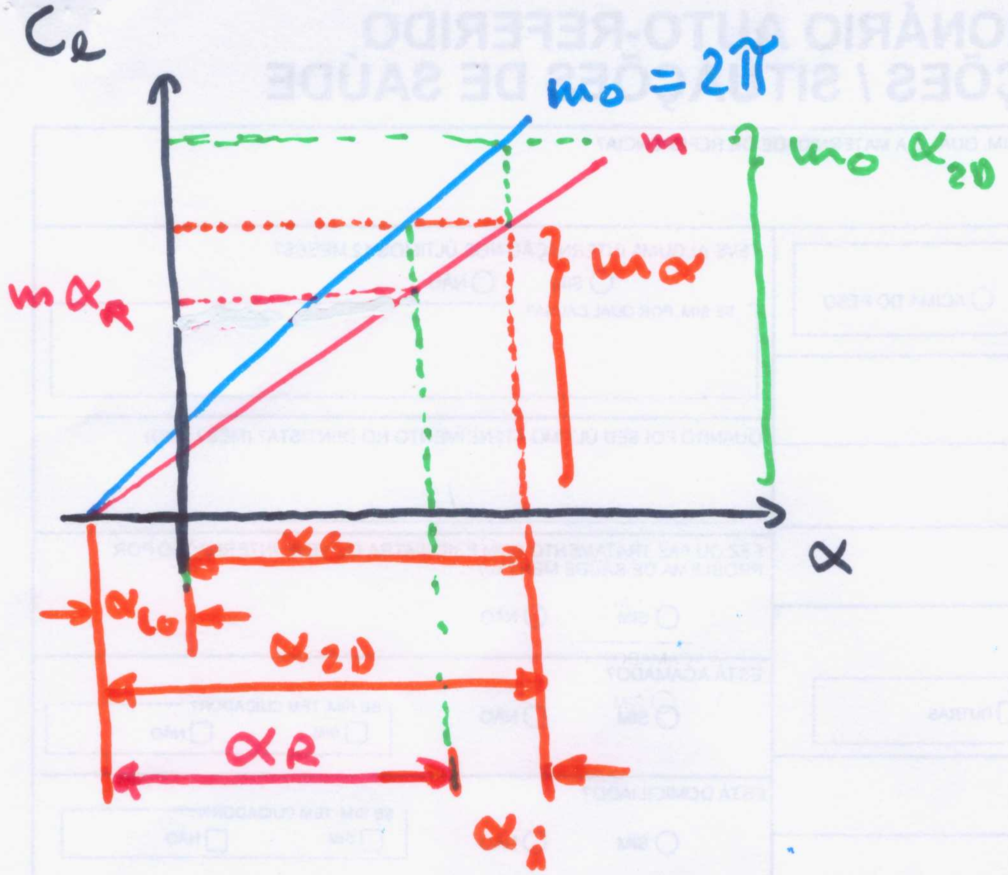
$$m = \frac{m_0}{1 + \alpha_i / \alpha_r} \Rightarrow m \leq m_0$$

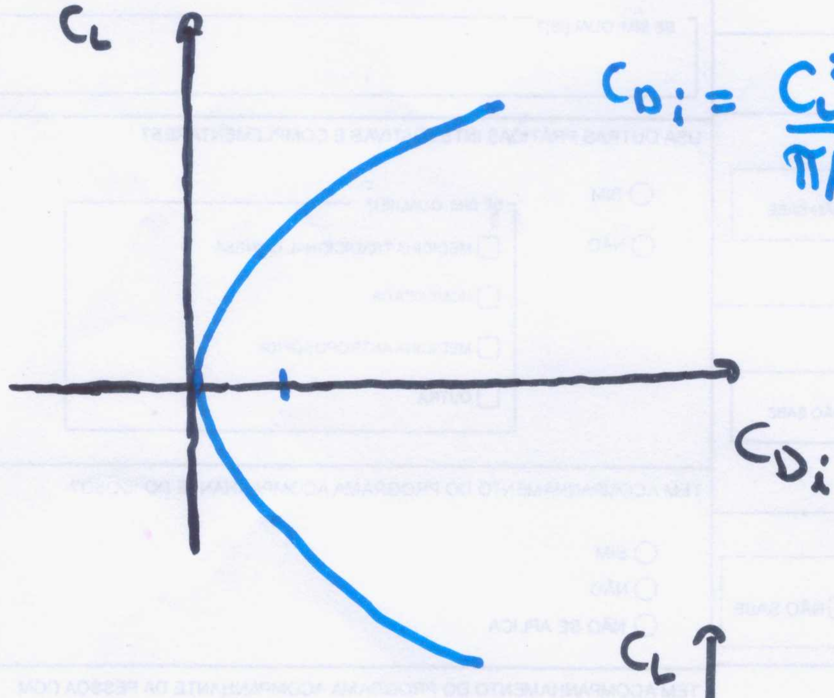
$$m \leq 2\pi$$

General formula for arbitrary wing loads:

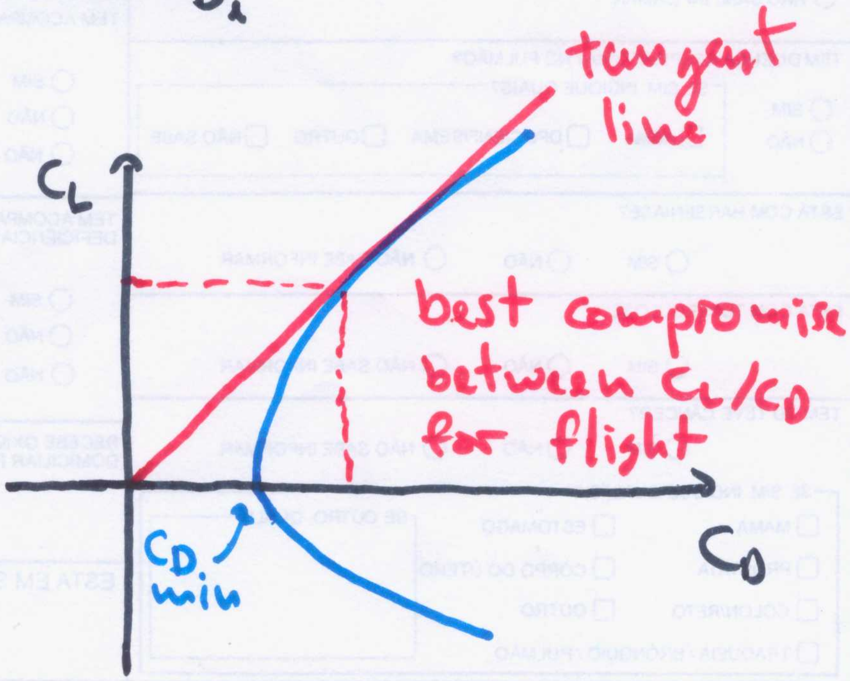
$$\boxed{m = \frac{m_0}{1 + \frac{m_0}{\pi AR}(1 + \tau)}}$$

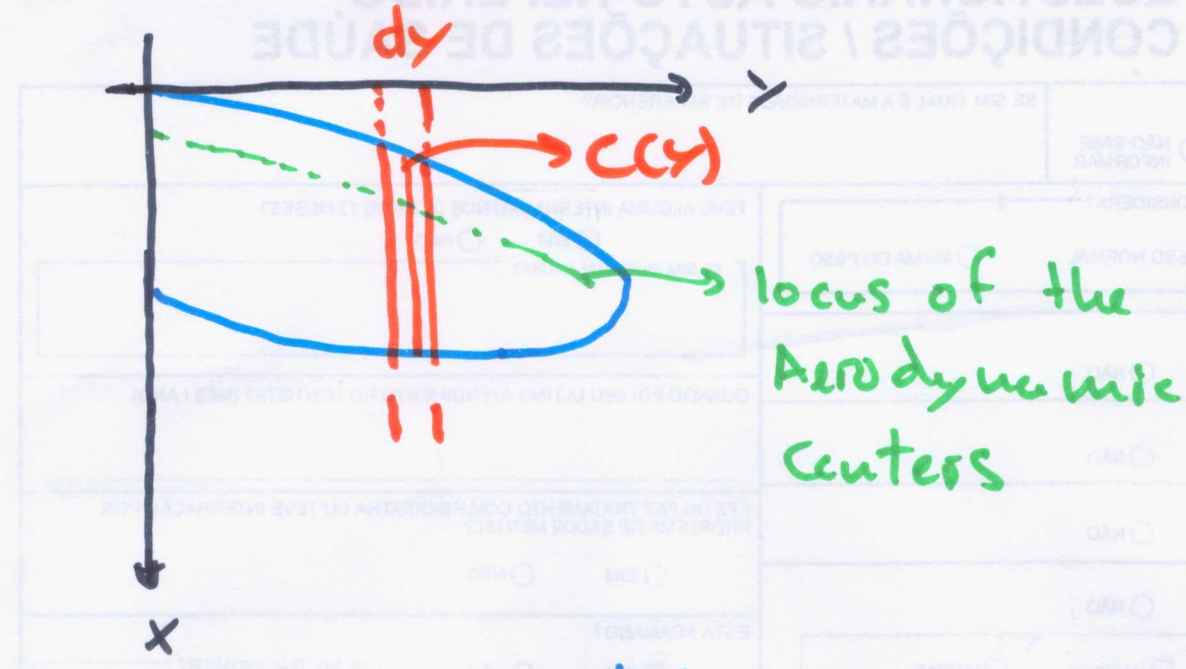
where τ depends on the departure the load has with respect to the elliptic





$$C_{D_i} = \frac{C_L^2}{\pi AR}(1+\delta)$$





$$\bar{c} = \frac{2}{S_w}\int_0^{b/2} c^2\, dy$$

Pressure Center

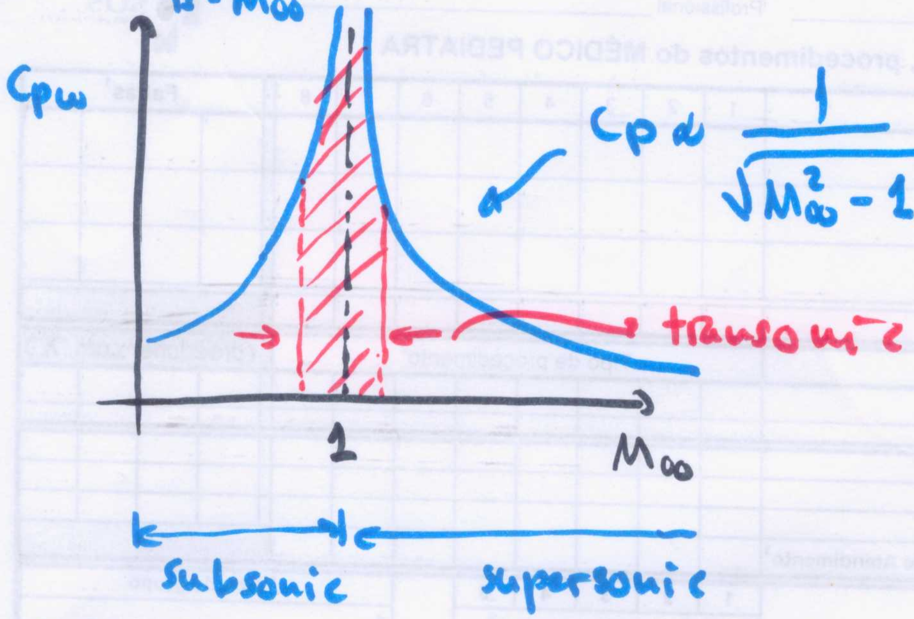
$$\begin{cases} \bar{x} = \dfrac{2}{C_L S_w}\displaystyle\int_0^{b/2} C_{l_a}\, c\, x\, dy \\[2ex] \bar{y} = \dfrac{2}{C_L S_w}\displaystyle\int_0^{b/2} C_{l_a}\, c\, y\, dy = \\[2ex] \bar{z} = \dfrac{2}{C_L S}\displaystyle\int_0^{b/2} C_{l_a}\, c\, z\, dy \end{cases}$$

$$C_{m_p} = -\int_{-1/2}^{1/2} \left[\frac{c_l x_{ac}\, c(y)}{\bar{c}^2} + \frac{c_{mac}(y)\, c^2(y)}{\bar{c}^2} \right] \frac{dy}{b}$$

02/07/2020

$$C_{p_\infty} \frac{1}{\sqrt{1-M_\infty^2}}$$

$$C_{p_\alpha} \frac{1}{\sqrt{M_\infty^2-1}}$$



$M_\infty < 1$	$M_\infty > 1$
$\beta^2 \phi'_{xx} + \phi'_{yy} = 0$	$\lambda^2 \phi'_{xx} - \phi'_{yy} = 0$
$\beta = \sqrt{1-M_\infty^2}$	$\lambda = \sqrt{M_\infty^2 - 1}$

Karman - Tsien $(M_\infty < 1)$

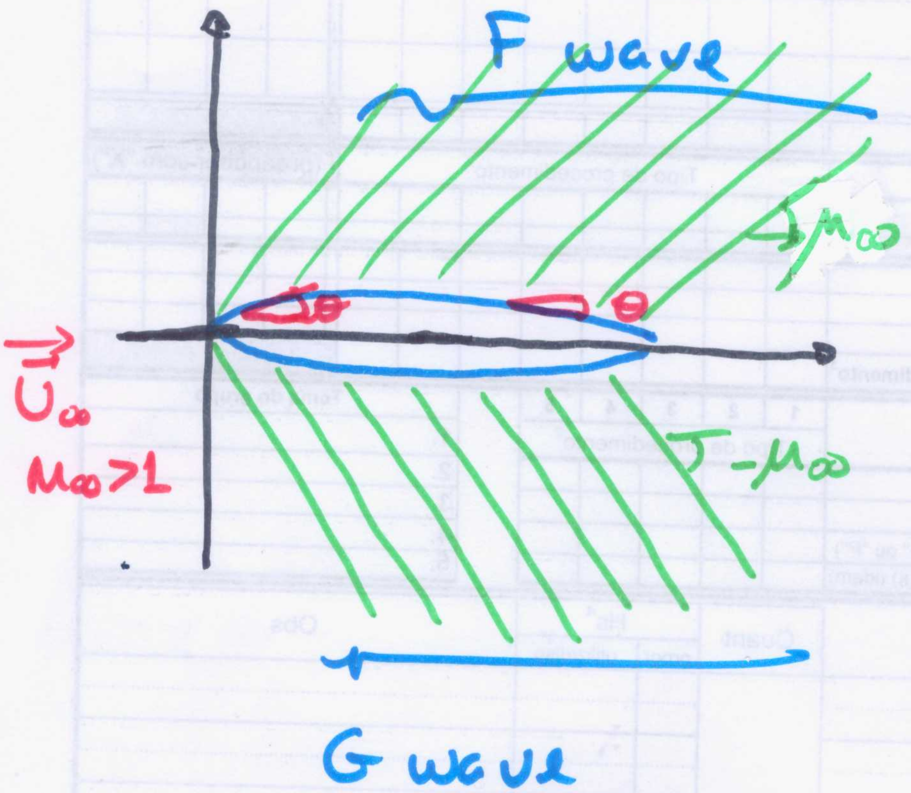
$$C_p = \frac{C_{p_{inc}}}{\sqrt{1-M_\infty^2} + \frac{C_{p_{in}}}{2}\left(\frac{M_\infty^2}{1+\sqrt{1-M_\infty^2}}\right)}$$

$M_\infty > 1$

$$\lambda^2 \phi'_{xx} - \phi'_{yy} = 0$$

$$\lambda = \sqrt{M_\infty^2 - 1}$$

$$\phi'_{xx} = \frac{\phi'_{yy}}{\lambda_2} \qquad \begin{cases} \phi' = F(x-\lambda y) + G(x+\lambda y) \\ \phi' = F(\xi) + G(\eta) \\ \xi \equiv x - \lambda y \ ; \ \eta \equiv x + \lambda y \end{cases}$$



$$\left.\frac{dy}{dx}\right|_\xi = +\frac{1}{\lambda} \Rightarrow \mu_\infty$$

$$\left.\frac{dy}{dx}\right|_\eta = -\frac{1}{\lambda} \Rightarrow -\mu_\infty$$

Mach angle: $\mu_\infty = \sin^{-1}\left(\frac{1}{M_\infty}\right) = \tan^{-1}\left(\frac{1}{\sqrt{M_\infty^2 - 1}}\right)$

For slender bodies in potential flow:

$$C_p = -\frac{2u'}{U_\infty} \qquad \begin{cases} \phi'_x = u' = F' \\ \phi'_y = v' = -\lambda F' \end{cases}$$

$$\tan\theta \cong \theta = \left.\frac{dy}{dx}\right|_s \qquad \text{Also } \tan\theta \cong \theta \cong \frac{v'}{U_\infty + u'}$$

$\theta << 1$

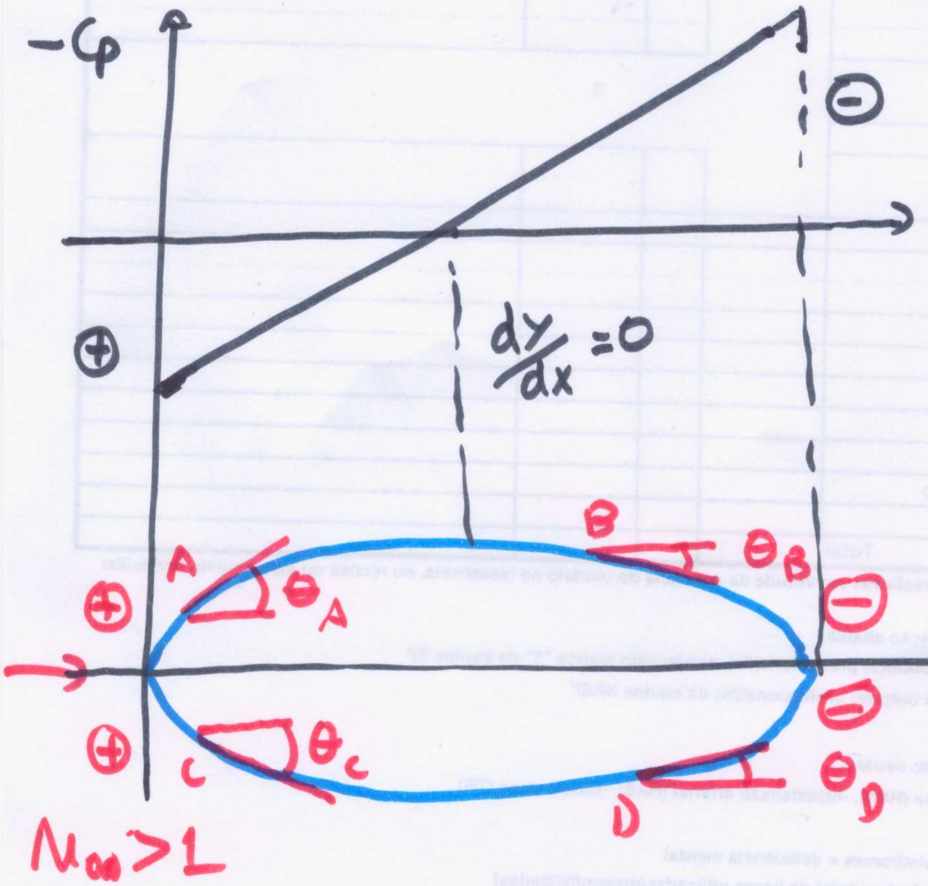
$U_\infty >> u', v'$

the wall tangency condition becomes:

$$\theta \cong \frac{v'}{U_\infty} = \frac{-u'\lambda}{U_\infty} \; ; \; C_p = -\frac{2u'}{U_\infty}$$

$$\boxed{\left. C_p \right|_F \cong \frac{2\theta}{\sqrt{M_\infty^2 - 1}} = \frac{2}{\sqrt{M_\infty^2 - 1}} \left. \frac{dy}{dx} \right|_s} \quad \text{F wave}$$

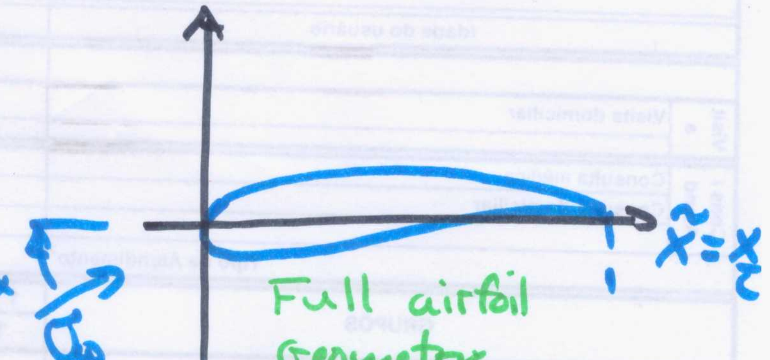
$$\boxed{\left. C_p \right|_G \cong \frac{-2\theta}{\sqrt{M_\infty^2 - 1}} = \frac{-2}{\sqrt{M_\infty^2 - 1}} \left. \frac{dy}{dx} \right|_s} \quad \text{G wave}$$



$$\theta = -\alpha + \left.\frac{dy}{dx}\right|_{camber} + \left.\frac{dy}{dx}\right|_{thickness}$$

$$C_L = \int_0^1 (C_{p_\ell} - C_{p_u})\, d\tilde{x}$$

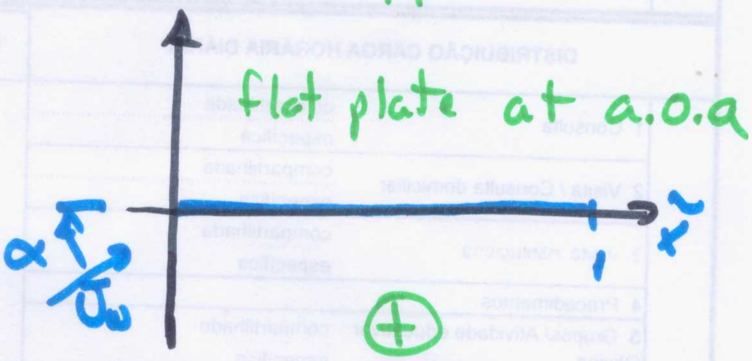
$M_\infty > 1$



Full airfoil Geometry

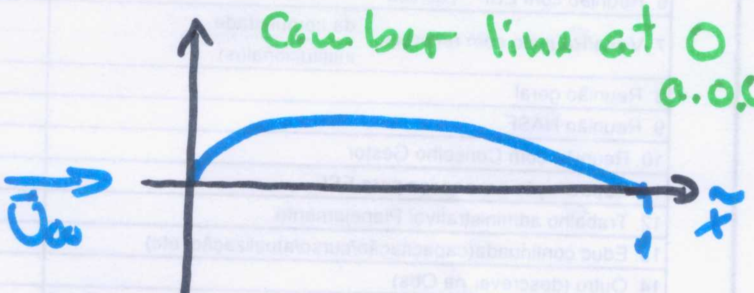
$\parallel$

flat plate at a.o.a

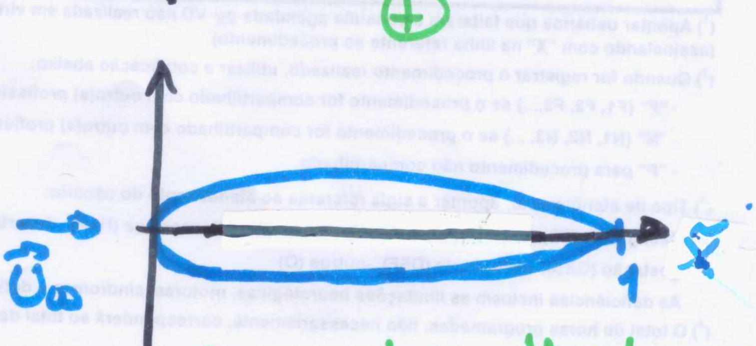


$\oplus$

Camber line at 0 a.o.a

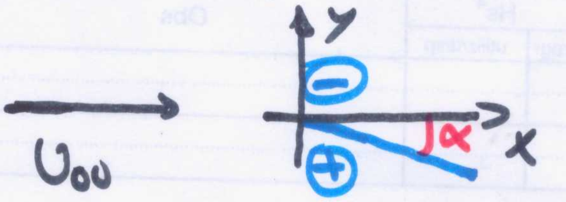


$\oplus$

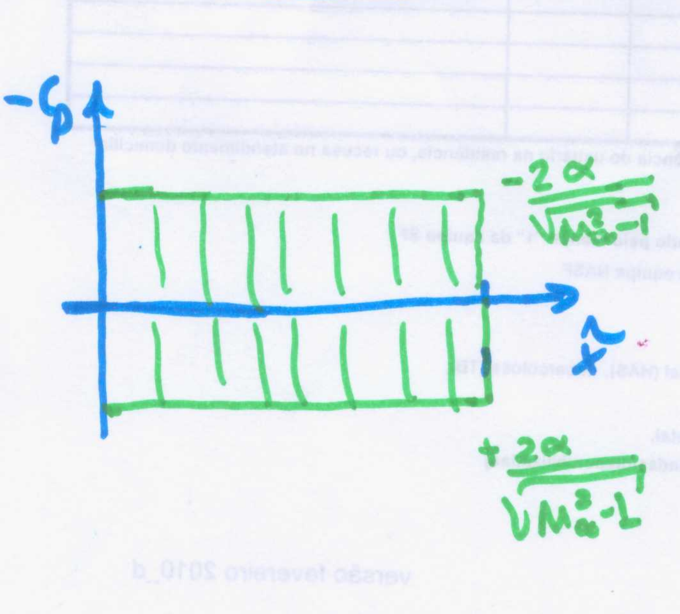


Symmetric thickness distribution at 0 a.o.a

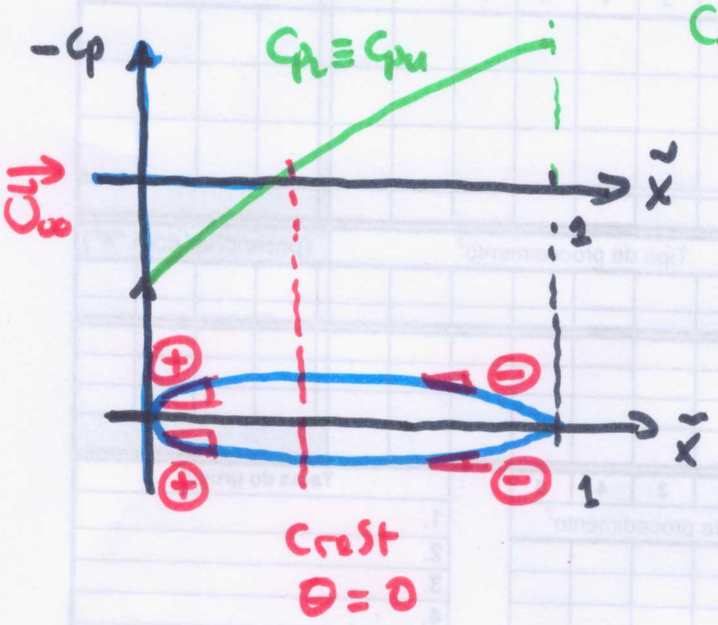
1) contribution: flat plate at α a.o.a.



$$\boxed{C_L = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}}$$ flat plate



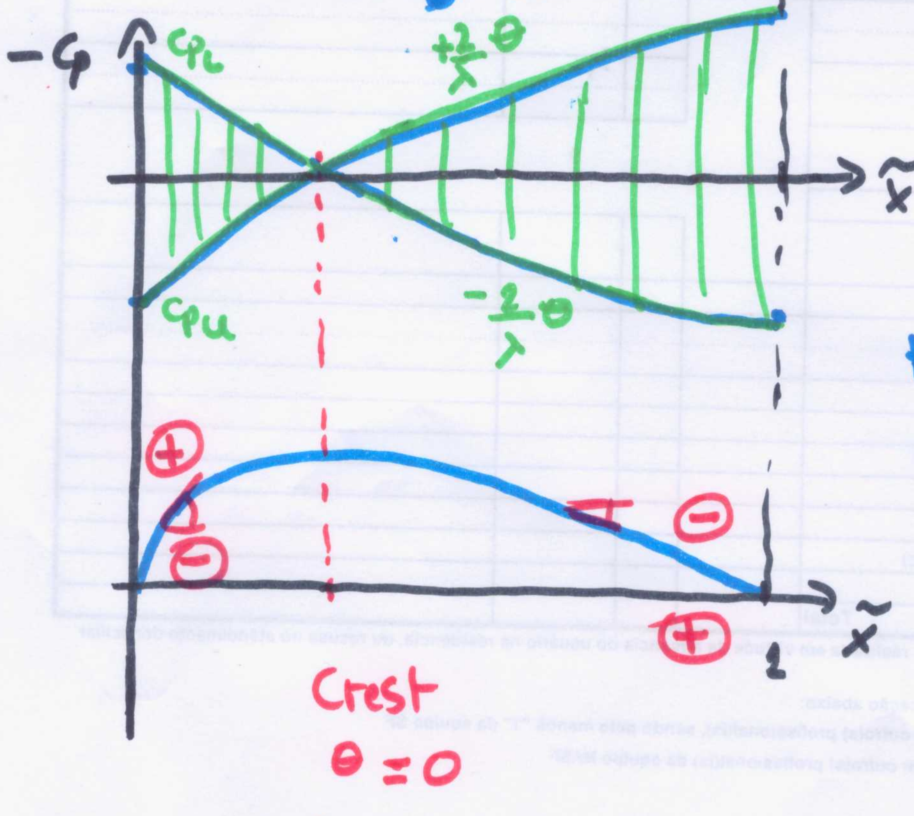
2) Contribution from the symmetric thickness distribution at 0 a.o.a.



$$C_{p_L}(\tilde{x}) - C_{p_u}(\tilde{x}) = 0$$

$$C_L = 0$$

3) Contribution from the camber line at 0 a.o.a.



fore: positive contribution

aft: negative contribution

the two C_p curves are antisymmetric

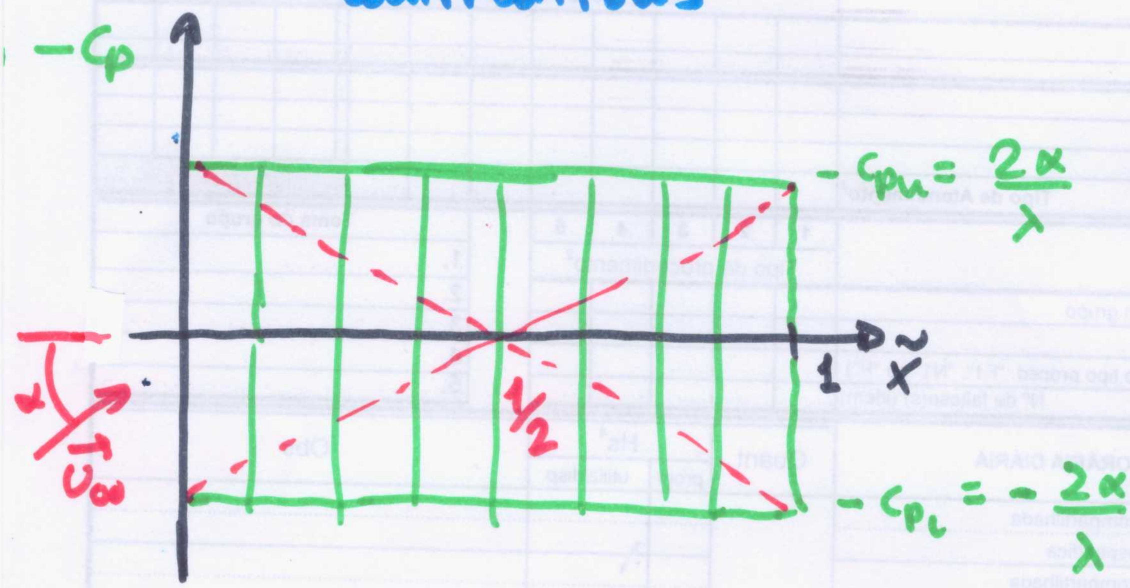
$$C_L = -\frac{2}{\lambda}\int_0^1 \left[\frac{dy_c}{d\tilde{x}}\bigg|_L - \frac{dy_c}{d\tilde{x}}\bigg|_u \right] d\tilde{x} = -\frac{4}{\lambda}\int_0^1 \frac{dy_c}{dx}\,dx = -\frac{4}{\lambda}\left[y_c(1) - y_c(0) \right]$$

$$\boxed{C_L = 0}$$

Pitching moment:

$$C_m\Big|_{\tilde{x}=0}^{LE} = \underbrace{-\frac{C_L}{2}}_{\text{Lift}} + \underbrace{\frac{4}{\sqrt{M_\infty^2-1}}\int_0^1 \tilde{x}\,\frac{dy_c}{d\tilde{x}}\,d\tilde{x}}_{\text{Camber}}$$

contributions



$$C_{pu} = \frac{-2\alpha}{\lambda}$$

$$C_{pL} = \frac{2\alpha}{\lambda}$$

Aerodynamic center: $\tilde{x} = \frac{x}{c} = \frac{1}{2}$

Wave drag:

$$C_{D_w} = -\int_0^1 \left[C_{p_L} \frac{dy}{d\tilde{x}}\Big|_L - C_{pu} \frac{dy}{d\tilde{x}}\Big|_u \right] d\tilde{x}$$

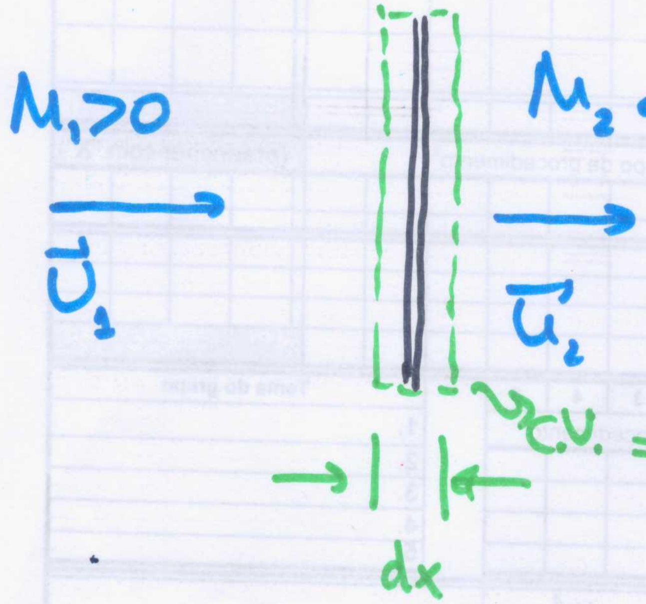
$$\boxed{C_{D_w} = \frac{4}{\sqrt{M_\infty^2-1}} \int_0^1 \left(\frac{dy}{d\tilde{x}}\right)^2 d\tilde{x}}$$

Ideal Gases

$$T_0 = T\left[1 + \frac{(\gamma-1)}{2} M^2\right] \qquad \gamma = \frac{c_p}{c_v}$$

$$P_0 = P\left[1 + \frac{(\gamma-1)}{2} M^2\right]^{\frac{\gamma}{(\gamma-1)}}$$

Rankine – Huggoniot (R-H) relations for a normal shock wave



$$\underbrace{\left(h + \frac{u^2}{2}\right)_1}_{h_{0,1}} = \underbrace{\left(h + \frac{u^2}{2}\right)_2}_{h_{02}}$$

$h_{0,1} = h_{02}$ stagnation enthalpy

C.V. ⇒ control volume of infinitesimal thickness.

On applying conservation of mass, momentum and energy to the control volume (steady flow) we get:

$$\eta = \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2}$$

$$\frac{P_2}{P_1} = 1 + \left(\frac{u^2}{P/\rho}\right)_1 \left(1 - \frac{1}{\eta}\right)$$

$$\frac{h_2}{h_1} = 1 + \left(\frac{u^2}{2h}\right)_1 \left(1 - \frac{1}{\eta^2}\right)$$

Ideal Gases:

$$\eta = \frac{(\gamma+1)M_1^2}{[2+(\gamma-1)M_1^2]}$$

$$\frac{P_2}{P_1} = \frac{1+\gamma(2M_1^2-1)}{(\gamma+1)}$$

$$M_2^2 = \frac{M_1^2(\gamma-1)+2}{2\gamma M_1^2-(\gamma-1)}$$

$$\frac{S_2-S_1}{R} = -\ln\left(\frac{P_{02}}{P_{01}}\right)$$

$h_{02} = h_{01}$, $P_{02} < P_{01}$

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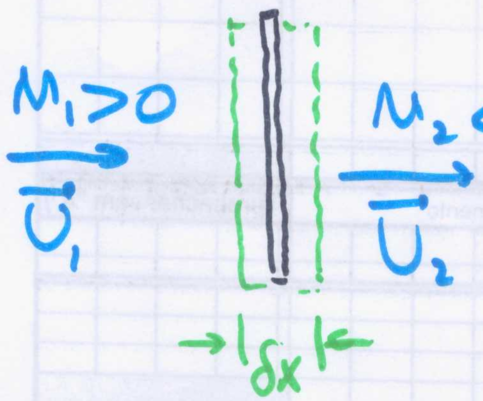
normal shock wave

R-H relations – Jump relations



1st Law (steady flow)

$$h_1 + \frac{U_1^2}{2} = h_2 + \frac{U_2^2}{2}$$

$$\Downarrow$$

$$h_{01} = h_{02}$$

2nd Law: entropy increase.

Ideal Gases:

$$Y = \frac{(\gamma+1) M_1^2}{[2 + (\gamma - 1) M_1^2]}$$

$$\frac{p_2}{p_1} = \frac{1 + \gamma (2M_1^2 - 1)}{(\gamma+1)}$$

$$M_2^2 = \frac{M_1^2(\gamma - 1) + 2}{2\gamma M_1^2 - (\gamma - 1)}$$

$$T_0 = T\left[1 + \frac{(\gamma-1)}{2} M^2\right]$$

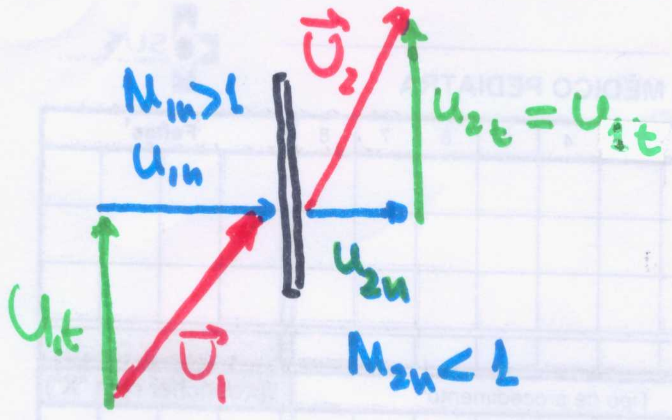
$$p_0 = p\left[1 + \frac{(\gamma-1)}{2} M^2\right]^{\frac{\gamma}{\gamma-1}}$$

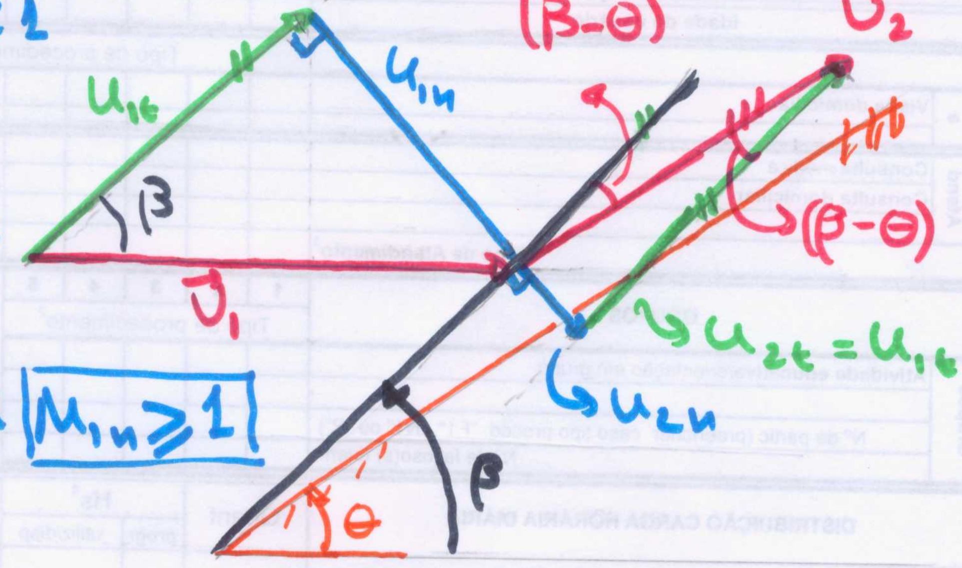
$$\gamma \equiv \frac{c_p}{c_v}$$

$$\frac{s_2 - s_1}{R} = -\ln\left(\frac{p_{02}}{p_{01}}\right) \Rightarrow \begin{cases} s_2 - s_1 \geq 0 \\ p_{02} \leq p_{01} \end{cases}$$

$s = \text{constant} \Leftrightarrow$ Mach wav

Oblique shock waves:





equality ⇒ acoustic Limit

$M_1 \geq 1$ and $\boxed{M_{1n} \geq 1}$

$M_{2n} \leq 1$

θ ⇒ Geometry wedge angle

β ⇒ thermal fluid dynamics oblique wave angle

$$\begin{cases} M_{1n} = M_1 \sin\beta \\ \tan\beta = \dfrac{U_{1n}}{U_{1t}} \\ \tan(\beta - \theta) = \dfrac{U_{2n}}{U_{1t}} \end{cases}$$

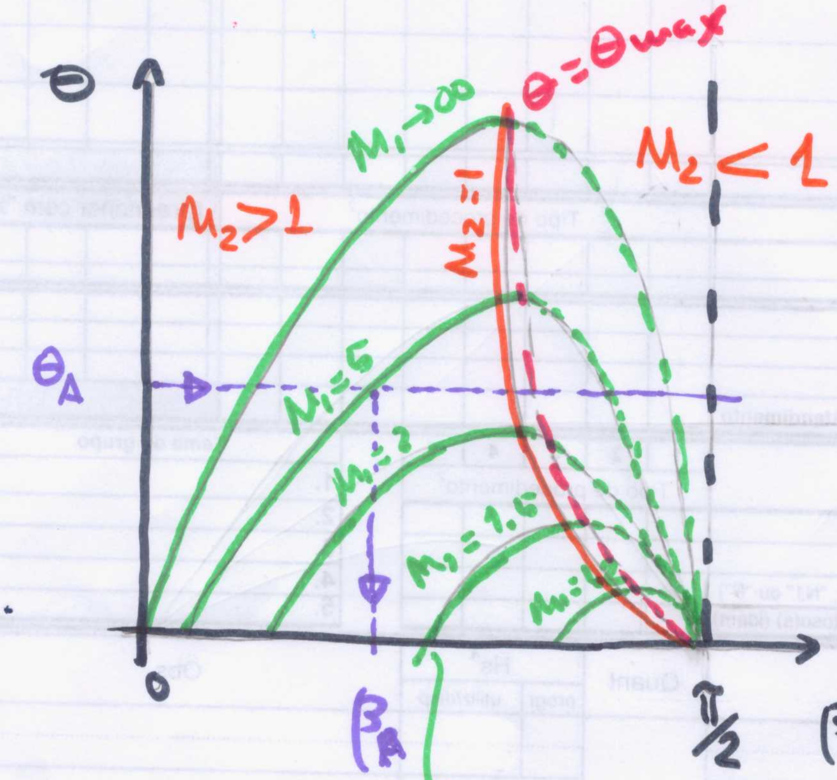
$$M_{1n} \geq 1 \Rightarrow M_{1n} = M_1 \sin(\beta) \geq 1$$

$$\beta \geq \sin^{-1}\left(\frac{1}{M_1}\right)$$

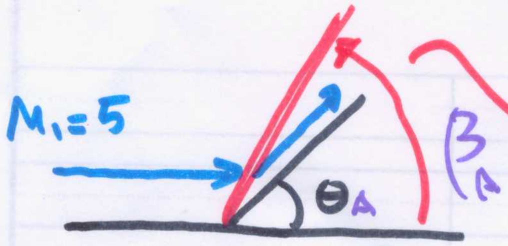
on the other hand $\beta \leq \frac{\pi}{2}$ ⇒ normal shock

Therefore: $\sin^{-1}\left(\frac{1}{M_1}\right) \leq \beta \leq \frac{\pi}{2}$

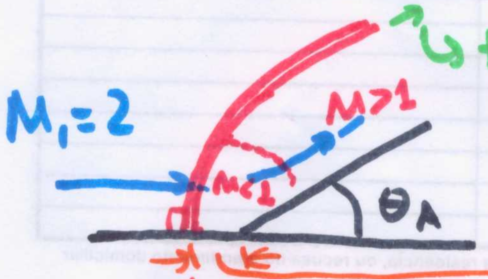
$$\tan\theta = 2\cot(\beta)\,\frac{\left(M_1^2\sin^2\beta - 1\right)}{2 + M_1^2\left[\cos(2\beta) + \gamma\right]}$$



Mach angle that corresponds to $\sin^{-1}\left(\frac{1}{M_1}\right)$



oblique (attached) shock

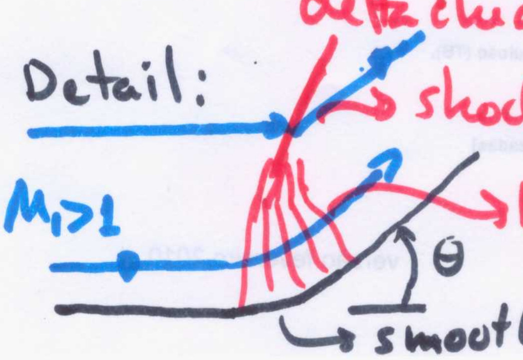


the local angle tends to the Mach angle

shock standoff distance

Bow (curved shock)

Detail:



detatched wave

shock wave

Mach waves

smooth turn.

in this case all angles over the curve are attained at some point.

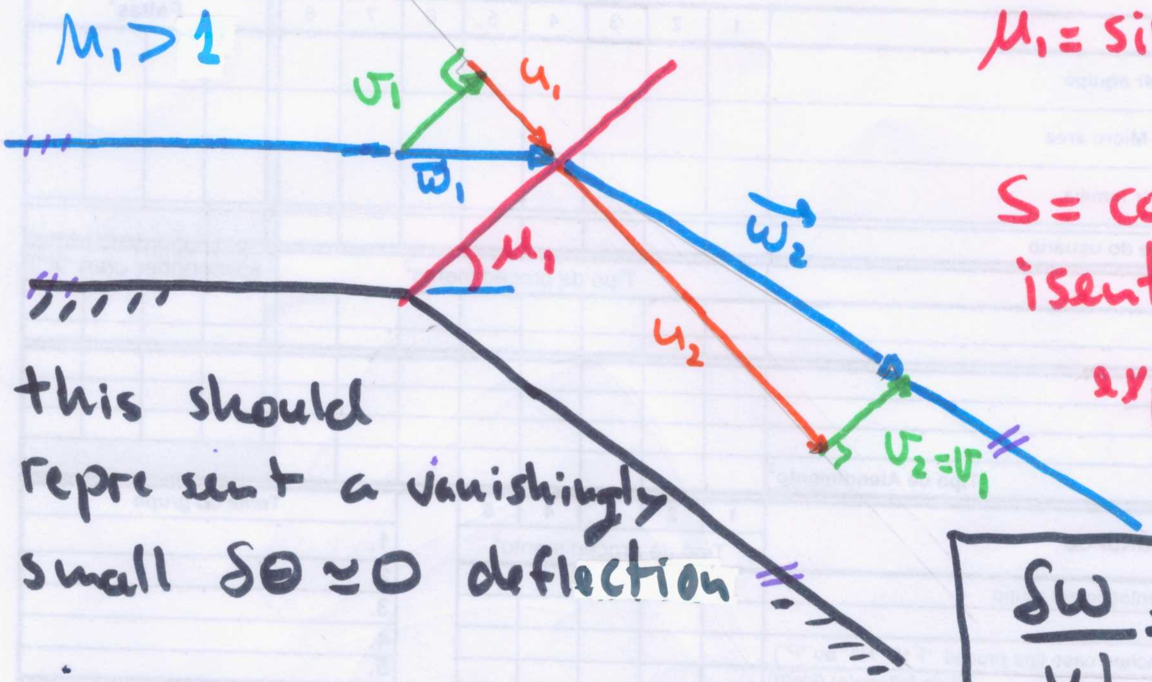
Prandtl - Meyer flow

$\mu_1 \Rightarrow$ Mach angle

$$\mu_1 = \sin^{-1}\left(\frac{1}{M_1}\right)$$

S = constant

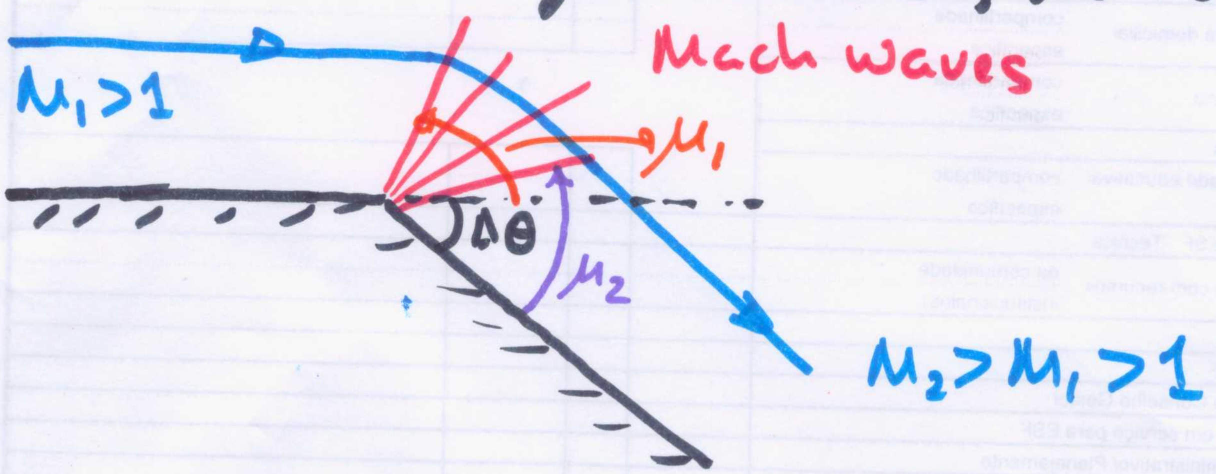
isentropic expansion



this should represent a vanishingly small $\delta\theta \simeq 0$ deflection

$$\boxed{\frac{\delta w}{w} = \frac{\delta\theta}{\sqrt{M^2-1}}}$$

For a finite angle deflection, one has:

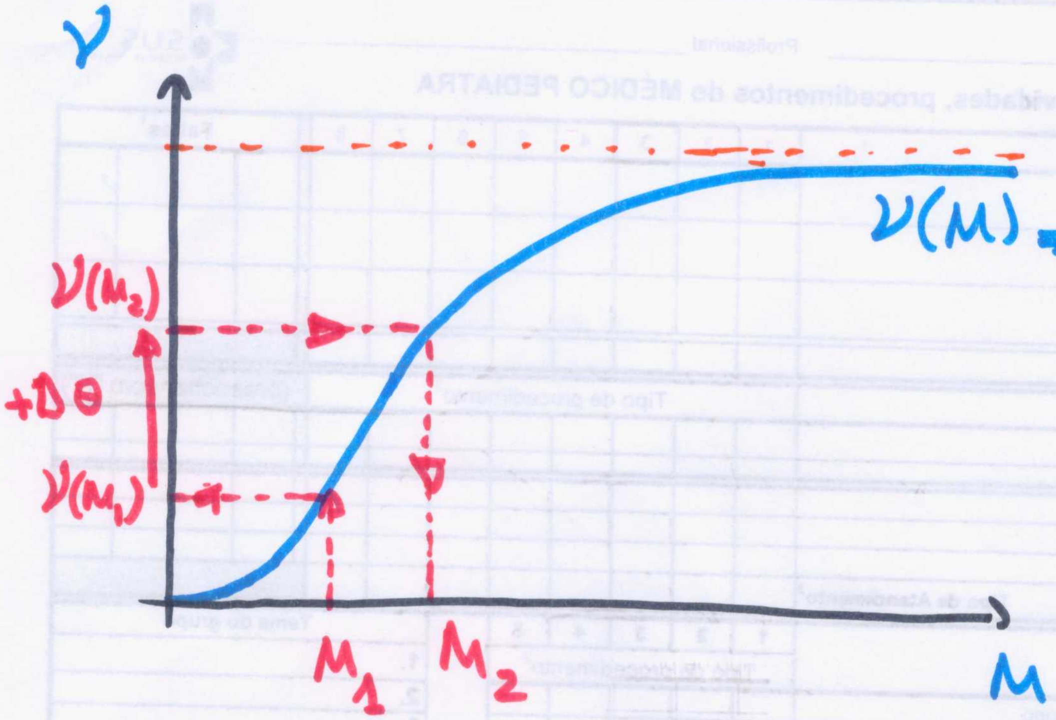


Prandtl - Meyer Function:

$$\nu(M) = \sqrt{\frac{(\gamma+1)}{(\gamma-1)}}\,\tan^{-1}\left[\sqrt{\frac{(\gamma-1)}{(\gamma+1)}(M^2-1)}\right] - \tan^{-1}\left[\sqrt{M^2-1}\right]$$

compression $\nu\downarrow$: $\nu = \nu_1(M_1) - |\theta - \theta_1|$

expansion $\nu\uparrow$: $\nu = \nu_1(M_1) + |\theta - \theta_1|$



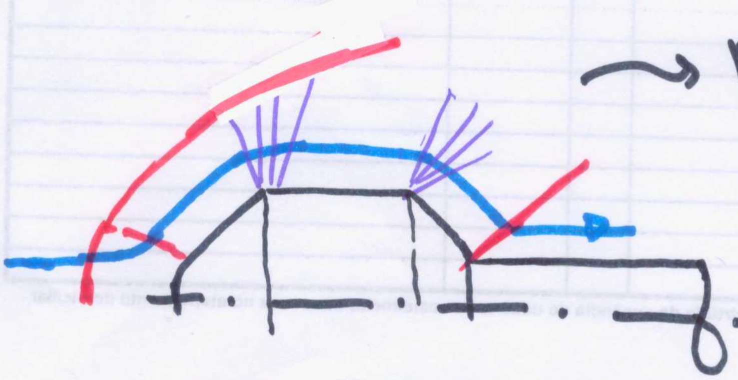
$\nu(M) \Rightarrow$ Prandtl-Meyer function

$$\nu(M_2) = \nu(M_1) + \Delta\theta$$

↳ expansion

$$\nu(M_2) = \nu(M_1) - \Delta\theta$$

↳ compression



→ Boat tail section.

↳ rocket fuzelage

— shock waves

— stream line

— Prandtl-Meyer fans