# Dinamica Non Lineare di Strutture e Sistemi Meccanici

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# Lezione 1

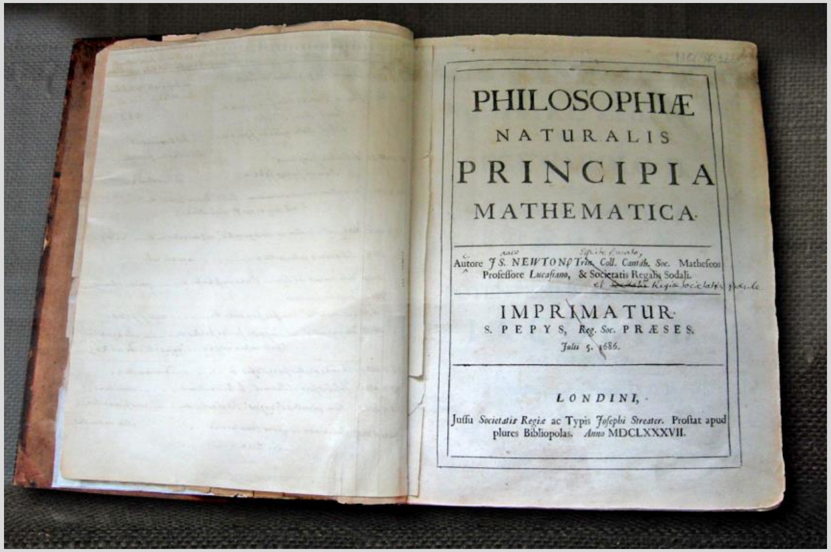
# Historical overview on dynamics

- Dynamics based on classic mechanics, whose fundamental laws are credited to Newton (1646-1727), ‘standing on giant’s shoulders”...
- Greeks: axiomatic reasoning disconnected from experimentation
  - Forces were necessarily caused by contact; what about field forces?
  - Aristotle (384BC-322BC): a force causes constant velocity?
  - Terrestrial mechanics *vs* celestial mechanics?
- Ptolemy (90-168): geocentric system vs Aristarco (310BC-230BC) heliocentric system (three centuries before)
- ...Galileo (1564-1642): ‘e pur si muove’
- Romans?

# Historical overview on dynamics

- Moslems: from VIII to XIV centuries (Alexandria, Iberic Peninsula)
    - Barakat (1080-1165) denied Aristotle: force causes velocity to change... Newton’s second law?
    - Alhazen (965-1040): body moves perpetually unless force obliges it to stop or change direction... Newton’s first law?
    - Avempace (1095-1138): to an action corresponds a reaction... Newton’s third law?
- Kepler, Copernicus and Galileo: celestial mechanics
- Galileo: terrestrial mechanics (displacement of a falling body proportional to the square of time)
- Newton: law of universal gravitation and much more...

# Historical overview on dynamics

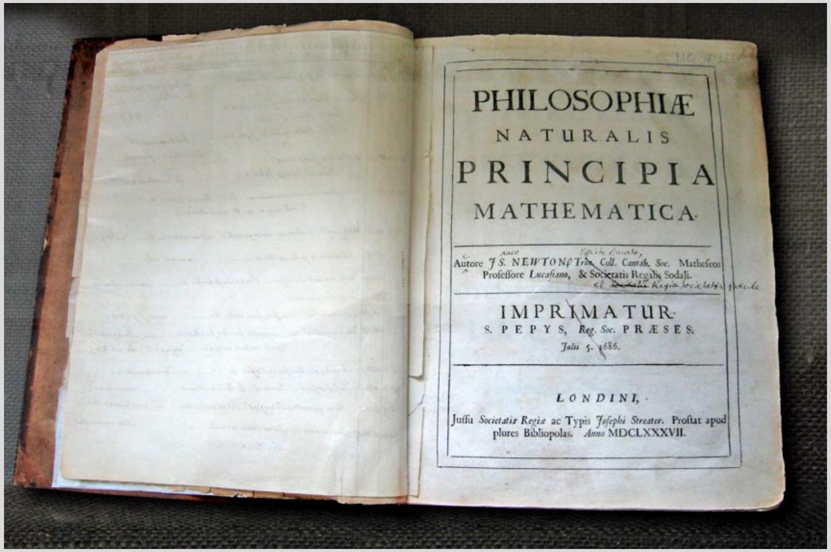



Newton’s laws

First law (inertia): there are priviledged observers, called inertial observers, with respect to whom isolated material points – that is, those subjected to null resultant force – are at rest or in uniform rectilinear motion.

*Lex I: Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare*

# Historical overview on dynamics



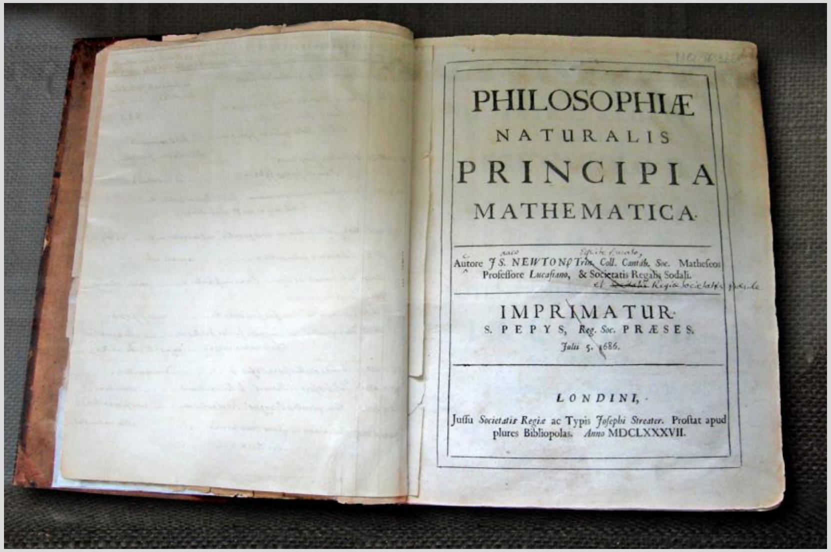

Newton’s laws

$$\vec{F}_\alpha = m_\alpha \frac{d^2 \vec{R}_\alpha}{dt^2}$$

Second law (fundamental): the resultant force of a mass point is proportional to its acceleration defined with respect to an inertial observer. The proportionality constant is termed mass, which is positive and it is a property of the material point.

*Lex II: Mutationem motus proportionalem esse vi motrici impressae, et eri secundum lineam rectam qua vis illa imprimitur*

# Historical overview on dynamics




Newton’s laws

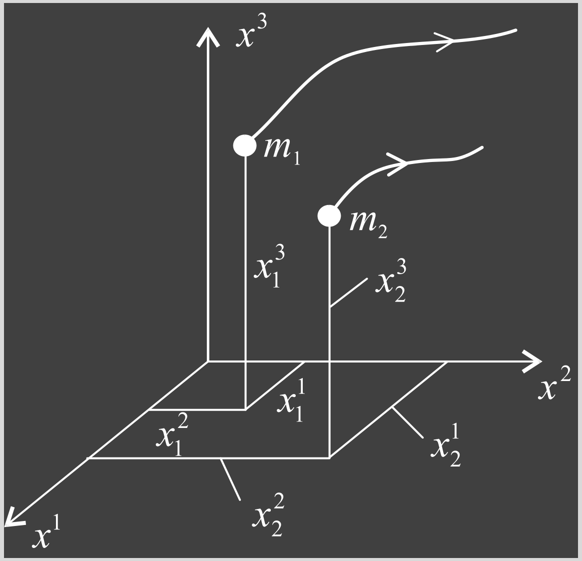
Third law (action and reaction): to every action of a material point upon another one corresponds a reaction of same intensity and direction, yet in oposite sense.

*Lex III: Actioni contrariam semper et aequalem esse reactionem: sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi*

# Historical overview on dynamics

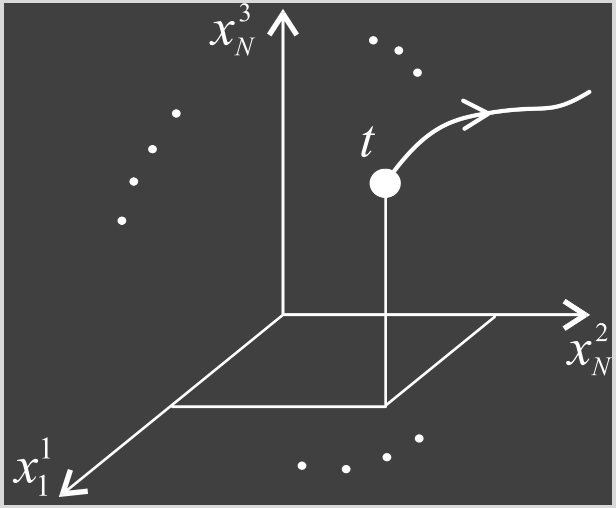
- Newton: differential and integral calculus
- Leibniz (1646-1716): independent development of differential calculus & fundamentals of analytical dynamics
- D’Alembert (1717-1783): principle...
- Lagrange (1736-1813): Mécanique Analytique and variational principles
- Hamilton (1805-1865): principle...
- ...

Physical space: affine Euclidian space of dimension 3




- $N$ material points $m_i$
- position of $m_i$ given by cartesian co-ordinates: $x_i^1$, $x_i^2$, $x_i^3$

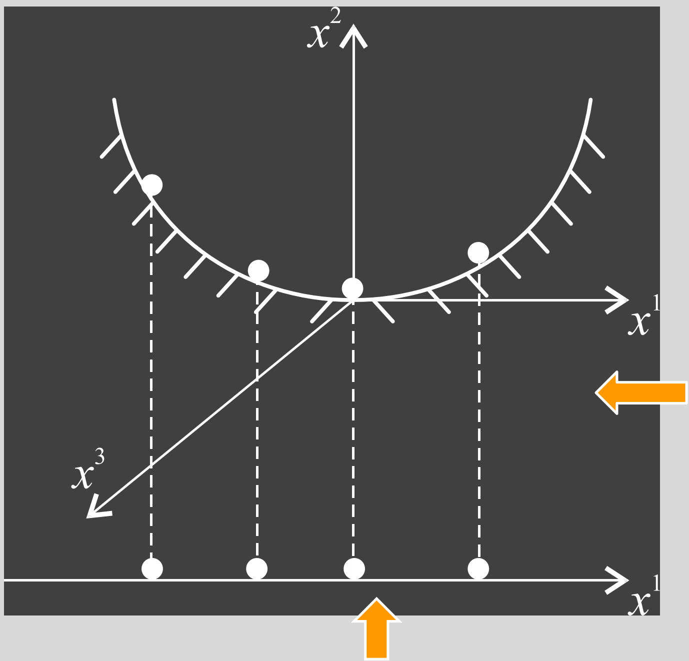
Configuration space: affine Euclidian space of dimension 3$N$ (provided the 3$N$ co-ordinates of the $N$ material points are independent)




- a "point" in this space caracterizes completely the configuration of the system of material points in a given time $t$ (co-ordinates of material points obtained by "projections")

- If there are <u>c</u> constraint equations relating these co-ordinates, it is possible to define another configuration space with dimension $n = 3N\text{-}c$ , termed “number of degrees of freedom” of the system

Example: a material point moving along a parabolic curve




$$x^3 = 0$$
$$x^2 = \beta (x^1)^2$$

$c = 2$ constraint equations

Original configuration space of dim $3N = 3$

Configuration space of dim $n = 3N\text{-}c=1$

- Generalised coordinates $Q_1(t)$, $Q_2(t)$, ... , $Q_n(t)$, $n$ = number of degrees of freedom, are scalars conveniently chosen, so that they uniquely define the original $3N$ physical coordinates of the system

$$\left.\begin{aligned} x_1^1 &= x_1^1\left(Q_1, Q_2, \ldots, Q_n, t\right) \\ x_1^2 &= x_1^2\left(Q_1, Q_2, \ldots, Q_n, t\right) \\ &\vdots \\ x_N^3 &= x_N^3\left(Q_1, Q_2, \ldots, Q_n, t\right) \end{aligned}\right\}$$

$3N$ holonomic constraint equations

- the functions $x_\alpha^i\left(Q_1, Q_2, \ldots, Q_n, t\right)$ are finite of class $C^1$

- Jacobian of the transformation is non-null

- Particular case of holonomic constraint: scleronomic constraint

$$x_\alpha^i = x_\alpha^i\left(Q_1, Q_2, \ldots, Q_n\right)$$

Example: a material point moving along a parabolic curve

$$x^1 = Q$$

$$x^2 = \beta\left(x^1\right)^2 = \beta Q^2$$

$$x^3 = 0$$

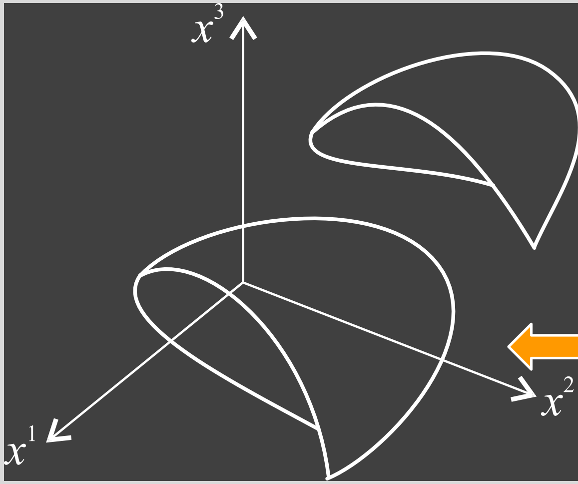
$\exists$ a transformation "matrix" (of order $n = 1$) with $\det T \neq 0$

$$\frac{\partial x^1}{\partial Q} = 1$$

Let it be $T = \left[\frac{\partial x^1}{\partial Q}\right]$

J= $\det T = 1$

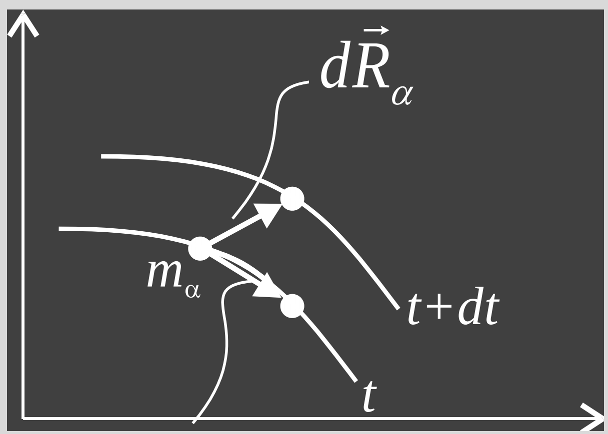
# Virtual displacements in holonomic constraints




$f_2(x^1, x^2, x^3, t_2) = 0$ constraint equation in $t_2$

$f_1(x^1, x^2, x^3, t_1) = 0$ constraint equation in $t_1$

real infinitesimal displacement




$\delta \vec{R}_\alpha \Leftrightarrow \delta x^i_\alpha \Leftrightarrow \delta Q_j$ virtual displacement

- Virtual displacements are kinematically admissible at a fixed time *t*, that is, they satisfy the constraint equations at that time *t*

- The class of real displacements doesn’t necessarily coincide with the class of real displacements for holonomic constraints

- For scleronomic constraints, however, since the constraint equations are independent of *t*, the class of real displacements coincides with the class of virtual displacements, that is, the real displacements are a particular case of virtual displacements

- Ideal (constraint) reactions are orthogonal to the virtual displacements at the points they are applied. Hence, the virtual work of ideal reactions is null.

$$\delta W = \vec{F}_{\alpha}^{vi} \cdot \delta \vec{R}_{\alpha} = 0$$

# D'Alembert's principle

Newton's 2nd law $\vec{F}_\alpha = m_\alpha \dfrac{d^2\vec{R}_\alpha}{dt^2}$ $\alpha = 1$ a $N$

$$\vec{F}_\alpha - m_\alpha \frac{d^2\vec{R}_\alpha}{dt^2} = \vec{0} \qquad \alpha = 1 \text{ a } N$$

the sum of the resultant force and the inertial force is the null vector

"closing" of the force polygon, as in statics

active non-ideal constraint

$$\vec{F}_\alpha = \vec{F}_\alpha^{\,a} + \vec{F}_\alpha^{\,vi} + \vec{F}_\alpha^{\,vn} = \text{resultant force}$$

ideal constraint

$$\vec{F}_\alpha^{\,I} = -m_\alpha \frac{d^2\vec{R}_\alpha}{dt^2} = \text{inertia force}$$

$$\sum_{\alpha=1}^{N} \left(\vec{F}_\alpha^{\,a} + \vec{F}_\alpha^{\,vi} + \vec{F}_\alpha^{\,vn} + \vec{F}_\alpha^{\,I}\right).\delta\vec{R}_\alpha = \sum_{\alpha=1}^{N} \vec{0}.\delta\vec{R}_\alpha = 0$$

$$\sum_{\alpha=1}^{N} \left(\vec{F}_\alpha^{\,a} + \vec{F}_\alpha^{\,vn} + \vec{F}_\alpha^{\,I}\right).\delta\vec{R}_\alpha = 0$$

Generalised D'Alembert's principle

- Remark 1 Effective force $\vec{F}_{\alpha}^{e} = \vec{F}_{\alpha}^{a} + \vec{F}_{\alpha}^{vn}$

- Remark 2 System with ideal constraints:

$$\sum_{\alpha=1}^{N} \left( \vec{F}_{\alpha}^{a} + \vec{F}_{\alpha}^{I} \right) . \delta \vec{R}_{\alpha} = 0$$

(it is not necessary to know a priori the reactions to write down the equations of equilibrium/motion)

- Remark 3 Principle of virtual displacements in statics is a particular case

$$\sum_{\alpha=1}^{N} \vec{F}_{\alpha}^{a} . \delta \vec{R}_{\alpha} = 0$$ ⟺ equilibrium

# Hamilton's principle

Newton's 2nd law ⟷ D'Alembert's principle ⟷ Hamilton's principle

$$\int_{t_1}^{t_2} \left( \delta T - \delta V + \delta W^{nc} \right) dt = 0$$

$\delta T$ = virtual variation of kinetic energy

$\delta V$ = virtual variation of potential energy

$\delta W^{nc}$ = virtual work of non-conservative forces

$$T = \text{kinetic energy} = \frac{1}{2} \sum_{\alpha=1}^{N} m_\alpha \left( \frac{d\vec{R}_\alpha}{dt} \cdot \frac{d\vec{R}_\alpha}{dt} \right)$$

$$\delta T = \sum_{\alpha=1}^{N} m_\alpha \left( \dot{\vec{R}}_\alpha . \delta \dot{\vec{R}}_\alpha \right)$$

notation $\dot{x} = \frac{d}{dt}(x)$

$$\delta V = -\sum_{\alpha=1}^{N} \vec{F}_\alpha^{\,c} . \delta \vec{R}_\alpha = -\ \text{virtual work of conservative forces}$$

$$\delta W^{nc} = \sum_{\alpha=1}^{N} \vec{F}_\alpha^{\,nc} . \delta \vec{R}_\alpha = \text{virtual work of non-conservative forces}$$

# Lagrange's equation

Hamilton's principle ⟺ Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}_i}\right) - \frac{\partial T}{\partial Q_i} = -\frac{\partial V}{\partial Q_i} + N_i$$

$$T = T\left(Q_1, Q_2, \ldots, Q_n, \dot{Q}_1, \dot{Q}_2, \ldots, \dot{Q}_n, t\right) \quad \text{kinetic energy}$$

$$V = V\left(Q_1, Q_2, \ldots, Q_n, t\right) \quad \text{potential energy}$$

$$N_i = \text{generalized non-conservative force} = \sum_{\alpha=1}^{N} \vec{F}_\alpha^{\,nc} \cdot \frac{\partial \vec{R}_\alpha}{\partial Q_i}$$

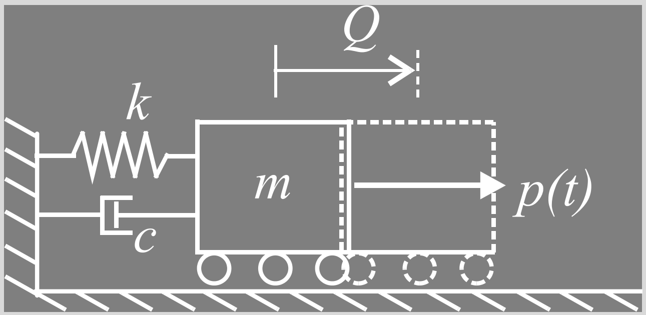
- Remark

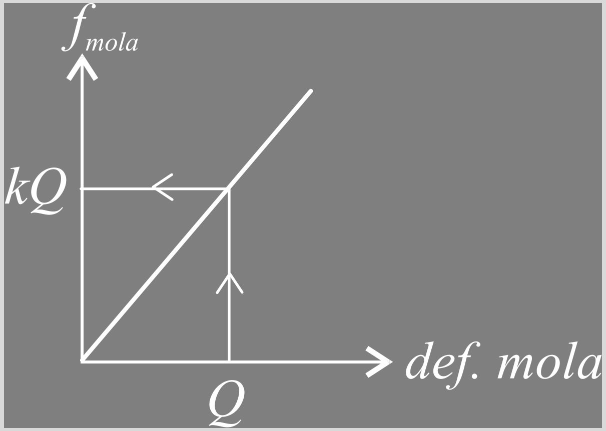
$$\sum_{i=1}^{n} N_i \delta Q_i = \sum_{\alpha=1}^{N} \vec{F}_\alpha^{\,nc} \cdot \delta \vec{R}_\alpha = \delta W^{nc}$$

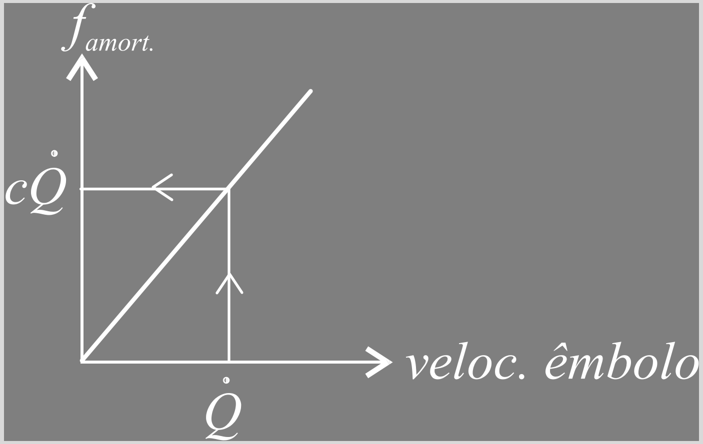
virtual work of the non-conservative forces

# Formulation of equations of motion

## Example 1: One-degree-of-freedom linear oscillator



Newton's 2nd law:

$$p(t) - kQ - c\dot{Q} = m\ddot{Q}$$

D'Alembert's principle:

$$p(t) - kQ - c\dot{Q} - m\ddot{Q} = 0$$

Generalised D'Alembert's principle:

$$\left[p(t) - kQ - c\dot{Q} - m\ddot{Q}\right]\delta Q = 0 \quad \forall \delta Q$$

$$m\ddot{Q} + c\dot{Q} + kQ = p(t)$$

Hamilton's principle: $$\int_{t_1}^{t_2} \left(\delta T - \delta V + \delta W^{nc}\right) dt = 0$$

$$T = \frac{1}{2} m \dot{Q}^2 \quad \Rightarrow \quad \delta T = m \dot{Q} \delta \dot{Q}$$

$$V = \frac{1}{2} k Q^2 \quad \Rightarrow \quad \delta V = k Q \delta Q$$

$$\delta W^{nc} = N \delta Q = \left(p(t) - c\dot{Q}\right) \delta Q$$

Substituting... $$\int_{t_1}^{t_2} m \dot{Q} \delta \dot{Q} dt + \int_{t_1}^{t_2} \left[-kQ + p(t) - c\dot{Q}\right] \delta Q dt = 0$$

integrating by parts

$$\delta Q(t_1) = \delta Q(t_2) = 0 \quad \Rightarrow \quad m \dot{Q} \delta Q \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[m \ddot{Q} + c\dot{Q} + kQ - p(t)\right] \delta Q dt = 0$$

$$\int_{t_1}^{t_2} \left[m \ddot{Q} + c\dot{Q} + kQ - p(t)\right] \delta Q dt = 0 \quad \forall \delta Q \quad \Rightarrow \quad m \ddot{Q} + c\dot{Q} + kQ = p(t)$$

Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}}\right) - \frac{\partial T}{\partial Q} = -\frac{\partial V}{\partial Q} + N$$

$$T = \frac{1}{2} m \dot{Q}^2 \quad \Rightarrow \quad \frac{\partial T}{\partial \dot{Q}} = m\dot{Q} \quad ; \quad \frac{\partial T}{\partial Q} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}}\right) = m\ddot{Q}$$

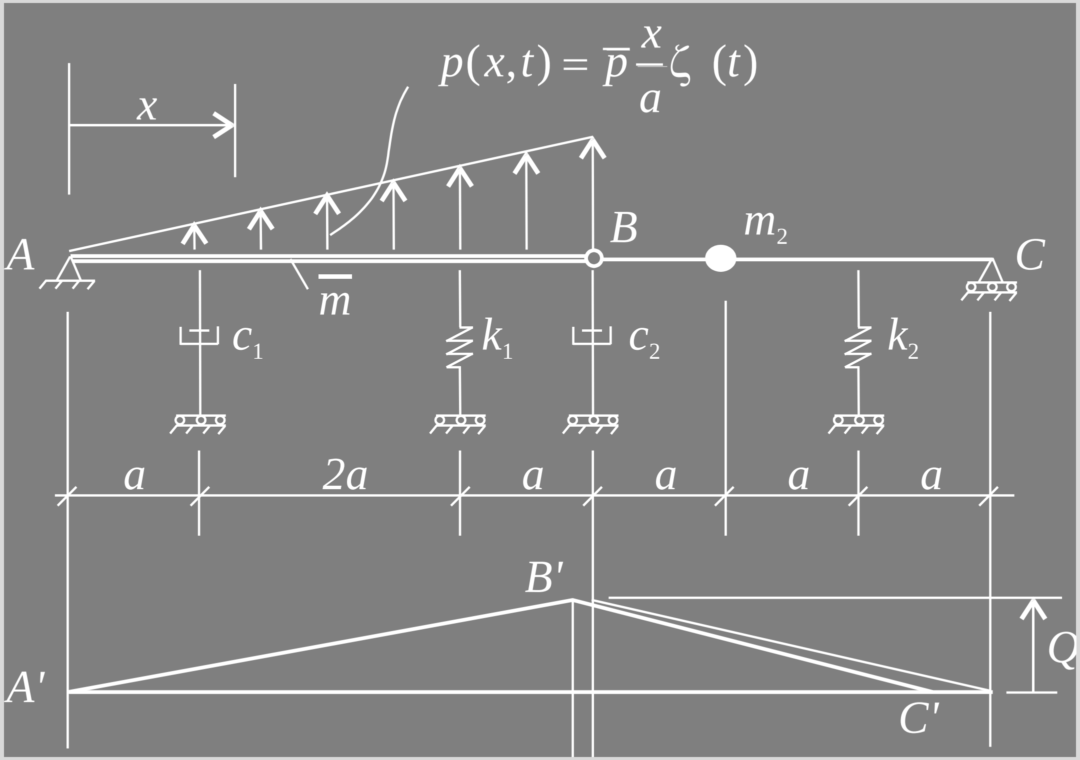
$$V = \frac{1}{2} k Q^2 \quad \Rightarrow \quad \frac{\partial V}{\partial Q} = kQ$$

$$\delta W^{nc} = N \delta Q = \left(p(t) - c\dot{Q}\right)\delta Q \quad \Rightarrow \quad N = p(t) - c\dot{Q}$$

Substituting...

$$m\ddot{Q} = -kQ + p(t) - c\dot{Q} \quad \Rightarrow \quad m\ddot{Q} + c\dot{Q} + kQ = p(t)$$

# Example 2: sistem of rigid rods




*AB* and *BC* rigid rods

*BC* massless rod

Linear dynamics: horizontal displacements of B and C are negligible for small Q

$$T = \frac{1}{2} m_2 \left( \frac{2}{3} \dot{Q} \right)^2 + \int_0^{4a} \frac{1}{2} \bar{m} \left( \frac{x}{4a} \dot{Q} \right)^2 dx = \frac{1}{2} m^* \dot{Q}^2$$

with $m^* = \frac{4}{9} m_2 + \frac{4}{3} \bar{m} a$

$$V = \frac{1}{2} k_1 \left( \frac{3}{4} Q \right)^2 + \frac{1}{2} k_2 \left( \frac{1}{3} Q \right)^2 + \int_0^{4a} \bar{m} g \left( \frac{x}{4a} Q \right) dx + m_2 g \left( \frac{2}{3} Q \right)$$

$$V = \frac{1}{2} k^* Q^2 - p_0^* Q$$

with $k^* = \frac{9}{16} k_1 + \frac{1}{9} k_2$

and $p_0^* = -\left( 2 \bar{m} a + \frac{2}{3} m_2 \right) g$

$$\delta W^{nc} = -c_1 \frac{\dot{Q}}{4}\frac{\delta Q}{4} - c_2 \dot{Q}\,\delta Q + \int_0^{4a} \bar{p}\,\frac{x}{a}\,\zeta(t)\left(\frac{x}{4a}\delta Q\right)dx = N\,\delta Q$$

$$N = -c^*\dot{Q} + p^*(t)$$

with $c^* = \frac{c_1}{16} + c_2$

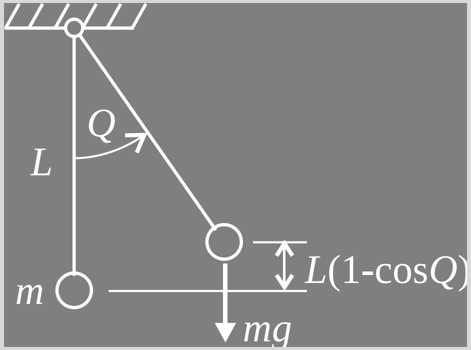
and $p^*(t) = \frac{16}{3}\bar{p}a\zeta(t)$

Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}}\right) - \frac{\partial T}{\partial Q} = -\frac{\partial V}{\partial Q} + N$$

$$m^*\ddot{Q} + c^*\dot{Q} + k^*Q = p_0^* + p^*(t)$$

# Example 3: Simple pendulum




$$T = \frac{1}{2} m \left( L \dot{Q} \right)^2$$

$$V = + m g L \left( 1 - \cos Q \right)$$

Lagrange's equation:

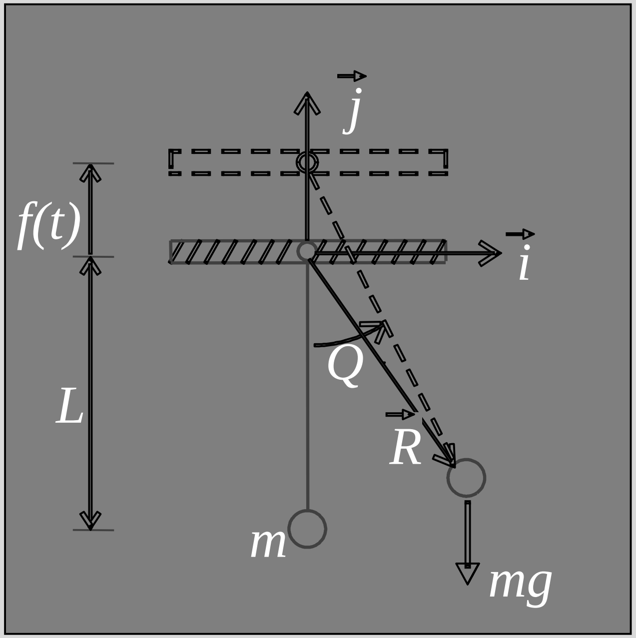
$$\frac{d}{dt}\left( \frac{\partial T}{\partial \dot{Q}} \right) - \frac{\partial T}{\partial Q} = - \frac{\partial V}{\partial Q} + N$$

$$m L^2 \ddot{Q} = - m g L \operatorname{sen} Q$$

$$\text{or} \quad \ddot{Q} + \frac{g}{L} \operatorname{sen} Q = 0 \quad \text{(non-linear analysis)}$$

$$\ddot{Q} + \frac{g}{L} Q = 0 \quad \text{(linear analysis)}$$

# Example 4: Simple pendulum subjected to support excitation




$$\vec{R} = L \operatorname{sen} Q \, \vec{i} + (f - L \cos Q) \, \vec{j}$$

$$\dot{\vec{R}} = L \dot{Q} \cos Q \, \vec{i} + (\dot{f} + L \dot{Q} \operatorname{sen} Q) \, \vec{j}$$

$$T = \frac{1}{2} m \left( L^2 \dot{Q}^2 \cos^2 Q + L^2 \dot{Q}^2 \operatorname{sen}^2 Q + 2 L \dot{f} \dot{Q} \operatorname{sen} Q + \dot{f}^2 \right)$$

$$V = m g \left[ f + L \left( 1 - \cos Q \right) \right]$$

$$T = \frac{1}{2} m L^2 \dot{Q}^2 + \frac{1}{2} m \dot{f}^2 + m L \dot{f} \dot{Q} \operatorname{sen} Q$$

Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}}\right) - \frac{\partial T}{\partial Q} = -\frac{\partial V}{\partial Q} + N$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}}\right) = mL^2\ddot{Q} + mL\ddot{f}\sin Q + mL\dot{f}\dot{Q}\cos Q$$

$$\frac{\partial T}{\partial Q} = mL\dot{f}\dot{Q}\cos Q$$

$$\frac{\partial V}{\partial Q} = mgL\sin Q$$

$$mL^2\ddot{Q} + mL\ddot{f}\sin Q + mL\dot{f}\dot{Q}\cos Q - mL\dot{f}\dot{Q}\cos Q = -mgL\sin Q$$

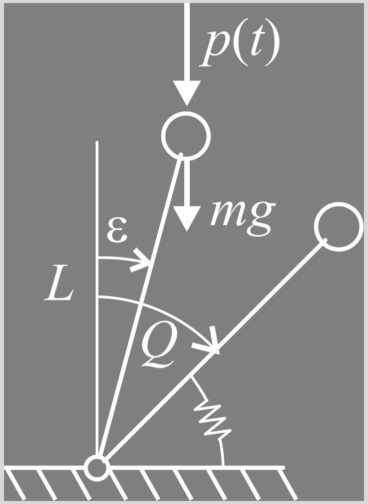
$$mL^2\ddot{Q} + mL(g + \ddot{f})\sin Q = 0$$

ou $\ddot{Q} + \frac{1}{L}(g + \ddot{f})\sin Q = 0$ (non-linear analysis)

$\ddot{Q} + \frac{1}{L}(g + \ddot{f})Q = 0$ (linear analysis)

# Example 5: Rigid rod with non-linear spring and geometric imperfection, subjected to static and dynamic loading

geometric imperfection $\varepsilon <<< 1$




Non-linear "constitutive" law

$$M(Q) = K(Q-\varepsilon)\left[1-(Q-\varepsilon)^2\right]$$

$$T = \frac{1}{2} m L^2 \dot{Q}^2$$

$$V = \int_0^{Q-\varepsilon} K\theta\left[1-\theta^2\right]d\theta - mgL(\cos\varepsilon - \cos Q) = K\left[\frac{(Q-\varepsilon)^2}{2} - \frac{(Q-\varepsilon)^4}{4}\right] - mgL(\cos\varepsilon - \cos Q)$$

$$\delta W^{nc} = N \delta Q = P(t) L \operatorname{sen} Q \delta Q$$

$$N = P(t) L \operatorname{sen} Q$$

Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}}\right) - \frac{\partial T}{\partial Q} = -\frac{\partial V}{\partial Q} + N$$

$$m L^2 \ddot{Q} + K(Q-\varepsilon)\left[1-(Q-\varepsilon)^2\right] = \left[m g + P(t)\right] L \sin Q$$