2.2 Restrições de igualdade

Consideremos o problema de minimização com restrições gerais de igualdade:

$$\begin{array}{c} \text{Minimizar } f(x) \\ h(x) = 0 \end{array} \tag{2.2.1}$$

onde $h : \mathbb{R}^n \to \mathbb{R}^m$. Como sempre, chamamos Ω ao conjunto factível do problema. Neste caso $\Omega = \{x \in \mathbb{R}^n \mid h(x) = 0\}$.

Definição 2.2.1 Se $x \in \Omega$, chamamos *conjunto tangente a* Ω *por* x (denotado por $M(x)$) ao conjunto dos vetores tangentes a curvas em Ω passando por x, ou seja:

$$M(x) = \{v \in I\!R^n \mid v = \gamma'(0) \text{ para alguma curva } \gamma \text{ passando por } x\}\ .$$

Utilizando a notação

$$h'(x) = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}(x) & \dots & \frac{\partial h_1}{\partial x_n}(x) \\ \vdots & & \\ \frac{\partial h_m}{\partial x_1}(x) & \dots & \frac{\partial h_m}{\partial x_n}(x) \end{pmatrix} = \begin{pmatrix} h'_1(x) \\ \vdots \\ h'_m(x) \end{pmatrix} = \begin{pmatrix} \nabla h_1(x)^T \\ \vdots \\ \nabla h_m(x)^T \end{pmatrix},$$

podemos relacionar $M(x)$ com o núcleo do Jacobiano de $h(x)$, denotado por $\mathcal{N}(h'(x))$, pelo seguinte lema:

Lema 2.2.2
Para todo $x \in \Omega$, $M(x) \subset \mathcal{N}(h'(x))$.

Prova: Seja $v \in M(x)$ e $\gamma : [-\varepsilon, \varepsilon] \to \Omega$ tal que $\gamma'(0) = v$, $\gamma(0) = x$. Definimos $\Phi(t) = h(\gamma(t))$, para todo $t \in [-\varepsilon, \varepsilon]$. Portanto, $\Phi(t) = 0$ para todo $t \in [-\varepsilon, \varepsilon]$. Logo, $\Phi'(t) \equiv (\Phi_1(t), \dots, \Phi_m(t))^T = 0$ para todo $t \in (-\varepsilon, \varepsilon)$. Mas, pela regra da cadeia, $\Phi'(t) = h'(\gamma(t))\gamma'(t)$, portanto

$$h'(\gamma(t))\gamma'(t) = 0$$

para todo $t \in (-\varepsilon, \varepsilon)$. Logo, $0 = h'(x)\gamma'(0) = h'(x)v$, ou seja, $v \in \mathcal{N}(h'(x))$.
QED

É natural que nos indaguemos sobre a validade da recíproca do Lema 2.2.2: $\mathcal{N}(h'(x)) \subset M(x)$? Em geral esta relação não é verdadeira, conforme ilustra o seguinte exemplo. Consideremos $h(x_1, x_2) = x_1 x_2$, $x = (\ 0,\ 0\ )^T$. Então $M(x) = \{v \in I\!R^2 \mid v_1 v_2 = 0\}$, mas $h'(x) = (0, 0)$ e, claramente, $\mathcal{N}(h'(x)) = I\!R^2$.

Definição 2.2.3
Dizemos que $x \in \Omega \equiv \{x \in I\!R^n \mid h(x) = 0\}$ é um *ponto regular* se o posto de $h'(x)$ é igual a m ($\{\nabla h_1(x), \dots, \nabla h_m(x)\}$ é um conjunto linearmente independente).

Teorema 2.2.4

(LICQ implica Abadie - restrições de igualdade)

Seja $\Omega = \{x \in I\!R^n \mid h(x) = 0\}$*,* $h \in C^k$*,* $x \in \Omega$ *um ponto regular. Então, para todo* $v \in \mathcal{N}(h'(x))$*, existe uma curva* γ *de classe* C^k *passando por* x *tal que* $\gamma'(0) = v$*. Portanto,* $M(x) = \mathcal{N}(h'(x))$*.*

Prova: Seja $v \in \mathcal{N}(h'(x))$. Então $h'(x)v = 0$. Queremos encontrar uma curva γ em Ω passando por x tal que $\gamma'(0) = v$. Consideramos o sistema de equações

$$h(x + tv + h'(x)^T u) = 0 \ , \tag{2.2.2}$$

Para x e v fixos, este é um sistema de m equações com $m+1$ variáveis ($u \in I\!R^m$ e $t \in I\!R$). Colocando $u = 0, t = 0$ temos uma solução particular deste sistema. O Jacobiano de (2.2.2) em relação a u em $t = 0$ é $h'(x)h'(x)^T \in I\!R^{m\times m}$ e é não singular pela regularidade de x. Logo, pelo Teorema da Função Implícita, existe $\bar{\gamma} \in C^k$, definida em $[-\varepsilon, \varepsilon]$, $\varepsilon > 0$, tal que (2.2.2) se verifica se e somente se $u = \bar{\gamma}(t)$. Portanto

$$h(x + tv + h'(x)^T \bar{\gamma}(t)) = 0 \text{ para todo } t \in [-\varepsilon, \varepsilon] \ . \tag{2.2.3}$$

Derivando (2.2.3) em relação a t, para $t = 0$ temos $h'(x)(v + h'(x)^T \bar{\gamma}'(0)) = 0$. Como $h'(x)v = 0$, segue que $h'(x)h'(x)^T \bar{\gamma}'(0) = 0$. Mas $h'(x)h'(x)^T$ é não singular, logo $\bar{\gamma}'(0) = 0$.
Em conseqüência, definindo $\gamma : [-\varepsilon, \varepsilon] \to \Omega$ por

$$\gamma(t) = x + tv + h'(x)^T \bar{\gamma}(t),$$

temos que

$$\gamma'(0) = v + h'(x)^T \bar{\gamma}'(0) = v.$$

Assim, γ é a curva procurada. Como v é arbitrário, temos que $\mathcal{N}(h'(x)) \subset M(x)$. Portanto, $M(x) = \mathcal{N}(h'(x))$. **QED**

5.2.2 *Optimality Conditions for Nonlinear Programs*

In the remainder of this section we consider nonlinear programming problems of the form

$$(NLP) \qquad \begin{array}{ll} \operatorname*{Min}_{x \in \mathbb{R}^n} & f(x) \\ \text{subject to} & g_i(x) = 0, \; i = 1, \dots, q, \\ & g_i(x) \le 0, \; i = q+1, \dots, p. \end{array} \qquad (5.99)$$

We assume that the objective function $f : \mathbb{R}^n \to \mathbb{R}$ and the constraint mapping $G := (g_1, \dots, g_p) : \mathbb{R}^n \to \mathbb{R}^p$ are twice continuously differentiable. We discuss now first and second order optimality conditions for the above problem (NLP).

Denote by $I(x)$ the set of inequality constraints active at x,

$$I(x) := \{i : g_i(x) = 0, \; i = q+1, \dots, p\}.$$

It is natural to associate with a feasible point $\bar{x}$ of the problem (NLP) the problem that is obtained by linearization of the objective function and of the equality and active inequality constraints at $\bar{x}$, that is,

$$
\begin{array}{lll}
\operatorname{Min}_{h \in \mathbb{R}^n} & Df(\bar{x})h, & \\
\text{subject to} & Dg_i(\bar{x})h = 0, & i = 1, \dots, q, \\
& Dg_i(\bar{x})h \le 0, & i \in I(\bar{x}).
\end{array}
\tag{5.100}
$$

There is another possible linearization, based on the following simple observation. If a point $\bar{x}$ is a locally optimal solution of (NLP), then $(\bar{x}, 0)$ is a locally optimal solution of the nonlinear programming problem

$$
\begin{array}{lll}
\operatorname*{Min}_{(x,z) \in \mathbb{R}^n \times \mathbb{R}} z & \text{subject to} & g_i(x) = 0,\ i = 1, \dots, q, \\
& & g_i(x) \le z,\ i = q+1, \dots, p, \\
& & f(x) - f(\bar{x}) \le z.
\end{array}
\tag{5.101}
$$

The corresponding problem linearized at $(\bar{x}, 0)$ is

$$
\begin{array}{lll}
\operatorname*{Min}_{(h,z) \in \mathbb{R}^n \times \mathbb{R}} z & \text{subject to} & Dg_i(\bar{x})h = 0,\ i = 1, \dots, q, \\
& & Dg_i(\bar{x})h \le z,\ i \in I(\bar{x}), \\
& & Df(\bar{x})h \le z.
\end{array}
\tag{5.102}
$$

The Lagrangian and generalized Lagrangian functions associated with problem (NLP) are defined, respectively, as follows:

$$
L(x, \lambda) := f(x) + \sum_{i=1}^{p} \lambda_i g_i(x), \quad L^g(x, \lambda_0, \lambda) := \lambda_0 f(x) + \sum_{i=1}^{p} \lambda_i g_i(x),
$$

where $\lambda_0 \in \mathbb{R}$ and $\lambda \in \mathbb{R}^p$.

Definition 5.46 *The set $\Lambda^g(\bar{x})$ of generalized Lagrange multipliers at a feasible point $\bar{x}$ of the problem (NLP) is defined as the set of nonzero vectors $(\lambda_0, \lambda) = (\lambda_0, \lambda_1, \dots, \lambda_p)$ satisfying the following first order optimality conditions:*

$$
D_x L^g(\bar{x}, \lambda_0, \lambda) = 0,\ \lambda_0 \ge 0, \text{ and } \lambda_i \ge 0,\ \lambda_i g_i(\bar{x}) = 0,\ i = q+1, \dots, p.
$$

If a generalized Lagrange multiplier (λ_0, λ) *is such that* $\lambda_0 = 0$, *then we say that* λ *is a* singular Lagrange *multiplier. If* $\lambda_0 = 1$, *then we say that* λ *is a* Lagrange multiplier. *The set of Lagrange multipliers is denoted by* $\Lambda(\bar{x})$.

It is not difficult to verify (see proposition 3.14) that, if the set of Lagrange multipliers $\Lambda(\bar{x})$ is nonempty, then the set of singular Lagrange multipliers, together with 0, forms the recession cone of $\Lambda(\bar{x})$.

Let $\bar{x}$ be a feasible point of (NLP). We say that the Mangasarian-Fromovitz (MF) constraint qualification holds at $\bar{x}$ if the following conditions are satisfied:

(i) The vectors $Dg_i(\bar{x})$, $i = 1, \dots, q$, are linearly independent,
(ii) $\exists \bar{h} \in \mathbb{R}^n : Dg_i(\bar{x})\bar{h} = 0,\ i = 1, \dots, q;\ Dg_i(\bar{x})\bar{h} < 0,\ i \in I(\bar{x})$.

Proposition 5.47 *Let $\bar{x}$ be a locally optimal solution of (NLP). Then the set $\Lambda^g(\bar{x})$ of generalized Lagrange multipliers is nonempty, and the following conditions are equivalent:* (i) *the Mangasarian-Fromovitz constraint qualification holds at $\bar{x}$,* (ii) *the set of singular Lagrange multipliers is empty,* (iii) *the set* $\Lambda(\bar{x})$ *of Lagrange multipliers is nonempty and bounded.*

Proof. Consider first the case where the vectors $Dg_i(\bar{x})$, $i = 1, \dots, q$, are not linearly independent. Then there exists a nonzero vector $\mu \in \mathbb{R}^q$ such that $\sum_{i=1}^q \mu_i Dg_i(\bar{x}) = 0$. Setting $\lambda_i := \mu_i$, $i = 1, \dots, q$, and $\lambda_i := 0$, $i = q+1, \dots, p$, we obtain that λ is a singular Lagrange multiplier.

Suppose now that vectors $Dg_i(\bar{x})$, $i = 1, \dots, q$, are linearly independent. We claim that the optimal value of the linearized problem (5.102) is zero. Indeed, if this is false, there would exist $h \in \mathbb{R}^n$ such that $Dg_i(\bar{x})h = 0$, $i = 1, \dots, q$, $Df(\bar{x})h < 0$, and $Dg_i(\bar{x})h < 0$, for all $i \in I(\bar{x})$. Since it is assumed that $Dg_i(\bar{x})$, $i = 1, \dots, q$, are linearly independent, the implicit function theorem implies that there exists a path $x(t) = \bar{x} + th + o(t)$, $t \geq 0$, such that $g_i(x(t)) = 0$, $i = 1, \dots, q$, for $t > 0$ small enough. We have for each $i \in I(\bar{x})$ that $g_i(x(t)) = tDg_i(\bar{x})h + o(t) < 0$, and hence $x(t)$ is feasible. At the same time, $f(x(t)) = f(\bar{x}) + tDf(\bar{x})h + o(t) < f(\bar{x})$, which contradicts the local optimality of $\bar{x}$.

Problem (5.102) is a linear programming problem, and its dual is given by

$$\begin{aligned} &\operatorname*{Max}_{(\lambda_0,\lambda)} \quad 0 \quad \text{subject to} \\ &D_x L^g(\bar{x}, \lambda_0, \lambda) = 0, \;\; \lambda_0 + \sum_{i \in I(\bar{x})} \lambda_i = 1, \;\; \lambda_i \geq 0, \;\; i \in I(\bar{x}) \cup \{0\}. \end{aligned}$$

Since zero is the optimal value of problem (5.102), we obtain by theorem 5.44 that the optimal value of its dual is also zero. Consequently, the feasible set of the dual problem is nonempty. It is not difficult to see that any feasible point (λ_0, λ) of the above dual problem is a generalized Lagrange multiplier, associated with $\bar{x}$, for problem (NLP). This shows that the set $\Lambda^g(\bar{x})$ is nonempty.

We prove now that the MF-constraint qualification implies that the optimal value of the linearized problem (5.100) is zero. Indeed, let h be a feasible point of problem (5.100) and $\bar{h}$ a vector satisfying condition (ii) of the MF-constraint qualification. For a given $\varepsilon > 0$ consider $h_\varepsilon := h + \varepsilon\bar{h}$. Since $Dg_i(\bar{x})h_\varepsilon < 0$ for all $i \in I(\bar{x})$ and $\varepsilon > 0$ small enough, by arguments similar to those used at the beginning of this proof we obtain existence of a path $x(t) = \bar{x} + th_\varepsilon + o(t)$ such that $g_i(x(t)) = 0$, $i = 1, \dots, q$, and $g_i(x(t)) \leq 0$, $i = q+1, \dots, p$, for $t > 0$ small enough, and hence this path is feasible for small enough $t > 0$. Since $\bar{x}$ is a local minimizer, it follows that

$$Df(\bar{x})h_\varepsilon = \lim_{t \downarrow 0} \frac{f(\bar{x} + th_\varepsilon) - f(\bar{x})}{t} \geq 0.$$

This being true for any $\varepsilon > 0$ small enough, it follows that $Df(\bar{x})h \geq 0$ for any feasible point h of problem (5.100). This proves that $h = 0$ is an optimal solution of (5.100).

Problem (5.100) is a linear programming problem, and its dual can be written in the form

$$\begin{aligned} &\operatorname{Max}_{\lambda} \quad 0 \\ &\text{subject to} \quad D_x L(\bar{x}, \lambda) = 0, \;\; \lambda_i \geq 0, \;\; i \in I(\bar{x}). \end{aligned}$$

If the optimal value of (5.100) is zero, then by theorem 5.44, the optimal value of its dual is also zero, and hence the feasible set of the dual is nonempty. We obtain that the MF-constraint qualification implies the existence of a Lagrange multiplier.

Finally, we apply proposition 5.45, in which x corresponds to h, a_i to $g_i(x)$, and b is zero. Relation (i) of this proposition is nothing but the MF-constraint qualification, while (5.97) characterizes generalized Lagrange multipliers (except if $\lambda = 0$). So equivalence of (i) and (ii) of this proposition means that the MF-constraint qualification holds iff the set of singular Lagrange multipliers is empty. If the latter is satisfied, then since the linearized problem (5.100) has value 0, by proposition 5.45 the set of Lagrange multipliers is nonempty and bounded. Conversely, let the set of Lagrange multipliers be nonempty and bounded. Then the corresponding linear program (5.100) has value 0, and it follows from proposition 5.45 again that the MF-constraint qualification holds. This completes the proof. □

We discuss now second order optimality conditions. The critical cone associated with a feasible point $\bar{x}$ of (NLP) can be written as follows:

$$C(\bar{x}) := \{h : Df(\bar{x})h \leq 0,\ Dg_i(\bar{x})h = 0,\ i \leq q,\ Dg_i(\bar{x})h \leq 0,\ i \in I(\bar{x})\}.$$

Its elements are called critical directions. Recall that if the set $\Lambda(\bar{x})$ of Lagrange multipliers is nonempty, then the inequality $Df(\bar{x})h \leq 0$ in the definition of the critical cone $C(\bar{x})$ can be replaced by the equation $Df(\bar{x})h = 0$, and for any $\bar{\lambda} \in \Lambda(\bar{x})$ the critical cone can be written in the form (5.77).

We denote by $I(\bar{x}, h)$ the set of constraints active at $\bar{x}$ that are also active, up to the first order, in a direction h, i.e.,

$$I(\bar{x}, h) := \{i \in I(\bar{x}) : Dg_i(\bar{x})h = 0\}.$$

Proposition 5.48 (Second order optimality conditions) *Let $\bar{x}$ be a feasible point of problem (NLP). Then the following holds.*

(i) *If $\bar{x}$ is a locally optimal solution of (NLP), then for every $h \in C(\bar{x})$ there exists a generalized Lagrange multiplier $(\lambda_0, \lambda) \in \Lambda^g(\bar{x})$ such that*

$$D^2_{xx}L^g(\bar{x}, \lambda_0, \lambda)(h, h) \geq 0. \tag{5.103}$$

(ii) *If for each $h \in C(\bar{x}) \setminus \{0\}$ there exists $(\lambda_0, \lambda) \in \Lambda^g(\bar{x})$ such that*

$$D^2_{xx}L^g(\bar{x}, \lambda_0, \lambda)(h, h) > 0, \tag{5.104}$$

then $\bar{x}$ is a locally optimal solution of (NLP) satisfying the quadratic growth condition.

Proof. (i) Let $h \in C(\bar{x})$ be a critical direction. Consider first the case where vectors $Dg_i(\bar{x})$, $i = 1, \dots, q$, are not linearly independent. Then there exists a singular Lagrange multiplier $\hat{\lambda}$ with $\hat{\lambda}_i = 0$, for all $i = q+1, \dots, p$. If (5.103) holds with $\lambda := \hat{\lambda}$, we are done, otherwise since $-\hat{\lambda}$ is another singular Lagrange multiplier, and

$$D^2_{xx}L^g(\bar{x}, 0, -\hat{\lambda})(h, h) = -D^2_{xx}L^g(\bar{x}, 0, \hat{\lambda})(h, h),$$

we have that (5.103) is satisfied for $\lambda := -\hat{\lambda}$.

We now discuss the case where vectors $Dg_i(\bar{x})$, $i = 1, \dots, q$, are linearly independent. Consider the linear program

$$\begin{array}{lll} \underset{(w,z)\in I\!R^n\times I\!R}{\text{Min}} & z \text{ s.t.} & Df(\bar{x})w + D^2 f(\bar{x})(h,h) \le z, \\ & & Dg_i(\bar{x})w + D^2 g_i(\bar{x})(h,h) = 0,\ i = 1,\dots,q, \\ & & Dg_i(\bar{x})w + D^2 g_i(\bar{x})(h,h) \le z,\ i \in I(\bar{x},h). \end{array} \tag{5.105}$$

The optimal value of this problem is nonnegative. Indeed, otherwise there exists w which satisfies

$$\begin{array}{lll} Df(\bar{x})w + D^2 f(\bar{x})(h,h) & < 0, & \\ Dg_i(\bar{x})w + D^2 g_i(\bar{x})(h,h) & = 0, & i = 1,\dots,q, \\ Dg_i(\bar{x})w + D^2 g_i(\bar{x})(h,h) & < 0, & i \in I(\bar{x},h). \end{array} \tag{5.106}$$

Since $\bar{x}$ is feasible and h is a critical direction, and hence $g_i(\bar{x}) = 0$ and $Dg_i(\bar{x})h = 0$ for $i = 1, \dots, q$, and by the second equation in (5.106) we have

$$g_i(\bar{x} + th + \tfrac{1}{2}t^2 w) = \tfrac{1}{2}t^2[Dg_i(\bar{x})w + D^2 g_i(\bar{x})(h,h)] + o(t^2) = o(t^2).$$

The Implicit Function Theorem implies then that there exists a path $x(t) = \bar{x} + th + \frac{1}{2}t^2 w + o(t^2)$ such that $g_i(x(t)) = 0$, $i = 1, \dots, q$, for $t > 0$ small enough. Then, by a second order Taylor expansion, we have for $t > 0$ small enough that

$$f(x(t)) = f(\bar{x}) + tDf(\bar{x})h + \tfrac{1}{2}t^2[Df(\bar{x})w + D^2 f(\bar{x})(h,h)] + o(t^2) < f(\bar{x}),$$

and similarly $g_i(x(t)) < 0$, for all $i \in I(\bar{x},h)$. If $i > q$ and $i \notin I(\bar{x},h)$, then either $g_i(\bar{x}) < 0$, or $g_i(\bar{x}) = 0$ and $Dg_i(\bar{x})h < 0$, and in both cases $g_i(x(t)) < 0$, for small enough $t > 0$. Therefore, for $t > 0$ small enough, $x(t)$ is feasible and $f(x(t)) < f(\bar{x})$, which contradicts the local optimality of $\bar{x}$. This proves that (5.105) has a nonnegative optimal value.

Since $Dg_i(\bar{x})$, $i = 1, \dots, q$, are linearly independent, the equality constraints of (5.105) have a feasible solution, and hence, since z can be made arbitrarily large, problem (5.105) is consistent. Therefore, (5.105) has a finite nonnegative optimal value. Since (5.105) is a linear programming problem, it follows that its dual has the same optimal value. The dual of (5.105) is

$$\begin{array}{ll} \underset{\lambda\in I\!R^p}{\text{Max}} & D^2_{xx}L^g(\bar{x},\lambda_0,\lambda)(h,h) \\ \text{subject to} & D_x L^g(\bar{x},\lambda_0,\lambda) = 0,\ \lambda_0 + \sum_{i\in I(\bar{x},h)} \lambda_i = 1, \\ & \lambda_0 \ge 0,\ \lambda_i \ge 0,\ i \in I(\bar{x},h),\ \lambda_i = 0,\ i > q,\ i \notin I(\bar{x},h). \end{array}$$

Since an optimal solution of this dual problem is a generalized Lagrange multiplier associated with $\bar{x}$, and the dual objective function is $D^2_{xx}L^g(\bar{x},\lambda)(h,h)$, assertion (i) follows.

Consider now assertion (ii). Suppose that the conclusion of assertion (ii) does not hold. Then there exists a sequence $x_k \to \bar{x}$ of feasible points such that $f(x_k) \le$

$f(\bar{x}) + o(\|x_k - \bar{x}\|^2)$. Set $t_k := \|x_k - \bar{x}\|$. Then

$$\limsup_{k\to\infty} \frac{f(x_k) - f(\bar{x})}{t_k^2} \le 0.$$

Extracting a subsequence if necessary, we can assume that $h_k := (x_k - \bar{x})/t_k$ converges to a vector $\hat{h}$ of unit norm, i.e., $x_k = \bar{x} + t_k\hat{h} + o(t_k)$ and $\|\hat{h}\| = 1$. Since $f(x_k) \le f(\bar{x}) + o(\|x_k - \bar{x}\|^2)$, and x_k are feasible, we obtain by a first order expansion of $f(x_k)$ and $g_i(x_k)$, $i = 1, \dots, p$, that $\hat{h}$ is a critical direction. Let $(\hat{\lambda}_0, \hat{\lambda})$ be a generalized Lagrange multiplier such that

$$\alpha := D^2_{xx} L^g(\bar{x}, \hat{\lambda}_0, \hat{\lambda})(\hat{h}, \hat{h}) > 0.$$

Then, since the components of $\hat{\lambda}$ corresponding to inequality constraints are nonnegative, and $D_x L^g(\bar{x}, \hat{\lambda}_0, \hat{\lambda}) = 0$, we have

$$\begin{aligned} \hat{\lambda}_0 f(x_k) &\ge L^g(x_k, \hat{\lambda}_0, \hat{\lambda}) = \hat{\lambda}_0 f(\bar{x}) + \tfrac{1}{2} t_k^2 D^2_{xx} L^g(\bar{x}, \hat{\lambda}_0, \hat{\lambda})(\hat{h}, \hat{h}) + o(t_k^2) \\ &\ge \hat{\lambda}_0 f(\bar{x}) + \tfrac{1}{2}\alpha t_k^2 + o(t_k^2). \end{aligned}$$

It follows that

$$\alpha \le 2\hat{\lambda}_0 \limsup_{k\to\infty} \frac{f(x_k) - f(\bar{x})}{t_k^2} \le 0,$$

which gives a contradiction. □

Remark 5.49 If the set $\Lambda(\bar{x})$ of Lagrange multipliers is nonempty, then the sufficient second order conditions of proposition 5.48(ii) are equivalent to the following (stronger) conditions: for any $h \in C(\bar{x}) \setminus \{0\}$ there exists $\lambda \in \Lambda(\bar{x})$ such that $D^2_{xx} L(\bar{x}, \lambda)(h, h) > 0$. These conditions can be also written in the following equivalent form:

$$\sup_{\lambda \in \Lambda(\bar{x})} D^2_{xx} L(\bar{x}, \lambda)(h, h) > 0, \quad \forall h \in C(\bar{x}) \setminus \{0\}, \tag{5.107}$$

where the above supremum can be $+\infty$. Indeed, let $(\lambda_0, \lambda) \in \Lambda^g(\bar{x})$ be such that (5.104) holds. Then either $\lambda_0 > 0$, and in that case $\lambda_0^{-1}\lambda$ satisfies (5.107), or λ is a singular Lagrange multiplier, and given any Lagrange multiplier $\bar{\lambda}$, we have that $\hat{\lambda} := \bar{\lambda} + t\lambda \in \Lambda(\bar{x})$ and $\hat{\lambda}$ satisfies (5.107) for large enough $t > 0$.