# MAP 2210 – Aplicações de Álgebra Linear

## 1º Semestre - 2020

**Prof. Dr. Luis Carlos de Castro Santos**

lsantos@ime.usp.br

## Objetivos

Formação básica de álgebra linear aplicada a problemas numéricos. Resolução de problemas em microcomputadores usando linguagens e/ou software adequados fora do horário de aula.

# NÃO ESQUEÇA DE INICIAR A GRAVAÇÃO

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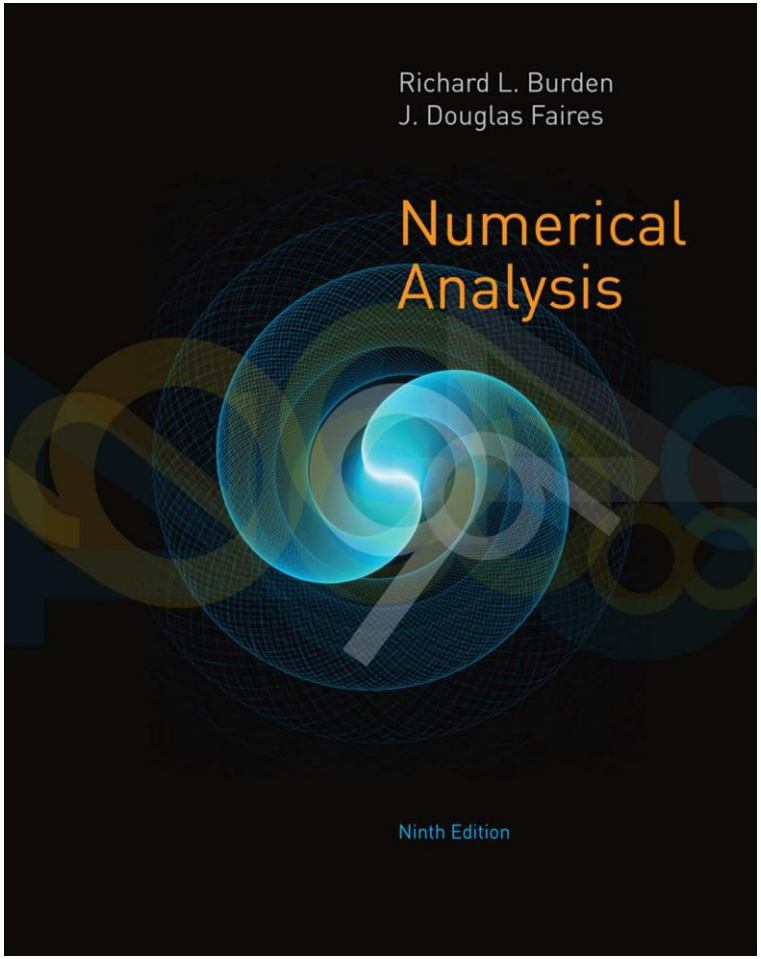


# Numerical Analysis

NINTH EDITION

**Richard L. Burden**
*Youngstown State University*

**J. Douglas Faires**
*Youngstown State University*

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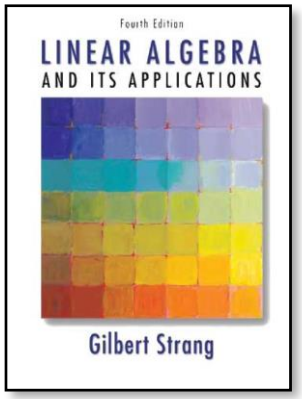




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# Conjugate gradient method

$\mathbf{r}_0 := \mathbf{b} - \mathbf{A}\mathbf{x}_0$
if $\mathbf{r}_0$ is sufficiently small, then return $\mathbf{x}_0$ as the result
$\mathbf{p}_0 := \mathbf{r}_0$
$k := 0$
repeat
$\alpha_k := \frac{\mathbf{r}_k^\mathsf{T}\mathbf{r}_k}{\mathbf{p}_k^\mathsf{T}\mathbf{A}\mathbf{p}_k}$
$\mathbf{x}_{k+1} := \mathbf{x}_k + \alpha_k\mathbf{p}_k$
$\mathbf{r}_{k+1} := \mathbf{r}_k - \alpha_k\mathbf{A}\mathbf{p}_k$
if $\mathbf{r}_{k+1}$ is sufficiently small, then exit loop
$\beta_k := \frac{\mathbf{r}_{k+1}^\mathsf{T}\mathbf{r}_{k+1}}{\mathbf{r}_k^\mathsf{T}\mathbf{r}_k}$
$\mathbf{p}_{k+1} := \mathbf{r}_{k+1} + \beta_k\mathbf{p}_k$
$k := k + 1$
end repeat
return $\mathbf{x}_{k+1}$ as the result

# The preconditioned conjugate gradient method

$\mathbf{r}_0 := \mathbf{b} - \mathbf{A}\mathbf{x}_0$
$\mathbf{z}_0 := \mathbf{M}^{-1}\mathbf{r}_0$
$\mathbf{p}_0 := \mathbf{z}_0$
$k := 0$
**repeat**
$\alpha_k := \frac{\mathbf{r}_k^\mathsf{T}\mathbf{z}_k}{\mathbf{p}_k^\mathsf{T}\mathbf{A}\mathbf{p}_k}$
$\mathbf{x}_{k+1} := \mathbf{x}_k + \alpha_k\mathbf{p}_k$
$\mathbf{r}_{k+1} := \mathbf{r}_k - \alpha_k\mathbf{A}\mathbf{p}_k$
**if** $\mathbf{r}_{k+1}$ is sufficiently small **then** exit loop **end if**
$\mathbf{z}_{k+1} := \mathbf{M}^{-1}\mathbf{r}_{k+1}$
$\beta_k := \frac{\mathbf{r}_{k+1}^\mathsf{T}\mathbf{z}_{k+1}}{\mathbf{r}_k^\mathsf{T}\mathbf{z}_k}$
$\mathbf{p}_{k+1} := \mathbf{z}_{k+1} + \beta_k\mathbf{p}_k$
$k := k + 1$
**end repeat**
The result is $\mathbf{x}_{k+1}$

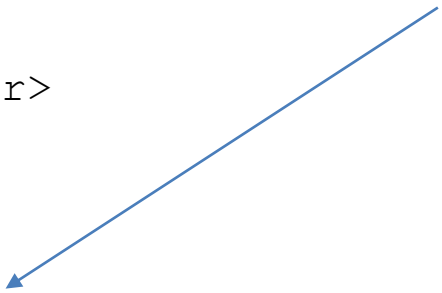
A matriz M deve ser tal que encontrar sua inversa, ou melhor resolver `Mz=r` tem baixo custo computacional

# Conjugate Gradient Method

```
Dados x(0), b, A, N

r = b- Ax
v = r
α = <v,v> = <r,r>

k = 1
Enquanto k ≤ N
        u = A.v
        t = α /<v,u>
        x = x + t*v
        r = r − t*u
        β = <r,r>
        se β < TOL saída x,r
        s = β/ α
        v = r + s*v
        α = β
        k = k + 1
Se k > N saída x,r,k
```



multiplicação matrix-vetor pode ser customizada para matrizes esparsas

# Preconditioned Conjugate Gradient Method

```
Dados x(0), b, A, N, P

r = b- Ax
w = P.r
v = P^T.w
α = <w,w>

k = 1
Enquanto k ≤ N
        u = A.v
        t = α /<v,u>
        x = x + t*v
        r = r − t*u
        w = P.r
        β = <w,w>
        se β < TOL saída x,r
        s = β/ α
        v = P^T.w + s*v
        α = β
        k = k + 1
Se k > N saída x,r,k
```

multiplicação matrix-vetor pode ser customizada para matrizes esparsas

$$P = M^{-1/2}$$

Gradiente Conjugado

$$\|\mathbf{e}_k\|_{\mathbf{A}} \leq 2\left(\frac{\sqrt{\kappa(\mathbf{A})}-1}{\sqrt{\kappa(\mathbf{A})}+1}\right)^k \|\mathbf{e}_0\|_{\mathbf{A}},$$

$$\frac{\sqrt{\kappa(\mathbf{A})}-1}{\sqrt{\kappa(\mathbf{A})}+1} \approx 1-\frac{2}{\sqrt{\kappa(\mathbf{A})}} \quad \text{for} \quad \kappa(\mathbf{A}) \gg 1.$$

Gauss-Seidel Jacobi

$$\approx 1-\frac{2}{\kappa(\mathbf{A})}.$$

### Theorem 7.27

Suppose that $\tilde{\mathbf{x}}$ is an approximation to the solution of $A\mathbf{x} = \mathbf{b}$, $A$ is a nonsingular matrix, and $\mathbf{r}$ is the residual vector for $\tilde{\mathbf{x}}$. Then for any natural norm,

$$\|\mathbf{x} - \tilde{\mathbf{x}}\| \leq \|\mathbf{r}\| \cdot \|A^{-1}\|$$

and if $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$,

$$\frac{\|\mathbf{x} - \tilde{\mathbf{x}}\|}{\|\mathbf{x}\|} \leq \|A\| \cdot \|A^{-1}\| \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}. \tag{7.20}$$

■

The **condition number** of the nonsingular matrix $A$ relative to a norm $\|\cdot\|$ is

$$K(A) = \|A\| \cdot \|A^{-1}\|.$$

■

For any nonsingular matrix $A$ and natural norm $\|\cdot\|$,

$$1 = \|I\| = \|A \cdot A^{-1}\| \leq \|A\| \cdot \|A^{-1}\| = K(A).$$

A matrix $A$ is **well-conditioned** if $K(A)$ is close to 1, and is **ill-conditioned** when $K(A)$ is significantly greater than 1. Conditioning in this context refers to the relative security that a small residual vector implies a correspondingly accurate approximate solution.

The next example illustrates the effect of preconditioning on a poorly conditioned matrix. In this example, we use $D^{-1/2}$ to represent the diagonal matrix whose entries are the reciprocals of the square roots of the diagonal entries of the coefficient matrix $A$. This is used as the preconditioner. Because the matrix $A$ is positive definite we expect the eigenvalues of $D^{-1/2}AD^{-1/2}$ to be close to 1, with the result that the condition number of this matrix will be small relative to the condition number of $A$.

M = D

**Example 3**

Use Maple to find the eigenvalues and condition number of the matrix

$$A = \begin{bmatrix} 0.2 & 0.1 & 1 & 1 & 0 \\ 0.1 & 4 & -1 & 1 & -1 \\ 1 & -1 & 60 & 0 & -2 \\ 1 & 1 & 0 & 8 & 4 \\ 0 & -1 & -2 & 4 & 700 \end{bmatrix}$$

and compare these with the eigenvalues and condition number of the preconditioned matrix $D^{-1/2}AD^{-1/2}$.

The preconditioned matrix is

$AH := CI.A.Transpose(CI)$

$$\begin{bmatrix} 1.000002 & 0.1118035 & 0.2886744 & 0.7905693 & 0 \\ 0.1118035 & 1 & -0.0645495 & 0.1767765 & -0.0188983 \\ 0.2886744 & -0.0645495 & 0.9999931 & 0 & -0.00975898 \\ 0.7905693 & 0.1767765 & 0 & 0.9999964 & 0.05345219 \\ 0 & -0.0188983 & -0.00975898 & 0.05345219 & 1.000005 \end{bmatrix}$$

The eigenvalues of $A$ and $AH$ are found with

$Eigenvalues(A); Eigenvalues(AH)$

Maple gives these as

$$Eigenvalues\ of\ A : 700.031, 60.0284, 0.0570747, 8.33845, 3.74533$$

$$Eigenvalues\ of\ AH : 1.88052, 0.156370, 0.852686, 1.10159, 1.00884$$

The condition numbers of $A$ and $AH$ in the $l_\infty$ norm are found with

$ConditionNumber(A); ConditionNumber(AH)$

which Maple gives as 13961.7 for $A$ and 16.1155 for $AH$. It is certainly true in this case that $AH$ is better conditioned that the original matrix $A$. ■

## Illustration

The linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 0.2 & 0.1 & 1 & 1 & 0 \\ 0.1 & 4 & -1 & 1 & -1 \\ 1 & -1 & 60 & 0 & -2 \\ 1 & 1 & 0 & 8 & 4 \\ 0 & -1 & -2 & 4 & 700 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

has the solution

$$\mathbf{x}^* = (7.859713071, 0.4229264082, -0.07359223906, -0.5406430164, 0.01062616286)^t.$$

Table 7.5 lists the results obtained by using the Jacobi, Gauss-Seidel, and SOR (with $\omega = 1.25$) iterative methods applied to the system with $A$ with a tolerance of 0.01, as well as those when the Conjugate Gradient method is applied both in its unpreconditioned form and using the preconditioning matrix described in Example 3. The preconditioned conjugate gradient method not only gives the most accurate approximations, it also uses the smallest number of iterations. □

**Table 7.5**

| Method | Number of Iterations | $\mathbf{x}^{(k)}$ | $\|\mathbf{x}^* - \mathbf{x}^{(k)}\|_\infty$ |
|---|---|---|---|
| Jacobi | 49 | $(7.86277141, 0.42320802, -0.07348669, -0.53975964, 0.01062847)^t$ | 0.00305834 |
| Gauss-Seidel | 15 | $(7.83525748, 0.42257868, -0.07319124, -0.53753055, 0.01060903)^t$ | 0.02445559 |
| SOR ($\omega = 1.25$) | 7 | $(7.85152706, 0.42277371, -0.07348303, -0.53978369, 0.01062286)^t$ | 0.00818607 |
| Conjugate Gradient | 5 | $(7.85341523, 0.42298677, -0.07347963, -0.53987920, 0.008628916)^t$ | 0.00629785 |
| Conjugate Gradient (Preconditioned) | 4 | $(7.85968827, 0.42288329, -0.07359878, -0.54063200, 0.01064344)^t$ | 0.00009312 |

Sobre o Trabalho

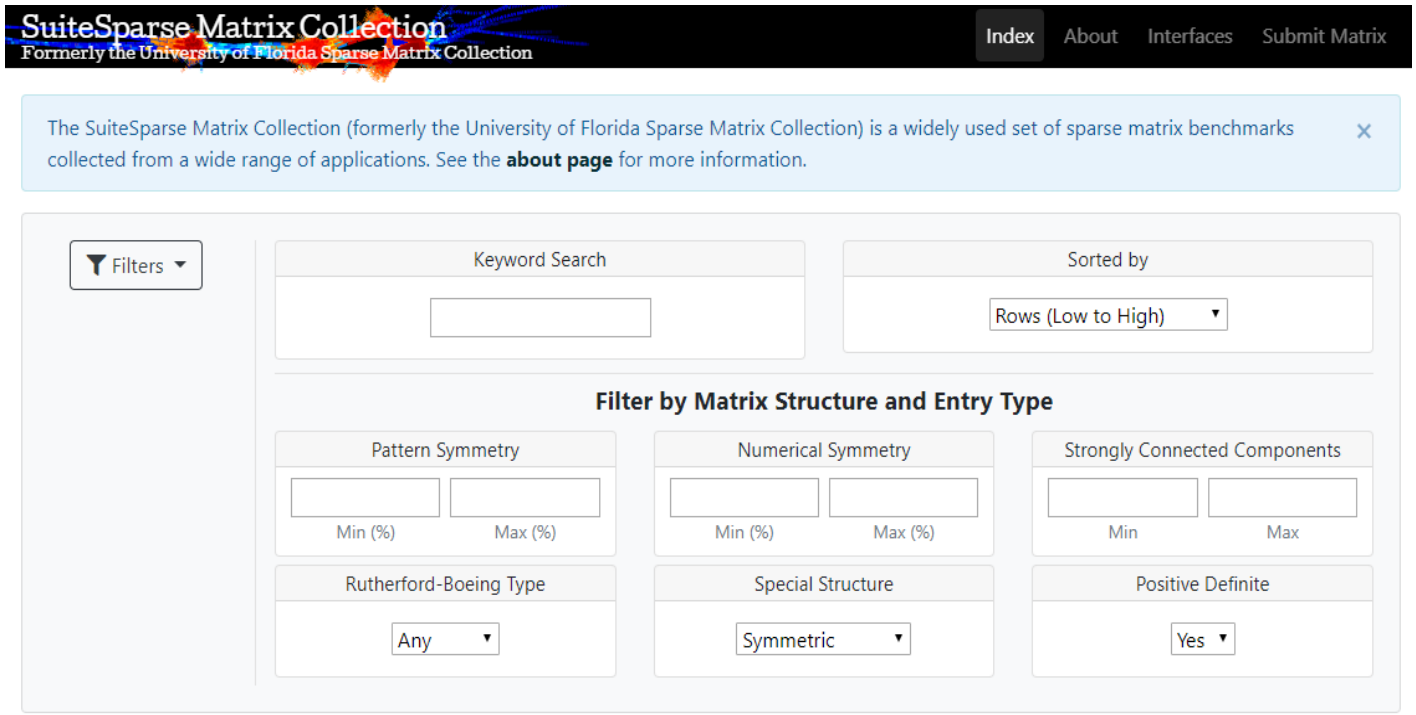
Matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 \\ 3 & 4 & 0 & 0 & 0 & 5 \\ 0 & 6 & 7 & 0 & 0 & 0 \\ 0 & 0 & 8 & 9 & 0 & 0 \\ 1 & 0 & 0 & 2 & 3 & 0 \\ 0 & 4 & 0 & 0 & 5 & 6 \end{bmatrix}$$

Sparse Storage

$$\rightarrow \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & \dots & 5 & 5 & 5 \\ 0 & 4 & 0 & 1 & 5 & \dots & 1 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & \dots & 4 & 5 & 6 \end{bmatrix} \begin{matrix} \text{rows} \\ \text{cols} \\ \text{vals} \end{matrix}$$

Os métodos iterativos tem seu desempenho diferenciado quando se tira proveito da esparsidade da matriz, para isso tanto o armazenamento da matriz quanto as multiplicações matriz-vetor não podem ser ingênuas.


SuiteSparse Matrix Collection
Formerly the University of Florida Sparse Matrix Collection
Index
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Interfaces
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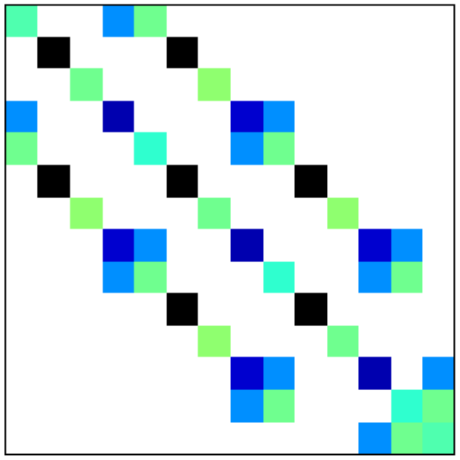
Pattern Symmetry
Min (%)
Max (%)
Numerical Symmetry
Min (%)
Max (%)
Strongly Connected Components
Min
Max
Rutherford-Boeing Type
Any
Special Structure
Symmetric
Positive Definite
Yes

MAP2210

| Id | Name | Group | Rows ▲ | Cols | Nonzeros | Kind | Date | Download File |
|---|---|---|---|---|---|---|---|---|
| 1440 | LFAT5 | Oberwolfach | 14 | 14 | 46 | Model Reduction Problem | 2004 | MATLAB Rutherford Boeing Matrix Market |
| 1438 | LF10 | Oberwolfach | 18 | 18 | 82 | Model Reduction Problem | 2004 | MATLAB Rutherford Boeing Matrix Market |
| 2203 | Trefethen_20b | JGD_Trefethen | 19 | 19 | 147 | Combinatorial Problem | 2008 | MATLAB Rutherford Boeing Matrix Market |
| 2204 | Trefethen_20 | JGD_Trefethen | 20 | 20 | 158 | Combinatorial Problem | 2008 | MATLAB Rutherford Boeing Matrix Market |
| 436 | ex5 | FIDAP | 27 | 27 | 279 | Computational Fluid Dynamics Problem | 1994 | MATLAB Rutherford Boeing Matrix Market |
| 873 | mesh1em1 | Pothen | 48 | 48 | 306 | Structural Problem | 2003 | MATLAB Rutherford Boeing Matrix Market |



LFAT5.mtx

```
%%MatrixMarket matrix coordinate real symmetric
%-------------------------------------------------------------------------------
% UF Sparse Matrix Collection, Tim Davis
% http://www.cise.ufl.edu/research/sparse/matrices/Oberwolfach/LFAT5
% name: Oberwolfach/LFAT5
% [Oberwolfach: linear 1D beam]
% id: 1440
% date: 2004
% author: J. Lienemann, A. Greiner, J. Korvink
% ed: E. Rudnyi
% fields: name title A id notes aux date author ed kind
% aux: M E B C
% kind: model reduction problem
%-------------------------------------------------------------------------------
% notes:
% Primary matrix in this model reduction problem is the Oberwolfach K matrix
%-------------------------------------------------------------------------------
14 14 30
1 1 1.57088
4 1 -94.2528
5 1 .78544
2 2 1.25664e7
6 2 -6.2832e6
3 3 .6088062015503876
7 3 -.3044031007751938
4 4 15080.447999999997
8 4 -7540.223999999998
9 4 94.2528
5 5 3.14176
```

4 matrizes escolhidas por vocês

Vetor b gerado de forma que x tenha todas as componentes unitárias

SOR com valores crescentes de ω para achar valor ótimo (tempo/iterações)

Repetir os 2 mais lentos com Gradiente Conjugado

Comparar e analisar

Entrega: 07/06/2020 23:59


AULA 13