MAP 2320 – MÉTODOS NUMÉRICOS EM EQUAÇÕES DIFERENCIAIS II

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Equações Diferenciais Parciais: Uma Introdução (Versão Preliminar)

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2.1 Teorema de Fourier

Teorema 2.1 (Fourier). *Seja L um número real maior que zero. Para toda função $f : [-L, L] \to \mathbb{R}$ contínua por partes tal que a sua derivada f' também seja contínua por partes, a série de Fourier de f*

$$S_f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \operatorname{sen}\frac{n\pi t}{L},$$

em que

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt \quad \text{para } n = 0, 1, 2, \ldots \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\operatorname{sen}\frac{n\pi t}{L}dt, \quad \text{para } n = 1, 2, \ldots \end{aligned}$$

converge para f nos pontos de $(-L, L)$ em que f é contínua. Ou seja, podemos representar f por sua série de Fourier:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \operatorname{sen}\frac{n\pi t}{L}, \quad \text{para } t \in (-L, L) \text{ em que } f \text{ é contínua.}$$

Exemplo 2.5. Seja L um número real maior que zero. Seja m um inteiro positivo. Seja

$$u_m : [-L, L] \to \mathbb{R} \quad \text{definida por}$$

$$u_m(t) = \cos \frac{m\pi t}{L}, \quad \text{para } t \in [-L, L]$$

Fazendo a mudança de variáveis $s = \frac{\pi t}{L}$,

$$a_0 = \frac{1}{L} \int_{-L}^{L} \cos \frac{m\pi t}{L} dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos msds = 0,$$

Vamos calcular os coeficientes a_n, para $n = 0, 1, 2 \ldots$ Fazendo a mudança de variáveis $s = \frac{\pi t}{L}$, para $n > 0$ e $n \neq m$ temos que,

$$a_n = \frac{1}{L} \int_{-L}^{L} \cos \frac{m\pi t}{L} \cos \frac{n\pi t}{L} dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos ms \cos ns \, ds$$

Usando o fato de que $\cos ms \cos ns = \frac{1}{2}[\cos(m+n)s + \cos(m-n)s]$, temos então que

$$\begin{aligned} a_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos(m+n)s + \cos(m-n)s] ds \\ &= \frac{1}{2\pi(m+n)} \operatorname{sen}(m+n)s \Big|_{-\pi}^{\pi} + \frac{1}{2\pi(m-n)} \operatorname{sen}(m-n)s \Big|_{-\pi}^{\pi} = 0 \end{aligned}$$

e para $n = m$,

$$a_m = \frac{1}{L} \int_{-L}^{L} \cos^2 \frac{m\pi t}{L} dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 ms \, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} [1 + \cos 2ms] ds = 1.$$

Aqui usamos o fato de que $\cos^2 ms = \frac{1}{2}[1 + \cos 2ms]$.

Vamos calcular os coeficientes b_n, para $n = 1, 2 \dots$

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} \cos\frac{m\pi t}{L} \operatorname{sen}\frac{n\pi t}{L} dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos ms \ \operatorname{sen} ns\, ds \end{aligned}$$

Usando o fato de $\cos ms \operatorname{sen} ns = \frac{1}{2}[\operatorname{sen}(m+n)s + \operatorname{sen}(m-n)s]$, temos que

$$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\operatorname{sen}(m+n)s + \operatorname{sen}(m-n)s]ds = 0$$

Nestas integrais usamos relações que podem ser obtidas somando-se ou subtraindo-se duas das relações abaixo.

$$\begin{aligned} \cos(m+n)s &= \cos ms \cos ns - \operatorname{sen} ms \operatorname{sen} ns \\ \cos(m-n)s &= \cos ms \cos ns + \operatorname{sen} ms \operatorname{sen} ns \\ \operatorname{sen}(m+n)s &= \operatorname{sen} ms \cos ns + \cos ms \operatorname{sen} ns \\ \operatorname{sen}(m-n)s &= \operatorname{sen} ms \cos ns - \cos ms \operatorname{sen} ns. \end{aligned}$$

Assim a série de Fourier de $u_m : [-L, L] \to \mathbb{R}$ é dada por

$$S_{u_m}(t) = \cos\frac{m\pi t}{L} = u_m(t), \ \text{para } m = 1, 2 \dots$$

Exemplo 2.6. Seja L um número real maior que zero. Seja m um inteiro positivo. Seja

$$v_m : [-L, L] \to \mathbb{R} \quad \text{definida por}$$

$$v_m(t) = \operatorname{sen} \frac{m\pi t}{L}, \quad \text{para } t \in [-L, L]$$

Vamos calcular os coeficientes a_n, para $n = 0, 1, 2 \ldots$ Fazendo a mudança de variáveis $s = \dfrac{\pi t}{L}$,

$$a_0 = \frac{1}{L} \int_{-L}^{L} \operatorname{sen} \frac{m\pi t}{L} dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{sen} ms \, ds = 0.$$

Fazendo a mudança de variáveis $s = \dfrac{\pi t}{L}$, temos que para $n = 1, 2, 3 \ldots$

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} \operatorname{sen} \frac{m\pi t}{L} \cos \frac{n\pi t}{L} dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{sen} ms \, \cos nsds \end{aligned}$$

Usando o fato de que $\operatorname{sen} ms \, \cos ns = \frac{1}{2}[\operatorname{sen}(m+n)s + \operatorname{sen}(m-n)s]$ obtemos que

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\operatorname{sen}(m+n)s + \operatorname{sen}(m-n)s] ds = 0$$

Vamos calcular os coeficientes b_n, para $n = 1, 2 \ldots$ Para $n \neq m$ temos que

$$b_n = \frac{1}{L} \int_{-L}^{L} \operatorname{sen} \frac{m\pi t}{L} \operatorname{sen} \frac{n\pi t}{L} dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{sen} ms \, \operatorname{sen} ns \, ds$$

Usando o fato de que $\operatorname{sen} ms \, \operatorname{sen} ns = \frac{1}{2}[-\cos(m+n)s + \cos(m-n)s]$ temos que

$$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}[-\cos(m+n)s + \cos(m-n)s]ds = 0$$

E para $n = m$,

$$b_m = \frac{1}{L}\int_{-L}^{L} \operatorname{sen}^2 \frac{m\pi t}{L} dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{sen}^2 ms \, ds = \frac{1}{2\pi}\int_{-\pi}^{\pi}[1 - \cos 2ms]ds = 1$$

Aqui usamos o fato de que $\operatorname{sen}^2 ms = \frac{1}{2}[1 - \cos 2ms]$.

Assim a série de Fourier de $v_m : [-L, L] \to \mathbb{R}$ é dada por

$$S_{v_m}(t) = \operatorname{sen} \frac{m\pi t}{L} = v_m(t), \text{ para } m = 1, 2 \ldots$$

Com os coeficientes das funções destes exemplos podemos determinar os coeficientes das séries de Fourier de várias funções que são combinações lineares delas. Isto por que os coeficientes das séries dependem linearmente das funções, ou seja,

Proposição 2.3. *Sejam $f, g : [-L, L] \to \mathbb{R}$. Se*

$$a_n(f, L) = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt,\ \ b_n(f, L) = \frac{1}{L}\int_{-L}^{L} f(t)\,\text{sen}\,\frac{n\pi t}{L}dt,$$

$$a_n(g, L) = \frac{1}{L}\int_{-L}^{L} g(t)\cos\frac{n\pi t}{L}dt,\ \ b_n(g, L) = \frac{1}{L}\int_{-L}^{L} g(t)\,\text{sen}\,\frac{n\pi t}{L}dt,$$

então para quaisquer números α e β,

$$a_n(\alpha f + \beta g, L) = \alpha a_n(f, L) + \beta a_n(g, L) \quad e \quad b_n(\alpha f + \beta g, L) = \alpha b_n(f, L) + \beta b_n(g, L).$$

Demonstração.

$$a_n(\alpha f + \beta g, L) = \frac{1}{L}\int_{-L}^{L}(\alpha f(t) + \beta g(t))\cos\frac{n\pi t}{L}dt = \alpha\frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt + \beta\frac{1}{L}\int_{-L}^{L} g(t)\cos\frac{n\pi t}{L}dt = \alpha a_n(f, L) + \beta a_n(g, L).$$

$$b_n(\alpha f + \beta g, L) = \frac{1}{L}\int_{-L}^{L}(\alpha f(t) + \beta g(t))\,\text{sen}\,\frac{n\pi t}{L}dt = \alpha\frac{1}{L}\int_{-L}^{L} f(t)\,\text{sen}\,\frac{n\pi t}{L}dt + \beta\frac{1}{L}\int_{-L}^{L} g(t)\,\text{sen}\,\frac{n\pi t}{L}dt = \alpha b_n(f, L) + \beta b_n(g, L).$$

Exemplo 2.7. Seja L um número real maior que zero. Seja n um inteiro positivo. Seja

$$f : [-L, L] \to \mathbb{R} \quad \text{definida por}$$

$$f(t) = 3 - 2\cos\frac{15\pi t}{L} + 4\,\text{sen}\,\frac{31\pi t}{L}, \quad \text{para } t \in [-L, L]$$

temos que a série de Fourier da função deste exemplo é ela própria:

$$S_f(t) = 3 - 2\cos\frac{15\pi t}{L} + 4\,\text{sen}\,\frac{31\pi t}{L} = f(t).$$

Exemplo 2.8. Considere a função $f : [-L, L] \to \mathbb{R}$ dada por

$$f(t) = f^{(1)}_{c,d}(t) = \begin{cases} t, & \text{se } cL < t \leq dL, \\ 0, & \text{caso contrário,} \end{cases}$$ para c e d fixos satisfazendo $-1 \leq c < d \leq 1$.

Vamos calcular a série de Fourier de $f^{(1)}_{cd}$. Fazendo a mudança de variáveis $s = \dfrac{n\pi t}{L}$ e integrando-se por partes obtemos

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_{cL}^{dL} f(t)dt = \frac{1}{L}\int_{cL}^{dL} t\,dt = \frac{L}{2}(d^2 - c^2) \\
a_n &= \frac{1}{L}\int_{cL}^{dL} f(t)\cos\frac{n\pi t}{L}dt = \frac{1}{L}\int_{cL}^{dL} t\cos\frac{n\pi t}{L}dt = \frac{L}{n^2\pi^2}\int_{n\pi c}^{n\pi d} s\cos s\,ds \\
&= \frac{L}{n^2\pi^2}\left(s \operatorname{sen} s + \cos s\right)\Big|_{n\pi c}^{n\pi d} \\
b_n &= \frac{1}{L}\int_{cL}^{dL} f(t)\operatorname{sen}\frac{n\pi t}{L}dt = \frac{1}{L}\int_{cL}^{dL} t\operatorname{sen}\frac{n\pi t}{L}dt = \frac{L}{n^2\pi^2}\int_{n\pi c}^{n\pi d} s\operatorname{sen} s\,ds \\
&= \frac{L}{n^2\pi^2}\left(-s\cos s + \operatorname{sen} s\right)\Big|_{n\pi c}^{n\pi d}
\end{aligned}$$

integral (93)

integral (99)

Logo

$$\begin{aligned}
S_f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \operatorname{sen}\frac{n\pi t}{L} \\
&= \frac{L(d^2 - c^2)}{4} + \frac{L}{\pi^2}\sum_{n=1}^{\infty} \frac{\left(s\operatorname{sen} s + \cos s\right)\Big|_{n\pi c}^{n\pi d}}{n^2}\cos\frac{n\pi t}{L} + \frac{L}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(-s\cos s + \operatorname{sen} s\right)\Big|_{n\pi c}^{n\pi d}}{n}\operatorname{sen}\frac{n\pi t}{L}.
\end{aligned}$$

Table of Integrals*

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$$\int x \cos x dx = \cos x + x \sin x \tag{93}$$

$$\int x \cos ax dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax \tag{94}$$

$$\int x^2 \cos x dx = 2x \cos x + \left(x^2 - 2\right) \sin x \tag{95}$$

$$\int x^2 \cos ax dx = \frac{2x \cos ax}{a^2} + \frac{a^2 x^2 - 2}{a^3} \sin ax \tag{96}$$

$$\int x^n cosx dx = -\frac{1}{2}(i)^{n+1} \left[\Gamma(n+1, -ix) + (-1)^n \Gamma(n+1, ix)\right] \tag{97}$$

$$\int x^n cosax dx = \frac{1}{2}(ia)^{1-n} \left[(-1)^n \Gamma(n+1, -iax) - \Gamma(n+1, ixa)\right] \tag{98}$$

$$\int x \sin x dx = -x \cos x + \sin x \tag{99}$$

$$\int x \sin ax dx = -\frac{x \cos ax}{a} + \frac{\sin ax}{a^2} \tag{100}$$

$$\int x^2 \sin x dx = \left(2 - x^2\right) \cos x + 2x \sin x \tag{101}$$

$$\int x^2 \sin ax dx = \frac{2 - a^2 x^2}{a^3} \cos ax + \frac{2x \sin ax}{a^2} \tag{102}$$

$$\int x^n \sin x dx = -\frac{1}{2}(i)^n \left[\Gamma(n+1, -ix) - (-1)^n \Gamma(n+1, -ix)\right] \tag{103}$$

1.4. Determine a série de Fourier da função $f_{c,d}^{(2)} : [-L, L] \to \mathbb{R}$ dada por

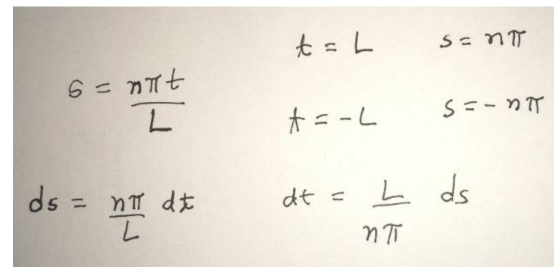
$$f_{c,d}^{(2)}(t) = \begin{cases} t^2, & \text{se } cL < t \leq dL, \\ 0, & \text{caso contrário,} \end{cases} \quad \text{para } c \text{ e } d \text{ fixos satisfazendo } -1 \leq c < d \leq 1.$$

$$S_f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \operatorname{sen} \frac{n\pi t}{L},$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi t}{L} dt \quad para\ n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \operatorname{sen} \frac{n\pi t}{L} dt, \quad para\ n = 1, 2, \ldots$$

+



+ Tabelas de integrais

+ Relações trigonométricas

1.4. Fazendo a mudança de variáveis $s = \frac{n\pi t}{L}$ e integrando-se por partes duas vezes obtemos

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_{cL}^{dL} f(t)dt = \frac{1}{L}\int_{cL}^{dL} t^2\, dt = \frac{L^2}{3}(d^3 - c^3) \\
a_n &= \frac{1}{L}\int_{cL}^{dL} f(t)\cos\frac{n\pi t}{L}dt = \frac{1}{L}\int_{cL}^{dL} t^2\cos\frac{n\pi t}{L}dt = \frac{L^2}{n^3\pi^3}\int_{n\pi c}^{n\pi d} s^2\cos s\, ds \\
&= \frac{L^2}{n^3\pi^3}\left(s^2 \operatorname{sen} s\Big|_{n\pi c}^{n\pi d} - 2\int_{n\pi c}^{n\pi d} s \operatorname{sen} s \right) \\
&= \frac{L^2}{n^3\pi^3}\left(\left(s^2 - 2\right)\operatorname{sen} s + 2s\cos s \right)\Big|_{n\pi c}^{n\pi d}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{L}\int_{cL}^{dL} f(t)\operatorname{sen}\frac{n\pi t}{L}dt = \frac{1}{L}\int_{cL}^{dL} t^2\operatorname{sen}\frac{n\pi t}{L}dt = \frac{L^2}{n^3\pi^3}\int_{n\pi c}^{n\pi d} s^2\operatorname{sen} s\, ds \\
&= \frac{L^2}{n^3\pi^3}\left(-s^2\cos s\Big|_{n\pi c}^{n\pi d} + 2\int_{n\pi c}^{n\pi d} s\cos s \right) \\
&= \frac{L^2}{n^3\pi^3}\left(2s\operatorname{sen} s + (2 - s^2)\cos s \right)\Big|_{n\pi c}^{n\pi d}
\end{aligned}
$$

$$
\begin{aligned}
S_{f_{c,d}^{(2)}}(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n\operatorname{sen}\frac{n\pi t}{L} \\
&= \frac{L^2}{6}(d^3 - c^3) + \frac{L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{\left((s^2-2)\operatorname{sen} s + 2s\cos s\right)\Big|_{n\pi c}^{n\pi d}}{n^3}\cos\frac{n\pi t}{L} \\
&\quad + \frac{L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{\left(2s\operatorname{sen} s + (2-s^2)\cos s\right)\Big|_{n\pi c}^{n\pi d}}{n^3}\operatorname{sen}\frac{n\pi t}{L}
\end{aligned}
$$

Os coeficientes obtidos a partir das soluções para funções básicas podem ser combinados para o cálculo de coeficientes de funções que são combinações lineares dessas funções básicas por trecho.

A tabela a seguir resume essas soluções e será utilizada intensamente no curso para a solução das equações diferenciais parciais de interesse (parabólica, elíptica e hiperbólica)

2.1.4 Tabela de Coeficientes de Séries de Fourier

Coeficientes das Séries de Fourier de Funções Elementares		
$f : [-L, L] \to \mathbb{R},\ -1 \le c < d \le 1$	$a_n(f, L) = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$	$b_n(f, L) = \frac{1}{L}\int_{-L}^{L} f(t)\,\text{sen}\,\frac{n\pi t}{L}dt$
$f_{c,d}^{(0)}(t) = \begin{cases} 1, & \text{se } t \in [cL, dL] \\ 0, & \text{caso contrário} \end{cases}$	$a_0(f_{c,d}^{(0)}, L) = d - c$ $a_n(f_{c,d}^{(0)}, L) = \frac{1}{n\pi}\,\text{sen}\,s\Big\vert_{n\pi c}^{n\pi d}$	$b_n(f_{c,d}^{(0)}, L) = -\frac{1}{n\pi}\cos s\Big\vert_{n\pi c}^{n\pi d}$
$f_{c,d}^{(1)}(t) = \begin{cases} t, & \text{se } t \in [cL, dL] \\ 0, & \text{caso contrário} \end{cases}$	$a_0(f_{c,d}^{(1)}, L) = \frac{L}{2}(d^2 - c^2)$ $a_n(f_{c,d}^{(1)}, L) =$ $\frac{L}{n^2\pi^2}(s\,\text{sen}\,s + \cos s)\Big\vert_{n\pi c}^{n\pi d}$	$b_n(f_{c,d}^{(1)}, L) =$ $\frac{L}{n^2\pi^2}(-s\cos s + \text{sen}\,s)\Big\vert_{n\pi c}^{n\pi d}$
$f_{c,d}^{(2)}(t) = \begin{cases} t^2, & \text{se } t \in [cL, dL] \\ 0, & \text{caso contrário} \end{cases}$	$a_0(f_{c,d}^{(2)}, L) = \frac{L^2}{3}(d^3 - c^3)$ $a_n(f_{c,d}^{(2)}, L) =$ $\frac{L^2}{n^3\pi^3}((s^2 - 2)\,\text{sen}\,s + 2s\cos s)\Big\vert_{n\pi c}^{n\pi d}$	$b_n(f_{c,d}^{(2)}, L) =$ $\frac{L^2}{n^3\pi^3}(2s\,\text{sen}\,s + (2 - s^2)\cos s)\Big\vert_{n\pi c}^{n\pi d}$

Proposição 2.3. *Sejam* $f, g : [-L, L] \to \mathbb{R}$. *Se*

$$a_n(f, L) = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt,\ \ b_n(f, L) = \frac{1}{L}\int_{-L}^{L} f(t)\,\text{sen}\,\frac{n\pi t}{L}dt,$$

$$a_n(g, L) = \frac{1}{L}\int_{-L}^{L} g(t)\cos\frac{n\pi t}{L}dt,\ \ b_n(g, L) = \frac{1}{L}\int_{-L}^{L} g(t)\,\text{sen}\,\frac{n\pi t}{L}dt,$$

então para quaisquer números α *e* β*,*

$$a_n(\alpha f + \beta g, L) = \alpha a_n(f, L) + \beta a_n(g, L) \quad e \quad b_n(\alpha f + \beta g, L) = \alpha b_n(f, L) + \beta b_n(g, L).$$

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Section 6: Alternative notation

- For a waveform $f(x)$ with period $2L = \frac{2\pi}{k}$, we have that $k = \frac{2\pi}{2L} = \frac{\pi}{L}$ and $nkx = \frac{n\pi x}{L}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

The corresponding Fourier coefficients are

STEP ONE
$$a_0 = \frac{1}{L} \int_{2L} f(x)\, dx$$

STEP TWO
$$a_n = \frac{1}{L} \int_{2L} f(x) \cos \frac{n\pi x}{L}\, dx$$

STEP THREE
$$b_n = \frac{1}{L} \int_{2L} f(x) \sin \frac{n\pi x}{L}\, dx$$

and integrations are over a single interval in x of $2L$

Exercise 1.

Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi \,. \end{cases}$$

a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$

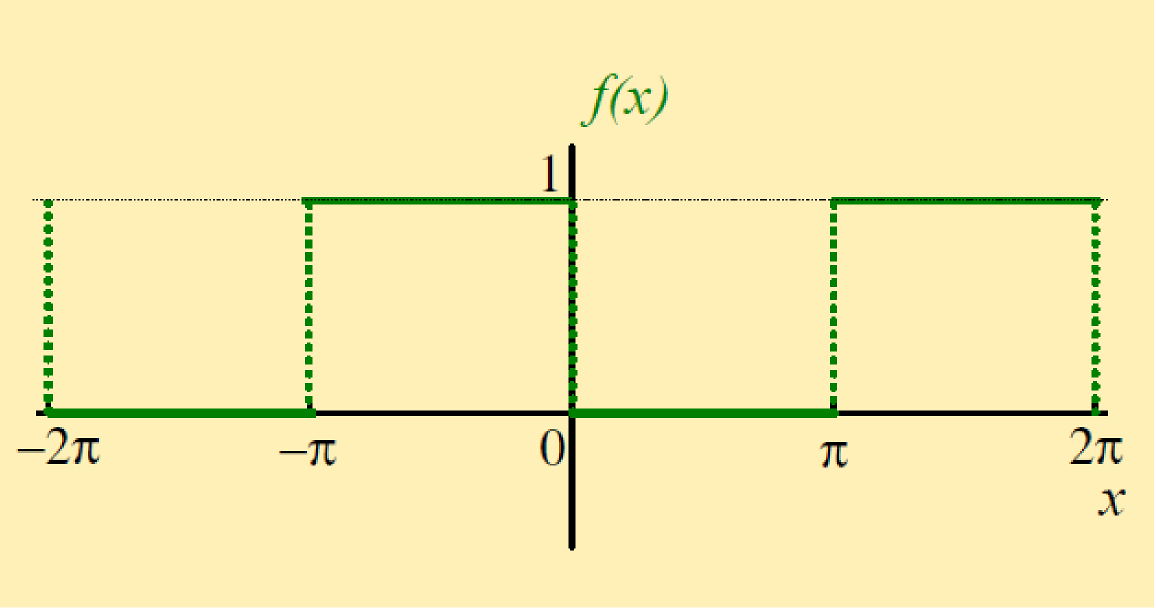
b) Show that the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$ is

$$\frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right]$$

Exercise 1.

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi, \end{cases} \quad \text{and has period } 2\pi$$

a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$\begin{aligned}
a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot \mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot \mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \mathrm{d}x \\
&= \frac{1}{\pi}\left[x\right]_{-\pi}^{0} \\
&= \frac{1}{\pi}\left(0-(-\pi)\right) \\
&= \frac{1}{\pi}\cdot(\pi) \\
\text{i.e. } a_0 &= 1\,.
\end{aligned}$$

STEP TWO

$$
\begin{aligned}
a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} = \frac{1}{n\pi}\left[\sin nx\right]_{-\pi}^{0} \\
&= \frac{1}{n\pi}\left(\sin 0 - \sin(-n\pi)\right) \\
&= \frac{1}{n\pi}(0 + \sin n\pi) \\
\text{i.e. } a_n &= \frac{1}{n\pi}(0+0) = 0.
\end{aligned}
$$

STEP THREE

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, \mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx \, \mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx \, \mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\sin nx \, \mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\sin nx \, \mathrm{d}x \end{aligned}$$

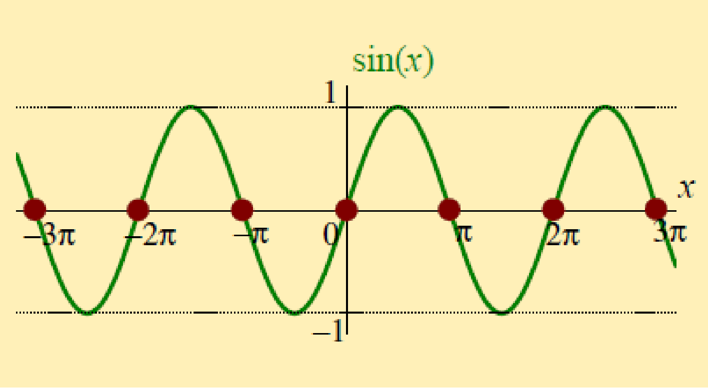
$$\begin{aligned} \text{i.e.}\quad b_n &= \frac{1}{\pi}\int_{-\pi}^{0} \sin nx \, \mathrm{d}x = \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} \\ &= -\frac{1}{n\pi}\left[\cos nx\right]_{-\pi}^{0} = -\frac{1}{n\pi}\left(\cos 0 - \cos(-n\pi)\right) \\ &= -\frac{1}{n\pi}(1-\cos n\pi) = -\frac{1}{n\pi}\left(1-(-1)^n\right), \ \text{see Trig} \end{aligned}$$

$$\text{i.e.}\quad b_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{n\pi} & , n \text{ odd} \end{cases} \quad , \text{since } (-1)^n = \begin{cases} 1 & , n \text{ even} \\ -1 & , n \text{ odd} \end{cases}$$

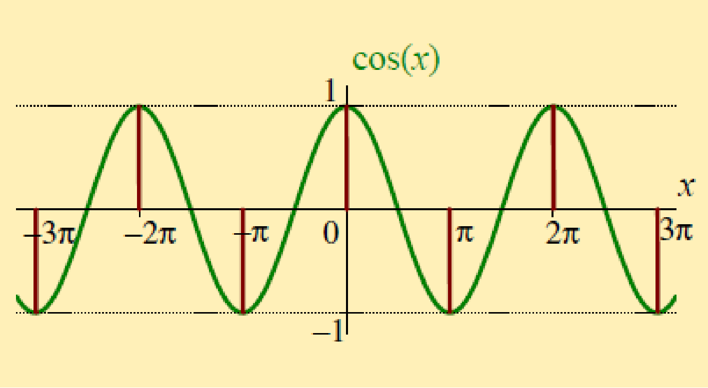
5. Useful trig results

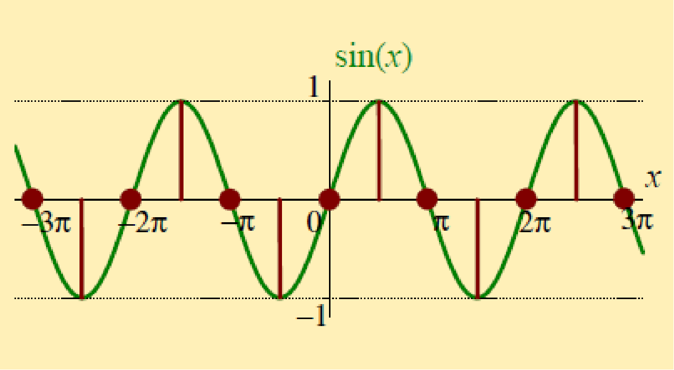
When calculating the Fourier coefficients a_n and b_n, for which $n = 1, 2, 3, \ldots,$ the following trig. results are useful. Each of these results, which are also true for $n = 0, -1, -2, -3, \ldots,$ can be deduced from the graph of $\sin x$ or that of $\cos x$

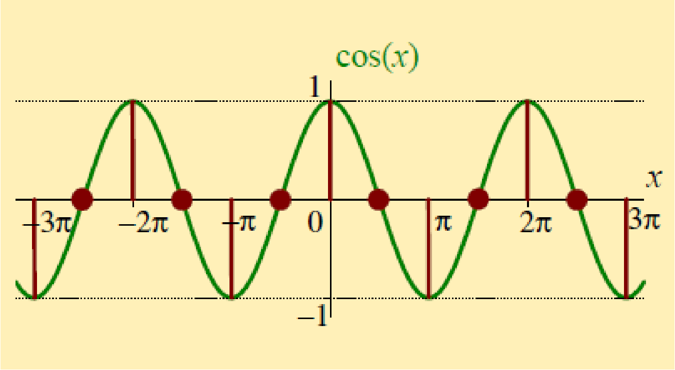
- $\sin n\pi = 0$



- $\cos n\pi = (-1)^n$



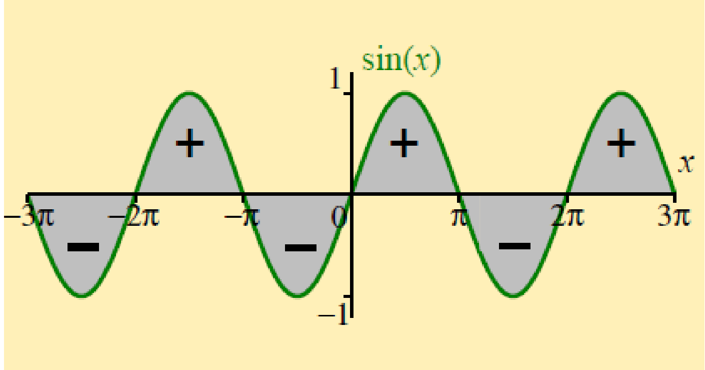




- $\sin n\dfrac{\pi}{2} = \begin{cases} 0 & , n \text{ even} \\ 1 & , n = 1, 5, 9, \ldots \\ -1 & , n = 3, 7, 11, \ldots \end{cases}$
- $\cos n\dfrac{\pi}{2} = \begin{cases} 0 & , n \text{ odd} \\ 1 & , n = 0, 4, 8, \ldots \\ -1 & , n = 2, 6, 10, \ldots \end{cases}$

Areas cancel when when integrating over whole periods

- $\int_{2\pi} \sin nx \, dx = 0$
- $\int_{2\pi} \cos nx \, dx = 0$



We now have that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

with the three steps giving

$$a_0 = 1,\ \ a_n = 0\,,\ \text{and}\ \ b_n = \begin{cases} 0 & ,\ n \text{ even} \\ -\frac{2}{n\pi} & ,\ n \text{ odd} \end{cases}$$

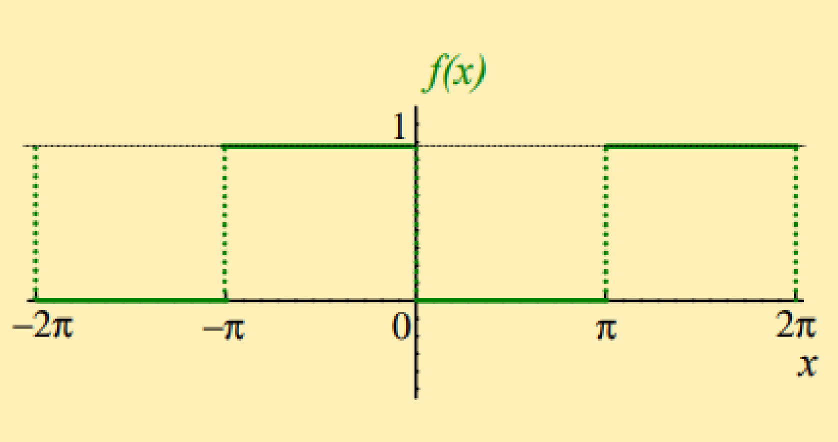
It may be helpful to construct a table of values of b_n

n	1	2	3	4	5
b_n	$-\frac{2}{\pi}$	0	$-\frac{2}{\pi}\left(\frac{1}{3}\right)$	0	$-\frac{2}{\pi}\left(\frac{1}{5}\right)$

Substituting our results now gives the required series

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right]$$

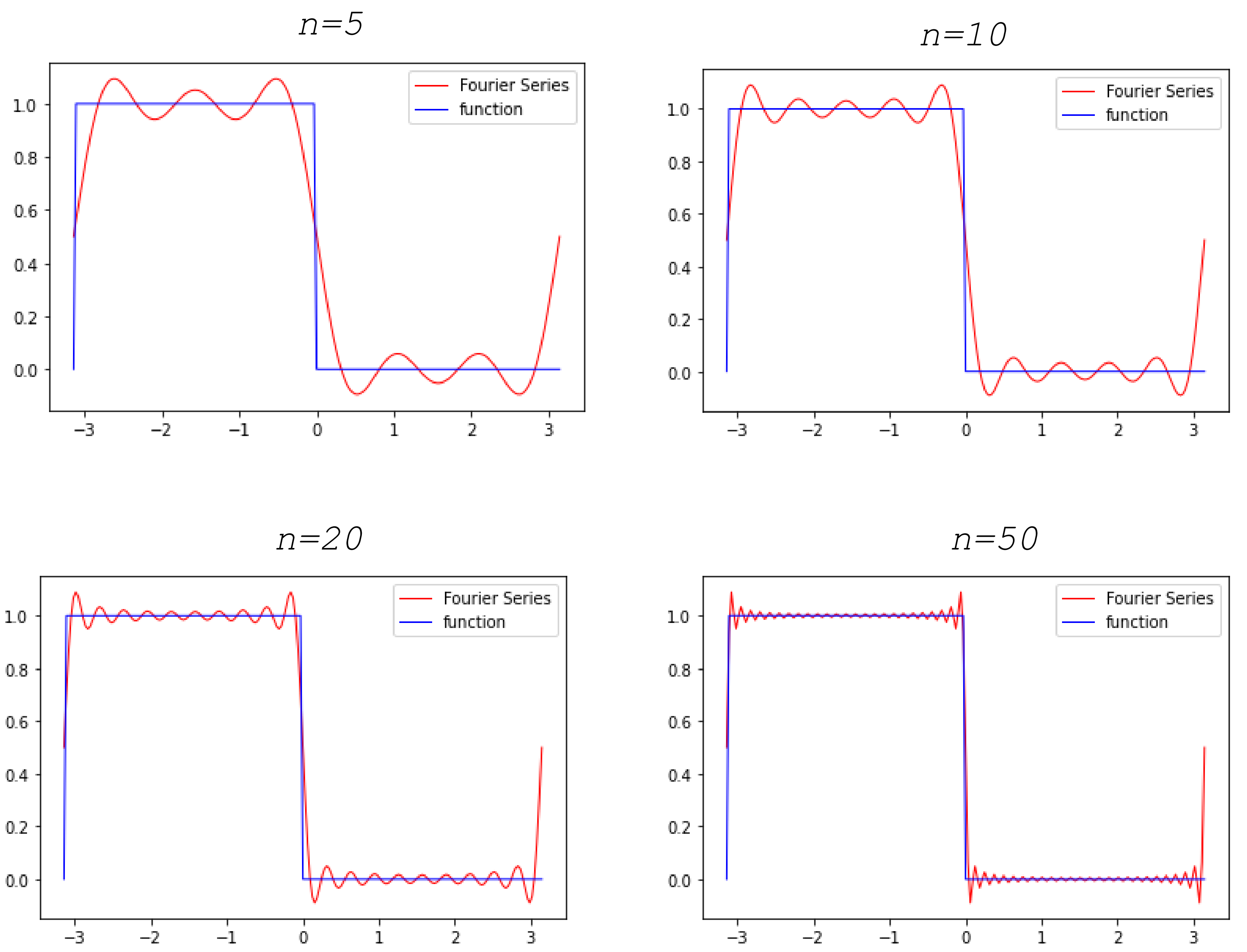
$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi, \end{cases}$$



Coeficientes das Séries de Fourier de Funções Elementares		
$f : [-L, L] \to \mathbb{R}$, $-1 \leq c < d \leq 1$	$a_n(f, L) = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$	$b_n(f, L) = \frac{1}{L}\int_{-L}^{L} f(t)\,\text{sen}\,\frac{n\pi t}{L}dt$
$f^{(0)}_{c,d}(t) = \begin{cases} 1, & \text{se } t \in [cL, dL] \\ 0, & \text{caso contrário} \end{cases}$	$a_0(f^{(0)}_{c,d}, L) = d - c$ $a_n(f^{(0)}_{c,d}, L) = \frac{1}{n\pi}\,\text{sen}\, s\Big\vert_{n\pi c}^{n\pi d}$	$b_n(f^{(0)}_{c,d}, L) = -\frac{1}{n\pi}\cos s\Big\vert_{n\pi c}^{n\pi d}$

Exercício 1

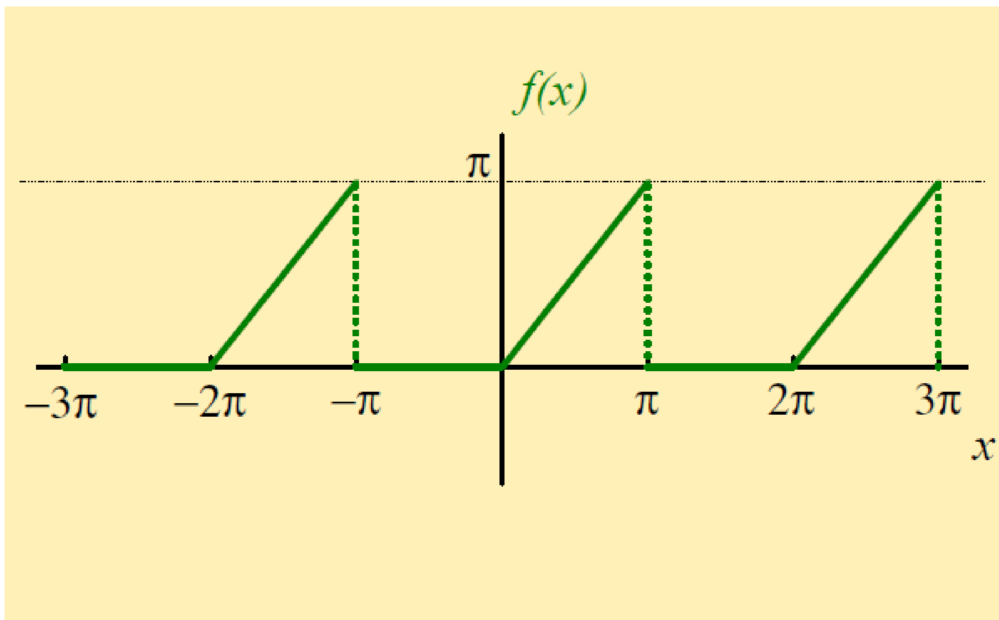
L = π , c = -1 , d = 0


n=5
Fourier Series
function
n=10
Fourier Series
function
n=20
Fourier Series
function
n=50
Fourier Series
function

Exercise 2.

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi, \end{cases} \quad \text{and has period } 2\pi$$

a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$



MAP 2320 – MÉTODOS NUMÉRICOS EM EQUAÇÕES DIFERENCIAIS II

2º Semestre - 2020

Roteiro do curso

- Introdução
- Séries de Fourier
- Método de Diferenças Finitas
- Equação do calor transiente (parabólica)
- Equação de Poisson (elíptica)
- Equação da onda (hiperbólica)