Variáveis Aleatórias Contínuas

1) X: renda (milhares de reais)

$$g(x) = \begin{cases} \frac{1}{10}x + \frac{1}{10}, & \text{se } 0 \leq x < 2 \\ -\frac{3}{40}x + \frac{9}{20}, & \text{se } 2 \leq x \leq 6 \\ 0, & \text{caso contrário} \end{cases}$$

a) Renda Média: $E(x) = \int_{-\infty}^{+\infty} x f(x) dx$

$$E(x) = \int_0^2 x\left[\frac{1}{10}x + \frac{1}{10}\right]dx + \int_2^6 x\left[-\frac{3}{40}x + \frac{9}{20}\right]dx$$

$$= \int_0^2 \left[\frac{x^2}{10} + \frac{x}{10}\right]dx + \int_2^6 \left[-\frac{3x^2}{40} + \frac{9x}{20}\right]dx$$

$$= \frac{1}{10}\int_0^2 x^2 dx + \frac{1}{10}\int_0^2 x\,dx - \frac{3}{40}\int_2^6 x^2 dx + \frac{9}{20}\int_2^6 x\,dx$$

$$= \frac{1}{10}\cdot\frac{x^3}{3}\Big|_0^2 + \frac{1}{10}\cdot\frac{x^2}{2}\Big|_0^2 - \frac{3}{40}\cdot\frac{x^3}{3}\Big|_2^6 + \frac{9}{20}\cdot\frac{x^2}{2}\Big|_2^6$$

$$= \frac{1}{10}\left(\frac{8}{3}\right) + \frac{1}{10}(2) - \frac{3}{40}\left(\frac{216}{3} - \frac{8}{3}\right) + \frac{9}{20}\left(\frac{36}{2} - \frac{4}{2}\right)$$

$$= 0{,}27 + 0{,}20 - 5{,}20 + 7{,}20 = \underline{2{,}47}$$

A renda média é de R$ 2.470,00

b) $P[\text{renda} > 3.000] = P[X > 3]$

$$P[X>3] = \int_3^6 \left[-\frac{3x}{40} + \frac{9}{20}\right]dx = -\frac{3}{40}\int_3^6 x\,dx + \frac{9}{20}\int_3^6 x^0 dx$$

$$= -\frac{3}{40}\cdot\frac{x^2}{2}\Big|_3^6 + \frac{9}{20}\cdot x\Big|_3^6 = -\frac{3}{40}\left(\frac{36}{2} - \frac{9}{2}\right) + \frac{9}{20}(6-3) = -1{,}0125 + 1{,}35$$

$$P[X>3] = \underline{0{,}3375}$$

c) Renda mediana: Mediana é o valor de renda cujo 50% da população está (ou recebe) abaixo deste valor e os outros 50% está (ou recebe) acima deste valor. Logo,

$$\int_{-\infty}^{m} g(x)\,dx = 0{,}5$$

Investigar a 1ª parte da função:

$$\int_0^2 \left[\frac{1}{10}x + \frac{1}{10}\right]dx = \frac{1}{10}\frac{x^2}{2}\Big|_0^2 + \frac{1}{10}x\Big|_0^2 = \frac{1}{10}(2) + \frac{1}{10}(2) = \underline{0{,}40}$$

Logo, o valor da renda mediana (m) será obtida na segunda parte da função.

$$\underbrace{\int_0^2 \left[\frac{1}{10}x + \frac{1}{10}\right]dx}_{0{,}40} + \int_2^m \left[-\frac{3}{40}x + \frac{9}{20}\right]dx = 0{,}5$$

$$\int_2^m \left[-\frac{3}{40}x + \frac{9}{20}\right]dx = 0{,}50 - 0{,}40$$

$$-\frac{3}{40}\cdot\frac{x^2}{2}\Big|_2^m + \frac{9}{20}\cdot x\Big|_2^m = 0{,}10$$

$$-\frac{3}{80}(m^2-4) + \frac{9}{20}(m-2) = 0{,}10$$

$$-\frac{3m^2}{80} + \frac{12}{80} + \frac{9m}{20} - \frac{18}{20} = 0{,}10$$

$$-\frac{3m^2}{80} + \frac{9m}{20} - \frac{15}{20} = 0{,}10$$

$$-0{,}0375m^2 + 0{,}45m - 0{,}85 = 0$$

$$\Delta = (0{,}45)^2 - 4(-0{,}0375)(-0{,}85) = 0{,}0750$$

$$m = \frac{-0{,}45 \pm \sqrt{0{,}0750}}{2(-0{,}0375)} = \begin{cases} m' = 9{,}65 \rightarrow \text{fora do intervalo} \\ m'' = 2{,}35 \rightarrow \text{OK} \end{cases}$$

A renda mediana é de R$ 2.350,00

2) $g(x) = \begin{cases} K(2-x), & \text{se } 0 \leq x \leq 1 \\ 0, & \text{caso contrário} \end{cases}$

a) Encontrar K para ser uma f.d.p.

$$\int_{-\infty}^{+\infty} g(x)\,dx = 1$$

$$\int_0^1 K(2-x)\,dx = 1 \Rightarrow \int_0^1 (2K - Kx)\,dx = 1 \Rightarrow \int_0^1 2K\,dx - \int_0^1 Kx\,dx = 1$$

$$\Rightarrow 2K\int_0^1 x^0\,dx - K\int_0^1 x\,dx = 1 \Rightarrow 2K\left(x\Big|_0^1\right) - K\left(\frac{x^2}{2}\Big|_0^1\right) = 1$$

$$\Rightarrow 2K(1) - K\left(\frac{1}{2}\right) = 1$$

$$\Rightarrow \frac{4K - K}{2} = 1 \Rightarrow 3K = 2 \Rightarrow \boxed{K = \frac{2}{3}}$$

b) Calcular a média e variância de x

$$E(X) = \int_{-\infty}^{+\infty} x\,g(x)\,dx = \int_0^1 x\left[\frac{2}{3}(2-x)\right]dx$$

$$= \frac{4}{3}\int_0^1 x\,dx - \frac{2}{3}\int_0^1 x^2\,dx$$

$$= \frac{4}{3}\cdot\frac{x^2}{2}\Big|_0^1 - \frac{2}{3}\cdot\frac{x^3}{3}\Big|_0^1 = \frac{4}{3}\left(\frac{1}{2}-0\right) - \frac{2}{3}\left(\frac{1}{3}-0\right) = \frac{2}{3} - \frac{2}{9}$$

$$= \underline{0{,}44}$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 g(x)\,dx = \int_0^1 x^2\left[\frac{2}{3}(2-x)\right]dx = \frac{4}{3}\int_0^1 x^2\,dx - \frac{2}{3}\int_0^1 x^3\,dx$$

$$= \frac{4}{3}\,\frac{x^3}{3}\Big|_0^1 - \frac{2}{3}\,\frac{x^4}{4}\Big|_0^1 = \frac{4}{9} - \frac{2}{12} = 0{,}2778$$

$$VAR(X) = E(X^2) - [E(X)]^2 = 0{,}2778 - [0{,}44]^2 = \underline{0{,}08}$$

c) $P[0{,}2 < X < 0{,}8] = \int_{0,2}^{0,8} \frac{2}{3}(2-x)\,dx$

$= \frac{4}{3}\int_{0,2}^{0,8} dx - \frac{2}{3}\int_{0,2}^{0,8} x\,dx$

$= \frac{4}{3} \cdot x \Big|_{0,2}^{0,8} - \frac{\cancel{2}}{3} \cdot \frac{x^2}{\cancel{2}} \Big|_{0,2}^{0,8}$

$= \frac{4}{3}(0{,}8 - 0{,}2) - \frac{1}{3}(0{,}8^2 - 0{,}2^2)$

$= 0{,}8000 - 0{,}2000 = \underline{0{,}6000}$

d) $P[X < 0{,}6] = \int_0^{0,6} \frac{2}{3}(2-x)\,dx = \frac{4}{3} x \Big|_0^{0,6} - \frac{\cancel{2}}{3}\frac{x^2}{\cancel{2}} \Big|_0^{0,6}$

$= \frac{4}{3}(0{,}6) - \frac{1}{3}(0{,}6^2) = 0{,}8000 - 0{,}1200 = \underline{0{,}6800}$

e) $P[X > 0{,}35] = \int_{0,35}^{1} \frac{2}{3}(2-x)\,dx = \frac{4}{3} x \Big|_{0,35}^{1} - \frac{1}{3} x^2 \Big|_{0,35}^{1}$

$= \frac{4}{3}(1 - 0{,}35) - \frac{1}{3}(1 - 0{,}35^2)$

$= 0{,}8667 - 0{,}2925 = \underline{0{,}5742}$

3) $$g(x) = \begin{cases} K(2x - x^2), & \text{se } 0 \leq x \leq 1 \\ 0, & \text{caso contrário} \end{cases}$$

a) Encontrar K para ser uma f.d.p.

$$\int_{-\infty}^{+\infty} g(x)\,dx = 1 \Rightarrow \int_0^1 K(2x - x^2)\,dx = 1 \Rightarrow$$

$$\int_0^1 2Kx - Kx^2\,dx = 1 \Rightarrow 2K\int_0^1 x\,dx - K\int_0^1 x^2\,dx = 1 \Rightarrow$$

$$2K \cdot \frac{x^2}{2}\Big|_0^1 - K \cdot \frac{x^3}{3}\Big|_0^1 = 1 \Rightarrow K(1^2 - 0) - K\left(\frac{1}{3} - 0\right) = 1 \Rightarrow$$

$$K - \frac{K}{3} = 1 \Rightarrow \frac{3K - K}{3} = 1 \Rightarrow \frac{2K}{3} = 1 \Rightarrow K = \frac{3}{2}$$

b) Média e Variância de X

$$E(X) = \int_{-\infty}^{+\infty} x\,g(x)\,dx = \int_0^1 x\left[\frac{3}{2}(2x - x^2)\right]dx = \int_0^1 3x^2 - \frac{3x^3}{2}\,dx$$

$$= 3 \cdot \frac{x^3}{3}\Big|_0^1 - \frac{3}{2} \cdot \frac{x^4}{4}\Big|_0^1 = 3\left(\frac{1}{3}\right) - \frac{3}{2}\left(\frac{1}{4}\right) = 1 - \frac{3}{8} = 0{,}6250$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 g(x)\,dx = \int_0^1 x^2\left[\frac{3}{2}(2x - x^2)\right]dx = \int_0^1 3x^3 - \frac{3x^4}{2}\,dx$$

$$= 3 \cdot \frac{x^4}{4}\Big|_0^1 - \frac{3}{2} \cdot \frac{x^5}{5}\Big|_0^1 = 3\left(\frac{1}{4}\right) - \frac{3}{2}\left(\frac{1}{5}\right) = 0{,}4500$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= 0{,}4500 - [0{,}6250]^2 = 0{,}059$$

c) $P[0<X<0,5] = \int_0^{0,5} \frac{3}{2}(2x-x^2)\,dx$

$$= \int_0^{0,5} 3x - \frac{3x^2}{2}\,dx = 3\int_0^{0,5} x\,dx - \frac{3}{2}\int_0^{0,5} x^2\,dx$$

$$= 3.\frac{x^2}{2}\Big|_0^{0,5} - \frac{\cancel{3}}{2}\frac{x^3}{\cancel{3}}\Big|_0^{0,5}$$

$$= 3\left(\frac{0,5^2}{2}\right) - \frac{1}{2}(0,5^3) = 0,3750 - 0,0625 = \underline{0,3125}$$

d) $P[0,1<X<0,9] = \int_{0,1}^{0,9} \frac{3}{2}(2x-x^2)\,dx = \int_{0,1}^{0,9} 3x - \frac{3x^2}{2}\,dx$

$$= 3.\frac{x^2}{2}\Big|_{0,1}^{0,9} - \frac{\cancel{3}}{2}\frac{x^3}{\cancel{3}}\Big|_{0,1}^{0,9} = \frac{3}{2}(0,9^2 - 0,1^2) - \frac{1}{2}(0,9^3 - 0,1^3)$$

$$= 1,2000 - 0,3640 = \underline{0,8360}$$

4) $g(x) = \begin{cases} 0, & \text{se } x<0 \\ 2e^{-2x}, & \text{se } x \geq 0 \end{cases}$

a) Mostrar que é uma f.d.p

$$\int_{-\infty}^{+\infty} g(x)\,dx = 1 \Rightarrow \int_0^{+\infty} 2e^{-2x}\,dx = 1 \Rightarrow 2\int_0^{+\infty} e^{\overset{u}{-2x}}\,dx = 1$$

Fazendo $u = -2x$ temos que: $\frac{du}{dx} = -2 \Rightarrow dx = -\frac{1}{2}du$

se $x=0 \Rightarrow u=0$

se $x=+\infty \Rightarrow u=-\infty$

Logo,

$$2\int_0^{-\infty} e^u\left(-\frac{1}{2}du\right) = 1 \Rightarrow -2\int_{-\infty}^{0} e^u\left(-\frac{1}{2}du\right) = 1 \Rightarrow \int_{-\infty}^{0} e^u\,du = 1$$

$$\Rightarrow e^u\Big|_{-\infty}^{0} = 1 \Rightarrow e^0 - e^{-\infty} = 1 \Rightarrow 1 - 0 = 1 \quad \underline{\text{c.q.d}}$$

b) $P[X > 10] = \int_{10}^{+\infty} 2e^{-2x}\,dx$ $\quad -2x \to u$

se $x = 10 \Rightarrow u = -20$

se $x = +\infty \Rightarrow u = -\infty$

Então, $\int_{-\infty}^{-20} e^{u}\,du \Rightarrow e^{u}\Big|_{-\infty}^{-20} = e^{-20} - \underbrace{e^{-\infty}}_{0} \approx \underline{0{,}0000}$

usando computador (R)

$= \underline{2{,}06 \cdot 10^{-9}}$