MAP 2210 – Aplicações de Álgebra Linear

1º Semestre - 2020

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Objetivos

Formação básica de álgebra linear aplicada a problemas numéricos. Resolução de problemas em microcomputadores usando linguagens e/ou software adequados fora do horário de aula.

NÃO ESQUEÇA DE INICIAR A GRAVAÇÃO



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MAP2210

Fourth Edition

LINEAR ALGEBRA
AND ITS APPLICATIONS

Gilbert Strang

Row Exchanges and Permutation Matrices

A permutation matrix P has the same rows as the identity (in some order). There is a single "1" in every row and column. The most common permutation matrix is $P = I$ (it exchanges nothing). The product of two permutation matrices is another permutation—the rows of I get reordered twice.

After $P = I$, the simplest permutations exchange two rows. Other permutations exchange more rows. **There are $n! = (n)(n-1)\cdots(1)$ permutations of size n.** Row 1 has n choices, then row 2 has $n-1$ choices, and finally the last row has only one choice. We can display all 3 by 3 permutations (there are $3! = (3)(2)(1) = 6$ matrices):

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \quad P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \quad P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \quad P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

Exemplo 5A: Aplicar o produtos da matrizes

$$P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \qquad P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$$

em sequencia a uma matriz genérica 3 x 3, e comparar com o produto pela matriz:

$$P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

Esse exemplo ilustra a equivalência entre a troca de linhas pela multiplicação por matrizes de permutação.

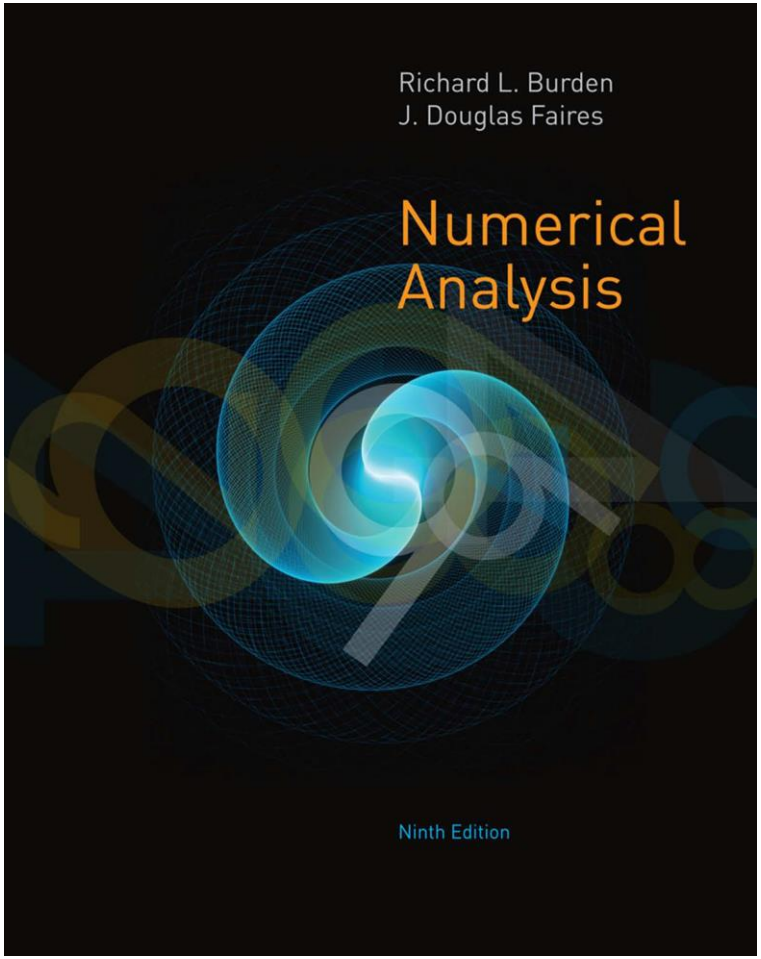
Exemplo 5B: Encontre a inversa da matriz

$$P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

Esse exemplo ilustra a propriedade.

P^{-1} is always the same as P^{T}.

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Numerical Analysis

NINTH EDITION

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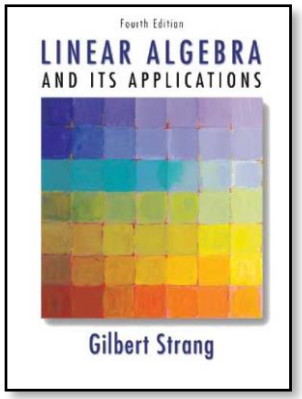
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Permutation Matrices

In the previous discussion we assumed that $A\mathbf{x} = \mathbf{b}$ can be solved using Gaussian elimination without row interchanges. From a practical standpoint, this factorization is useful only when row interchanges are not required to control the round-off error resulting from the use of finite-digit arithmetic. Fortunately, many systems we encounter when using approximation methods are of this type, but we will now consider the modifications that must be made when row interchanges are required. We begin the discussion with the introduction of a class of matrices that are used to rearrange, or permute, rows of a given matrix.

An $n \times n$ **permutation matrix** $P = [p_{ij}]$ is a matrix obtained by rearranging the rows of I_n, the identity matrix. This gives a matrix with precisely one nonzero entry in each row and in each column, and each nonzero entry is a 1.

The matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is a 3×3 permutation matrix. For any 3×3 matrix A, multiplying on the left by P has the effect of interchanging the second and third rows of A:

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

Similarly, multiplying A on the right by P interchanges the second and third columns of A. □

Two useful properties of permutation matrices relate to Gaussian elimination, the first of which is illustrated in the previous example. Suppose $k_1, \cdots, k_n$ is a permutation of the integers $1, \cdots, n$ and the permutation matrix $P = (p_{ij})$ is defined by

$$p_{ij} = \begin{cases} 1, & \text{if } j = k_i, \\ 0, & \text{otherwise.} \end{cases}$$

Then

- PA permutes the rows of A; that is,

$$PA = \begin{bmatrix} a_{k_1 1} & a_{k_1 2} & \cdots & a_{k_1 n} \\ a_{k_2 1} & a_{k_2 2} & \cdots & a_{k_2 n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k_n 1} & a_{k_n 2} & \cdots & a_{k_n n} \end{bmatrix}.$$

- P^{-1} exists and $P^{-1} = P^t$.

At the end of Section 6.4 we saw that for any nonsingular matrix A, the linear system $A\mathbf{x} = \mathbf{b}$ can be solved by Gaussian elimination, with the possibility of row interchanges. If we knew the row interchanges that were required to solve the system by Gaussian elimination, we could arrange the original equations in an order that would ensure that no row interchanges are needed. Hence there *is* a rearrangement of the equations in the system that permits Gaussian elimination to proceed *without* row interchanges. This implies that for any nonsingular matrix A, a permutation matrix P exists for which the system

$$PA\mathbf{x} = P\mathbf{b}$$

can be solved without row interchanges. As a consequence, this matrix PA can be factored into

$$PA = LU,$$

where L is lower triangular and U is upper triangular. Because $P^{-1} = P^t$, this produces the factorization

$$A = P^{-1}LU = (P^tL)U.$$

The matrix U is still upper triangular, but P^tL is not lower triangular unless $P = I$.

Example 3 Determine a factorization in the form $A = (P^t L)U$ for the matrix

$$A = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix}.$$

Solution The matrix A cannot have an LU factorization because $a_{11} = 0$. However, using the row interchange $(E_1) \leftrightarrow (E_2)$, followed by $(E_3 + E_1) \rightarrow (E_3)$ and $(E_4 - E_1) \rightarrow (E_4)$, produces

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Then the row interchange $(E_2) \leftrightarrow (E_4)$, followed by $(E_4 + E_3) \rightarrow (E_4)$, gives the matrix

$$U = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

The permutation matrix associated with the row interchanges $(E_1) \leftrightarrow (E_2)$ and $(E_2) \leftrightarrow (E_4)$ is

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

and

$$PA = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & 2 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Gaussian elimination is performed on PA using the same operations as on A, except without the row interchanges. That is, $(E_2 - E_1) \rightarrow (E_2)$, $(E_3 + E_1) \rightarrow (E_3)$, followed by $(E_4 + E_3) \rightarrow (E_4)$. The nonzero multipliers for PA are consequently,

$$m_{21} = 1, \quad m_{31} = -1, \quad \text{and} \quad m_{43} = -1,$$

and the LU factorization of PA is

$$PA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} = LU.$$

Multiplying by $P^{-1} = P^t$ produces the factorization

$$A = P^{-1}(LU) = P^t(LU) = (P^tL)U = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}. \blacksquare$$

Aparentemente temos que realizar a eliminação duas vezes para obter a fatoração, mas na verdade não é necessário se carregarmos os multiplicadores temporariamente na matriz L enquanto realizamos a eliminação, e aplicarmos as permutações **apenas** aos multiplicadores nessa matriz modificada.

O exemplo a seguir ilustra esse procedimento:

Exemplo 5C:

Encontrar a fatoração PA=LU da matriz A: $\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$

Dada a matriz A

$$\mathrm{A} = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$$

Apesar de **não** ser a boa prática numérica vamos trocar a Linha 3 pela Linha 1:

$$A_1 = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 4 & -4 \\ 2 & 1 & 5 \end{bmatrix}$$

A matriz de permutação equivalente é:

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

ou seja: $A_1 = P_1 A$

Para eliminar os elementos abaixo da diagonal da coluna 1:

- Multiplica-se a Linha 1 por (4) e subtrai-se da Linha 2
- Multiplica-se a Linha 1 por (2) e subtrai-se da Linha 3

Esses valores entre parênteses são exatamente os multiplicadores: $\mathrm{m}_{21} = 4, \mathrm{m}_{31} = 2$

E a matriz L <u>temporária</u> pode começar a ser construída

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Aplicadas as operações anteriores na matriz A_1

$$A_2 = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -8 & -8 \\ 0 & -5 & 3 \end{bmatrix}$$

Novamente apesar de **não** ser a boa prática numérica vamos trocar a Linha 2 pela Linha 3:

$$A_3 = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & 3 \\ 0 & -8 & -8 \end{bmatrix}$$

A matriz de permutação pode ser atualizada diretamente logo:

$$P_{21} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

ou seja: $P_{21} = P_2 P_1$

Para evitar a necessidade de repetir o processo para identificar a matriz L correta aplica-se a troca de linhas apenas nos multiplicadores da matriz anterior:

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

Finalmente para eliminar os elementos abaixo da diagonal da coluna 2:

- Multiplica-se a Linha 2 por $(\frac{-8}{-5})$ e subtrai-se da Linha 3

$$m_{32} = \frac{8}{5}$$

Esse multiplicador entra na sua posição correta:

$$L = L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & \frac{8}{5} & 1 \end{bmatrix}$$

Eliminando o elemento abaixo da diagonal da Coluna 2:

$$U = A_4 = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & 3 \\ 0 & 0 & -\frac{64}{5} \end{bmatrix}$$

E a matriz permutação já havia sido finalizada:

$$P = P_{21} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Logo: $PA = LU$ e: $A = P^T LU$

onde: $P^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

test_LUdecomp.nb - Wolfram Mathematica 12.0

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```
In[3]:= A = {{2, 1, 5}, {4, 4, -4}, {1, 3, 1}}

Out[3]= {{2, 1, 5}, {4, 4, -4}, {1, 3, 1}}

In[5]:= {lu, p, c} = LUDecomposition[A]
                     decomposição LU

Out[5]= {{{1, 3, 1}, {2, -5, 3}, {4, 8/5, -64/5}}, {3, 1, 2}, 0}

In[6]:= A[[p]]

Out[6]= {{1, 3, 1}, {2, 1, 5}, {4, 4, -4}}

In[8]:= l = luSparseArray[{i_, j_} /; j < i → 1, {3, 3}] + IdentityMatrix[3]
          array esparso                                    matriz identidade

Out[8]= {{1, 0, 0}, {2, 1, 0}, {4, 8/5, 1}}

In[9]:= MatrixForm[l]
        forma de matriz

Out[9]//MatrixForm=
( 1   0   0 )
( 2   1   0 )
( 4   8/5 1 )

In[11]:= u = luSparseArray[{i_, j_} /; j ≥ i → 1, {3, 3}];
          array esparso

         MatrixForm[u]
         forma de matriz

Out[12]//MatrixForm=
( 1   3    1  )
( 0  -5    3  )
( 0   0  -64/5 )
```

Vetor permutação de linhas

Exemplo 5D:

Se a fatoração $P^TLU = A$ estiver disponível encontre a solução do sistema $Ax = b$ com:

$$b=\begin{bmatrix}5\\0\\6\end{bmatrix}$$

Como a decomposição está completa

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & \frac{8}{5} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & 3 \\ 0 & 0 & -\frac{64}{5} \end{bmatrix}$$

O sistema pode ser facilmente resolvido

$Ax = \mathrm{b}$ $\qquad Ax = P^T LUx = b$ $\qquad LUx = \mathrm{P}b$

$\mathrm{Pb} = b^* = \begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix}$, logo $\qquad Ly = b^*$, portanto $\qquad y = \begin{bmatrix} 6 \\ -7 \\ -\frac{64}{5} \end{bmatrix}$

consequentemente: $\qquad Ux = y$, portanto $\qquad x = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

9. Obtain factorizations of the form $A = P^tLU$ for the following matrices.

a. $A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 3 & -6 & 9 & 3 \\ 2 & 1 & 4 & 1 \\ 1 & -2 & 2 & -2 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 1 & -2 & 3 & 1 \\ 1 & -2 & 2 & -2 \\ 2 & 1 & 3 & -1 \end{bmatrix}$

E verifique



AULA 05