Exercicio 34 Integração.

Determine uma formula de integração do tipo

$$\int_0^1 f(x)\,dx = a f(1/4) + b f(1/2) + c f(1)$$

que seja exata para todo polinomio de grau ≤ 2.
Utilize-a para aproximar a integral de $f(x) = \frac{\sin(x)}{x}$ no intervalo $[0,1]$.

Solução: $p(x) = \alpha x^2 + \beta x + \gamma$ de grau $= 2$

$$\int_0^1 p(x) = \left[\alpha \frac{x^3}{3} + \beta \frac{x^2}{2} + \gamma x\right]_0^1 = \frac{\alpha}{3} + \frac{\beta}{2} + \gamma = a\,p\left(\tfrac{1}{4}\right) + b\,p\left(\tfrac{1}{2}\right) + c\,p(1)$$

$$= a\left(\alpha\left(\tfrac{1}{4}\right)^2 + \beta\tfrac{1}{4} + \gamma\right)$$
$$+ b\left(\alpha\left(\tfrac{1}{2}\right)^2 + \beta\tfrac{1}{2} + \gamma\right)$$
$$+ c\left(\alpha + \beta + \gamma\right)$$
$$= \alpha\underbrace{\left[\frac{a}{16} + \frac{b}{4} + c\right]}_{=\frac{1}{3}} + \beta\underbrace{\left[\frac{a}{4} + \frac{b}{2} + c\right]}_{=\frac{1}{2}} + \gamma\underbrace{\left[a+b+c\right]}_{=1}$$

Obtemos 3 equações

$$\begin{cases} \frac{a}{16} + \frac{b}{4} + c = \frac{1}{3} \\ \frac{a}{4} + \frac{b}{2} + c = \frac{1}{2} \\ a + b + c = 1 \end{cases} \Leftrightarrow \begin{pmatrix} \frac{1}{16} & \frac{1}{4} & 1 \\ \frac{1}{4} & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ 1 \end{pmatrix} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ \frac{1}{4} & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{16} & \frac{1}{4} & 1 & \frac{1}{3} \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ \frac{1}{4} & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{16} & \frac{1}{4} & 1 & \frac{1}{3} \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{3}{16} & \frac{15}{16} & \frac{1}{3}-\frac{1}{16}=\frac{13}{48} \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 15 & \frac{13}{3} \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 6 & \frac{4}{3} \end{array}\right)$$

$$\Leftrightarrow \begin{cases} 6c = \frac{4}{3} \\ b+3c = 1 \\ a+b+c = 1 \end{cases} \Leftrightarrow \begin{cases} c = \frac{2}{9} \\ b = 1-\frac{2}{3} = \frac{1}{3} \\ a = 1-\frac{1}{3}-\frac{2}{9} = \frac{9-3-2}{9} \\ \quad = \frac{4}{9} \end{cases}$$

Então $\displaystyle\int_0^1 f(x)\,dx \simeq \frac{4}{9} f\left(\frac{1}{4}\right) + \frac{1}{3} f\left(\frac{1}{2}\right) + \frac{2}{9} f(1)$

Exemplo: $f(x) = 2x^2+1$

$$\int_0^1 f(x) = \left[\frac{2}{3}x^3 + x\right]_0^1 = \frac{2}{3}+1 = \frac{5}{3}$$

$$\frac{4}{9}\left(\frac{2}{16}+1\right)+\frac{1}{3}\left(\frac{2}{4}+1\right)+\frac{2}{9}(3) = \frac{\cancel{4}}{\cancel{9}}\,\frac{\cancel{18}\,8}{\cancel{16}\,42} + \frac{1}{\cancel{3}}\,\frac{\cancel{6}\,2}{\cancel{4}\,2} + \frac{2}{3}$$

$$= \frac{1}{2}+\frac{1}{2}+\frac{2}{3} = \frac{5}{3} \quad \text{ok}$$

- Agora vamos usar essa formula de integração para calcular $\int_0^1 \frac{\sin(x)}{x}\,dx$

$$\int_0^1 \frac{\sin(x)}{x}\,dx \simeq \frac{4}{9}\,\frac{\sin\left(\frac{1}{4}\right)}{1/4} + \frac{1}{3}\,\frac{\sin(1/2)}{1/2} + \frac{2}{9}\sin(1)$$

$$\simeq 0{,}9464398$$

(Usando um outro programa: $\int_0^1 \frac{\sin(x)}{x}\,dx \simeq 0{,}946083$)

Erro $\simeq 3{,}77 \times 10^{-4}$