17. Spontaneous Symmetry Breaking (SSB)   17.1   25/10/19

Many classical and quantum systems are such that the lowest energy state does not exhibit the symmetries of its Lagrangian. For instance, a ferromagnetic material at temperature $T < T_{Curie}$ has the rotational symmetry spontaneously broken.

17.1 Fabri-Picasso theorem: (1966)

Let's assume that $\mathcal{L}$ is invariant under a 1-parameter continuous transformation whose conserved current is J^μ.

$Q = \int d^3x\, J^0(x)$ (1) is conserved and the associated transformation is $U(\lambda) = e^{i\lambda Q}$. Fabri and Picasso showed that there are 2 possibilities:

i) $Q|0\rangle = 0$ and $|0\rangle$ is an eigen-state of Q with eigenvalue 0. In this case we say that the symmetry is manifest or realized in the Wigner-Weyl way.

ii) $Q|0\rangle$ does not exist since its norm is infinite! Usually this case is described as $Q|0\rangle \neq 0$.

Proof. for an internal symmetry $[P^\mu, Q] = 0$. (2)

we want $\langle 0|QQ|0\rangle = \int d^3x\, \langle 0|J^0(x)\, Q|0\rangle$

However, $\langle 0|J^0(x)\, Q|0\rangle = \langle 0|e^{iP_\mu x^\mu} J^0(0)\, e^{-iP_\mu x^\mu} Q|0\rangle$

momentum operator.

$$\overset{(2)}{\underset{P_\mu|0\rangle=0}{=}} \langle 0|\, J^0(0)\, Q\, e^{-iP_\mu x^\mu} |0\rangle = \langle 0|J^0(0)\, Q|0\rangle$$

$$\Rightarrow \langle 0|Q\,Q|0\rangle = \int d^3x\, \langle 0|J^0(x)\, Q|0\rangle = \int d^3x\, \langle 0|J^0(0)\, Q|0\rangle \quad (3)$$

therefore, either $Q|0\rangle = 0$ or $Q|0\rangle$ has an infinite norm!

Curious theorem by Coleman (J. Math. Phy. 7(1966) 787):

If $Q(t) = \int d^3x\, J^0(x)$ is such that $Q|0\rangle = 0$

$\Rightarrow \partial_\mu J^\mu = 0$ (!!) and that Q is independent of time

and the symmetry is realized by unitary operators $U = e^{i\alpha Q}$

Notice the vacuum is playing a big role!

Intuitive idea: If $H|0\rangle = E_0|0\rangle$

the state $|\alpha\rangle = e^{i\alpha Q}|0\rangle$ is degenerate with $|0\rangle$:

$$H|\alpha\rangle = \underbrace{H\, e^{i\alpha Q}}_{\text{commute}}|0\rangle = e^{i\alpha Q} H|0\rangle = E_0\, e^{i\alpha Q}|0\rangle = E_0|\alpha\rangle$$

So, there is a family of degenerate states with the vacuum! An excitation along this direction does not require ~~an~~ extra energy $\Rightarrow$ the associated particle is massless! Usually, we choose $E_0 \equiv 0$.

Let's prove this!

17.2 The Goldstone theorem (1961)

In the case $Q|0\rangle = 0$, the spectrum reflects the symmetry with the appearance of multiplets. However, in case ii of Fabri-Picasso theorem there will be a massless particle called the Nambu-Goldstone boson.

Proof: Given the charge Q (eq'n (1)) we consider that we are in case (ii), ie, "$Q|0\rangle \neq 0$".

Suppose that $\phi(y)$ is not invariant under the symmetry generated by Q

$$\langle 0|[Q,\phi(y)]|0\rangle = \langle 0|\phi'(y)|0\rangle$$

$\neq 0$ since Q does not annihilate $|0\rangle$

Notice that ϕ' has a non-vanishing vacuum expectation value!

So, $$0 \neq \langle 0|\left[\int d^3x \, j^0(x), \phi(y)\right]|0\rangle \qquad (4)$$

Moreover,

$$\frac{\partial}{\partial x^0} \langle 0|\left[\int d^3x \, j^0(x), \phi(y)\right]|0\rangle \overset{\partial_\mu J^\mu = 0}{=} \langle 0|\left[\int d^3x \, \nabla \cdot \vec{J}(x), \phi(y)\right]|0\rangle$$

$$\overset{\text{Stokes}}{=} -\langle 0| \int dS\hat{n} \cdot [\vec{J}(x), \phi(y)]|0\rangle \qquad (5)$$

If the fields fall off fast enough in the infinity (or any other boundary condition), eq'n (5) tells us that

$$\langle 0|\left[\int d^3x \, j^0(x), \phi(y)\right]|0\rangle \quad \text{is independent of } x^0!$$

We can construct this state by

$$|\pi(\vec{p})\rangle = -\frac{2i}{F}\int d^3_x \, e^{i\vec{p}\cdot\vec{x}} \, J^0(x)\,|0\rangle$$

$$P^j\,|\pi(\vec{p})\rangle = -\frac{2i}{F}\int d^3_x \, e^{i\vec{p}\cdot\vec{x}} \left(P^j J^0 - \underbrace{J^0 P^j}_{=0}\right)|0\rangle$$

$$= -\frac{2i}{F}\int d^3_x \, e^{i\vec{p}\cdot\vec{x}} \,[P^j, J^0]\,|0\rangle$$

$$= -\frac{2i}{F}\int d^3_x \, e^{i\vec{p}\cdot\vec{x}} (-i)\,\partial^j J^0 \,|0\rangle$$

$$= \vec{P}\,|\pi(\vec{p})\rangle$$

oh!

More over,

$$H\,|\pi(0)\rangle = -\frac{2i}{F}\, H \underbrace{\int d^3x\, J^0(x)}_{Q}\,|0\rangle = -\frac{2i}{F}\, H\, Q|0\rangle$$

~~[illegible]~~ 0 since ~~[illegible]~~ it is degenerate with the vacuum!

$\Rightarrow |\pi(\vec{p})\rangle$ is massless!

Now we introduce a complete set of states into (4):

$$0 \neq \int d^3x \sum_n \left\{ \langle 0|J^0(x)|n\rangle\langle n|\phi(y)|0\rangle - \langle 0|\phi(y)|n\rangle\langle n|J^0(x)|0\rangle \right\}$$

$$\underbrace{J^0(x) = e^{iP_\mu x^\mu} J^0(0)\, e^{-iP_\mu x^\mu}}$$

$$= \int d^3x \sum_n \left\{ \langle 0|J^0(0)|n\rangle\langle n|\phi(y)|0\rangle e^{-iP_n\cdot x} - \langle 0|\phi(y)|n\rangle\langle n|J^0(0)|0\rangle e^{+iP_n\cdot x} \right\}$$

$\int d^3x$ $\Downarrow$

$$0 \neq \sum_n \delta^{(3)}(\vec{p}_n) \left\{ \langle 0|J^0(0)|n\rangle\langle n|\phi(y)|0\rangle e^{-iP_{n0}x_0} - \langle 0|\phi(y)|n\rangle\langle n|J^0(0)|0\rangle e^{iP_{n0}x_0} \right\} \qquad (6)$$

Notice that the right-hand side of (6) is time independent. For states with $\vec{p}_n = 0$ we have $P_{n0} = m_{n0}$ (its mass). In order to (6) hold, ~~we~~ ~~need~~ there must exist a state $|n\rangle$ that is massless and for which $\langle 0|J^0(0)|n\rangle \neq 0$. This is the Goldstone theorem!

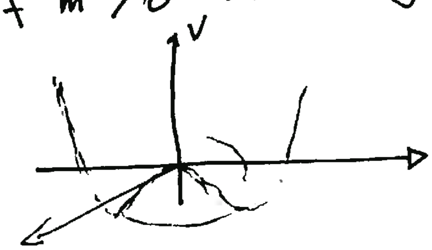
17.3 Example: Linear sigma model

Consider a complex field φ with

$$\mathcal{L} = \partial_\mu\varphi^* \partial^\mu\varphi + m^2\varphi^*\varphi - \frac{\lambda}{4}(\varphi^*\varphi)^2 \qquad (7)$$

(7) is invariant under $\varphi \to \varphi' = e^{i\alpha}\varphi$ for constant α.

If $m^2 > 0$ the theory is unstable around $\varphi = 0$:



At lowest order the effective potential is given by

$$V_{eff} = -m^2|\varphi_c|^2 + \frac{\lambda}{4}|\varphi_c|^4 \qquad (8)$$

- From page (10.19), we know that V_{eff} is the energy ~~of a the~~ density of the system in a state whose expectation value of φ is φ_c.

- The vacuum minimizes $V_{eff} \Rightarrow |\varphi_c|^2 = \frac{2m^2}{\lambda}$ (8)

$\Rightarrow$ there is a collection of vacua $|\Omega_\theta\rangle \Rightarrow \langle\Omega_\theta|\varphi|\Omega_\theta\rangle = \sqrt{\frac{2m^2}{\lambda}}\, e^{i\theta}$

- All vacua are equivalent so we choose the one with $\theta=0$ $|\Omega_0\rangle = |\Omega\rangle$

$$\langle\Omega|\varphi|\Omega\rangle = \sqrt{\frac{2m^2}{\lambda}} \equiv v \quad (9)$$

- It is convenient to parametrize φ as:

$$\varphi(x) = \left(\sqrt{\frac{2m^2}{\lambda}} + \frac{1}{\sqrt{2}}\sigma(x)\right) e^{i\frac{\pi(x)}{F_\pi}} \quad (10)$$

real

Notice that σ and π have no vacuum expectation value.

- (10) into (7) leads to

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \left(\sqrt{\frac{2m^2}{\lambda}} + \frac{1}{\sqrt{2}}\sigma\right)^2 \frac{1}{F_\pi^2}(\partial_\mu\pi)^2$$
$$- \left(-\frac{m^4}{\lambda} + m^2\sigma^2 + \frac{1}{2}\sqrt{\lambda}\, m\,\sigma^3 + \frac{\lambda}{16}\sigma^4\right) \quad (11)$$

- π is canonically normalized for $F_\pi = \frac{2m}{\sqrt{\lambda}}$

- the spectrum includes:
  - massless ~~state~~ field π ← Goldstone boson.
  - massive field σ

- ~~After the breaking of~~ the symmetry $\varphi \to \varphi e^{i\alpha}$, can be written as ~~[illegible]~~

$$\sigma \to \sigma \qquad \pi \to \pi + F_\pi\,\theta \qquad (12)$$

Consistency between (6) and (14):

According to our notation

$$\Pi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left\{ a(\vec{p})\, e^{-ip x} + a^\dagger(\vec{p})\, e^{ip\cdot x} \right\}$$

With
$$[a(\vec{p}), a^\dagger(\vec{p}')] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')$$

and
$$|\vec{p}\rangle = \sqrt{2E_p}\, a^\dagger(\vec{p})|0\rangle \Rightarrow \langle\vec{p}|\vec{p}'\rangle = (2\pi)^3\, 2E_p\, \delta^{(3)}(\vec{p}-\vec{p}')$$

The one-particle states participating e.in $\mathbb{1} = \sum_n |n\rangle\langle n|$ is

$$\int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} |\vec{k}\rangle\langle\vec{k}|$$

Now, $\langle 0|J^\mu|\vec{p}'\rangle \overset{(13)}{=} F_\pi \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \langle 0|\, a(\vec{p})e^{-ip\cdot x}(-i p^\mu) + ip^\mu \underbrace{a^\dagger(\vec{p})e^{ip\cdot x}|\vec{p}'\rangle}_{0}$

$$= -ip^\mu F_\pi e^{-ip\cdot x}. \qquad (14).$$

Focusing only on the one-particle intermediate states in (6) (the other do not contribute), we write

$$0 \neq \int \frac{d^3p_n}{(2\pi)^3 2E_p} \delta^3(\vec{p}_n) \{ \underbrace{\langle 0|\, J^0(0)|\vec{p}_n\rangle}_{-iE_p F_\pi} \langle n|\phi(y)|0\rangle e^{-ip_n^0 x^0} - \langle 0|\phi(y)|\vec{p}_n\rangle\langle\vec{p}_n|J^0(0)|0\rangle e^{i}$$

So the factors E_p cancels out! The resulting expression is non-vanishing!

- The Noether current is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \pi} \frac{\delta \pi}{\delta \theta} = F_\pi \partial_\mu \pi \qquad (13)$$

that leads to $\langle \Omega | J^\mu(x) | \pi(p) \rangle = -i p^\mu F_\pi e^{-ipx}$. (14).

$\Rightarrow$ see 17.6A

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17.4 Number of ~~masters~~ Goldstone bosons

To know the number of Goldstone bosons let's consider a set of hermitian scalar fields φ_n and that $\mathcal{D}\phi$ and $S[\phi]$ are invariant under

$$\varphi_n \longrightarrow \varphi_n + i\epsilon \sum_m t_{nm} \varphi_m(x) \qquad (15)$$

The effective action is invariant under (15) meaning that

$$i\Gamma = \sum_{n,m} \int d^4x \, \frac{\delta \Gamma[\varphi]}{\delta \varphi_n(x)} \underbrace{t_{nm}\varphi_m(x)}_{\approx \delta \varphi_n} = 0 \qquad (16)$$

In the case of theories with translation invariance φ_n is constant and we can use the effective potential $\Gamma = -(\text{volume spacetime})\, V_{eff}$:

$$\sum_{nm} \frac{\delta V_{eff}}{\delta \varphi_n} t_{nm} \varphi_m = 0 \qquad (17)$$

now $\frac{\delta(17)}{\delta \varphi_l} \Rightarrow \sum_n \frac{\delta V_{eff}}{\delta \varphi_n} t_{nl} + \sum_{nm} \frac{\delta^2 V_{eff}}{\delta \varphi_n \delta \varphi_l} t_{nm} \varphi_m = 0 \quad (18)$

For the minimum of V_{eff}, ie, at the vacuum expectation value (vev) $\bar{\varphi}$ we have

$\frac{\delta V_{eff}}{\delta \varphi_n}\Big|_{\bar\phi} = 0$ and (18) leads to

$$\sum_{n,m} \frac{\delta^2 V_{eff}(\varphi)}{\partial \phi_n \partial \phi_e}\Big|_{\bar\phi} t_{nm} \bar\varphi_m = 0 \quad (19)$$

Let's interprete (19):

$\frac{\delta^2 \Gamma}{\delta\phi(y)\,\delta\phi(x)}$ is the inverse of the propagator $\underset{p=0}{\Longrightarrow}$ $\frac{\partial^2 V_{eff}}{\partial\phi_n \partial\phi_e}$ is the mass $\overset{\text{squared}}{\text{matrix}}$ for the fields φs

$$M^2_{ne} = \frac{\partial^2 V_{eff}}{\partial\varphi_n \partial\varphi_e} \quad (20)$$

So, (19) reads

$$\sum_{nm} M^2_{ne}\, t_{nm} \bar\Phi_m = 0 \quad (21)$$

$\Rightarrow \sum_m t_{nm}\bar\Phi_m$ is a vanishing eigenvalue of the mass matrix, i.e, a Goldstone boson! The number of Goldstone bosons is dimension of the vector space generated by $t\phi$. So there is a Goldstone boson for every independent broken symmetry!

<u>Example</u>: Let's consider N real scalar fields φ_n

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\varphi_n\partial^\mu\varphi_n - \frac{\mu^2}{2}\varphi_n\varphi_n - \frac{\lambda}{4}(\varphi_n\varphi_n)^2 \quad (22)$$

In the tree approximation, $V_{eff} \cong \frac{\mu^2}{2}\varphi_n\varphi_n + \frac{\lambda}{4}(\varphi_n\varphi_n)^2 \quad (23)$

The minimum of V_{eff} follows from

$$\frac{\partial V_{eff}}{\partial \varphi_k}\bigg|_{\bar\varphi} = 0 \Rightarrow \left[\mu^2 + \lambda \bar\varphi_n \bar\varphi_n\right]\bar\varphi_k = 0$$

For $\mu^2 < 0 \Rightarrow \sum_n \bar\varphi_n \bar\varphi_n = -\frac{\mu^2}{\lambda}$ (24)

Notice that $\mathcal{L}$ exhibits an $O(N)$ symmetry:

$\Lambda \in O(N) \qquad \varphi_n \to \Lambda_{nm} \varphi_m$

The mass matrix in this case is

$$M^2_{nm} = \frac{\partial^2 V_{eff}}{\partial \varphi_n \partial \varphi_m}\bigg|_{\bar\varphi} = \mu^2 \delta_{nm} + \lambda \delta_{nm} \sum_k \bar\varphi_k \bar\varphi_k + 2\lambda \bar\varphi_n \bar\varphi_m$$

$$= 2\lambda \bar\varphi_n \bar\varphi_m$$

This matrix has just one non-vanishing eigenvalue

$$m^2 = 2\lambda \bar\varphi_n \bar\varphi_n = 2|\mu^2|$$

and $(N-1)$ zero eigenvalues orthogonal to it! So the symmetry breaking pattern is

$$O(N) \longrightarrow O(N-1)$$

$\swarrow$ # generators

$\frac{1}{2}N(N-1)$ $\qquad$ $\frac{1}{2}(N-1)(N-1)$ $\qquad \Longrightarrow$ # broken generators $\frac{1}{2}N(N-1) - \frac{1}{2}(N-1)(N-2) = N-1$

17.5 Goldstone bosons in QCD

$\boxed{17.8\,A}$

Let's consider the QCD Lagrangian containing only massless up and down quarks:

$$\mathcal{L} = -\frac{1}{4}(G^a_{\mu\nu})^2 + i\bar{u}\not{D}u + i\bar{d}\not{D}d \qquad (25\text{-}A)$$

$$= -\frac{1}{4}(G^a_{\mu\nu}) + \overline{\begin{pmatrix} u_L \\ d_L \end{pmatrix}}_L i\not{D}\begin{pmatrix} u_L \\ d_L \end{pmatrix}_L + \overline{\begin{pmatrix} u_R \\ d_R \end{pmatrix}}_R i\not{D}\begin{pmatrix} u_R \\ d_R \end{pmatrix} \qquad (25\text{-}B)$$

where we used that $\bar{f}\not{D}f = \bar{f}\not{D}(P_R+P_L)f = \bar{f}_L\not{D}f_L + \bar{f}_R\not{D}f_R$. This Lagrangian is invariant ~~under~~ under the global transformations:

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \to U_L\begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_R \\ d_R \end{pmatrix} \to U_R\begin{pmatrix} u_R \\ d_R \end{pmatrix} \qquad (25\text{-}C)$$

where $U_L \in SU(2)_L$ and $U_R = SU(2)_R$ (~~[illegible]~~)

just to distinguish the transformations

Moreover, (25.A) is also invariant under

$$q = \begin{pmatrix} u \\ d \end{pmatrix} \longrightarrow e^{i\theta_V + i\theta_A\gamma_5}\begin{pmatrix} u \\ d \end{pmatrix} \qquad (25\text{-}D)$$

So, the Lagrangian is invariant under $SU(2)_L \otimes SU(2)_R \otimes U_V(1) \otimes U_A(1)$

We will not treat $U(1)_A$ now since it's broken by quantum effects (the chapter 2... will symmetries are anomalies!). The conserved currents associated to this symmetries

$$J^a_\mu = \bar{q}\sigma^a\gamma^\mu q \qquad J^{5a}_\mu = \bar{q}\sigma^a\gamma^\mu\gamma^5 q \qquad J_\mu{}^V = \bar{q}\gamma^\mu q \qquad J_\mu^A = \bar{q}\gamma^\mu\gamma^5 q \qquad (2\ldots$$

It's convinient to introduce the scalar (matrixed) operators

$$\Sigma_{ij} = \bar{q}_{Lj}q_{Ri} \qquad (25\text{-}F)$$

Since $q'_{Ri} = (U_R)_{ik}\, q_{Rk}$ 17.8B

$$q'_{Lj} = (U_L)_{jk} \Longrightarrow \bar{q}'_{Lj} = (U_L^\dagger)_{kj}\, \bar{q}_{Lk} \tag{25-G}$$

we have that

$$\Sigma \longrightarrow \Sigma' = U_R \Sigma U_L^\dagger \tag{25-H}$$

$$\Sigma^\dagger_{ij} = \bar{\psi}_{Rj}\psi_{Li} \longrightarrow \Sigma'^\dagger = U_L \Sigma^\dagger U_R$$

We know empirically that

$$\langle 0|\Sigma_{ij}|0\rangle \neq 0 \quad (\approx \Lambda^3_{QCD}) \tag{25-I}$$

Even without explaining (25-I) we can obtain useful information!
In addition,

$$\langle 0|\Sigma_{ij}|0\rangle = \langle 0|\Sigma^\dagger_{ij}|0\rangle \tag{25-J}$$

that implies that parity is not spontaneously broken:

$$\langle 0|\underbrace{\bar{q}_i \gamma_5 q_j}_{\downarrow P \atop -\bar{q}_i\gamma_5 q_j}|0\rangle = \langle 0|(\Sigma - \Sigma^\dagger)_{ji}|0\rangle = 0.$$

Since $\langle 0|\Sigma|0\rangle$ is an hermitian matrix, it can be written as

$$\langle 0|\Sigma|0\rangle = v\,\mathbb{1} + v_3\,\sigma^3 \tag{25-K}$$

We want the chiral symmetry to be broken ~~to~~ from $SU(2)_L \otimes SU(2)_R$ to $SU(2)_{L+R}$, so we choose $v \neq 0$ and $v_3 = 0$. With this choice (dictated by nature)

$\langle 0|\Sigma|0\rangle$ is invariant under $U_L = U_R$. Since we broke

$$SU(2)_L \times SU(2)_R \longrightarrow SU(2)_{L+R}$$

we have 3 Goldstone bosons that are the pions!

eferences up to here:

- Aitchison "An informal introduction to gauge field theories", sections 6.1 to 6.7
- Itzykson - Zuber, sections 11.1 and 11.2
- Weinberg (volume II) sections 19.1 and 19.2
- Schwartz sections 28.1 and 28.2
- Pokorski sections 9.4 to 9.6

01/11/19

17.5 Abelian Higgs Model

There is a remarkable exception to the Goldstone theorem[*]: when we have local gauge symmetry, ~~the~~ massless states can disappear from the spectrum!

Let's consider the following Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu\varphi)^* D^\mu\varphi + m^2 |\varphi|^2 - \frac{\lambda}{4} |\varphi|^4 \tag{26}$$

where $D_\mu = \partial_\mu + igA_\mu$. For $m^2 > 0$, we have seen, that the ground state is such that

$$|\langle\varphi\rangle| = \frac{v}{\sqrt{2}} = \sqrt{\frac{2m^2}{\lambda}}$$

*) Goldstone's theorem requires Lorentz invariance and Hilbert space with positive-definite scalar products. Gauge theories do not meet both requirements simultaneously!

So, writing φ as in (10) 17.10

$$\varphi(x) = \frac{1}{\sqrt{2}}(v+\sigma(x))\, e^{i\pi(x)/F_\pi} \qquad (10')$$

with $F_\pi = \sqrt{2}\, v$. Substituting (10') into (26) leads to

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left[(\partial_\mu - igA_\mu)\left(\frac{v+\sigma}{\sqrt{2}}\right)e^{-i\pi/F_\pi}\right]\left[(\partial^\mu + igA^\mu)\frac{v+\sigma}{\sqrt{2}}\, e^{i\pi/F_\pi}\right]$$

$$- \left(-\frac{m^4}{\lambda} + m^2\sigma + \frac{1}{2}\sqrt{\lambda}\, m\sigma^3 + \frac{1}{16}\lambda\sigma^4\right) \qquad (27)$$

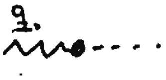
The quadratic part of (27) contains

$\rightarrow\ + \frac{1}{2}g^2v^2 A_\mu A^\mu$

$\rightarrow$ mixing A^μ-π : $2g\left(\frac{v}{\sqrt{2}}\right)^2\frac{1}{F_\pi} A_\mu \partial^\mu \pi$ (28)

$\Rightarrow$ still not clear what are the propagating degrees of freedom

$\Downarrow$

q  $\equiv g v q_\mu$

First choice: unitary gauge

$$\varphi \longrightarrow \varphi' = e^{-i\frac{\pi}{F_\pi}}\varphi = \frac{1}{\sqrt{2}}(v+\sigma)$$

$$A_\mu \longrightarrow A'_\mu = A_\mu + \frac{1}{g}\partial_\mu\left(\frac{\pi}{F_\pi}\right) \qquad (29)$$

leading to, (up to a constant)

$$\mathcal{L} = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}g^2(\sigma+v)^2A'_\mu A'^\mu - \frac{1}{4}\lambda(\sigma^2+2\sigma v)^2 \qquad (30)$$

$\Rightarrow$ vector A'_μ with mass $m_A = g v$

real scalar σ with mass $\sqrt{2\lambda v^2} = 2m^2$

Problem of this gauge choice:

- unitarity is manifest order by order in perturbation theory
- analyses of renormalizability is problematic due to the vector field propagator

$$\frac{-i}{k^2-M_A^2}\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{M_A^2}\right) \xrightarrow[|k|]{\text{large}} \approx \frac{k^\mu k^\nu}{k^2 M_A^2} \sim \text{constant!}$$

- the ghost propagator is a constant!

$$\text{----} \;\; \frac{i}{m_{gh}^2} \quad \text{where } m_{gh}^2 \text{ is an arbitrary parameter!}$$

- counting of degrees of freedom:

before SSB

massless $A_\mu \to 2$

complex $\varphi \to 2$

total 4

after SSB

massive $A_\mu \to 3$

massive real $\sigma \to 1$

total 4

<u>Second choice R_ξ gauge</u>: here, the goal is to get rid of the mixing A^μ-

So, we choose the gauge fixing function to be

$$F(A_\mu,\chi) = -\frac{1}{2\xi}\left(\partial^\mu A_\mu - \xi g v \chi\right)^2 \qquad (31)$$

$$\text{where} \quad \varphi = \frac{1}{\sqrt{2}}\left(v+\sigma+i\chi\right) \qquad (32)$$

In this gauge, the propagators are given by (homework)

$$A_\mu : \quad \frac{-i}{p^2 - M_A^2 + i\epsilon}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2 - \xi M_A^2} \right]$$

$$\chi : \quad \frac{i}{p^2 - \xi M_A^2 + i\epsilon}$$

$$\sigma : \quad \frac{i}{p^2 - 2\lambda v^2 + i\epsilon}$$

Notice the following ~~speci~~ values of ξ:

- $\xi = 0 \equiv$ Landau gauge
- $\xi = 1$ 't Hooft - Feynman gauge
- $\xi \to \infty$ unitary gauge.

The unphysical poles at $p^2 = \xi M_A^2$ cancel in ~~the~~ S-Matrix elements
The renormalized S-matrix is independent of ξ.

17.6 SSB in Non-Abelian Gauge Theories

Let's consider

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \frac{1}{2}(\underbrace{D_\mu H^a}_{\partial_\mu + ig\frac{\sigma^a}{2}A^a_\mu})^*(D^\mu H^a)^a + \mu^2 H^{a*}H^a - \lambda (H^{a*}H^a)^2 \quad (33)$$

where the gauge symmetry is SU(2) and H^a is a ~~complex~~ dublet of SU(2) of complex scalar fields.

$\boxed{17.15}$

At tree level, V_{eff} is

$$V_{eff}^{tree} = -\mu^2 M^{a*} M^a + \lambda (M^{a*} M^a)^2$$

$$\left.\frac{\partial V_{eff}^{tree}}{\partial M^b}\right|_{\bar{M}^b} = 0 \underset{\mu^2>0}{\Longrightarrow} \langle 0|\bar{M}^b M^b|0\rangle = \frac{\mu^2}{2\lambda} = \frac{v^2}{2} \qquad (34)$$

Choosing $\langle M\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$ we have that all SU(2) generators are broken:

$$\sigma^a \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \neq \quad \text{for } a=1,2,3!$$

Now, we write

$$M = \exp\left(i\,\frac{\xi^a \sigma^a}{v\;2}\right)\begin{pmatrix} 0 \\ \frac{v+\eta}{\sqrt{2}} \end{pmatrix} \qquad (35)$$

with the would-be Goldstone boson fields ξ^a. In the unitary gauge we eliminate the ξ^a fields, like in (29),

$$A^a_\mu \longrightarrow A'^a_\mu \qquad (36)$$

$$M \longrightarrow M' = e^{-i\frac{\xi^a \sigma^a}{v\;2}} M = \begin{pmatrix} 0 \\ \frac{v+\eta}{\sqrt{2}} \end{pmatrix}$$

and we obtain that the quadratic part of the Lagrangian is

$$\mathcal{L}_{quad} = \frac{1}{2}\left(\partial_\mu \eta \partial^\mu \eta - 2\lambda v^2 \eta^2\right) - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \frac{g^2 v^2}{8}\left((A^1_\mu - i A^2_\mu)(A^{1\mu} + i A^{2\mu}) + A^3_\mu A^{3\mu}\right) \quad (3\ldots$$

Therefore, we end up with 1 massive real scalar and 3 massive vector fields! Degrees of freedom $\begin{cases} \text{initially} \to 3\times 2+4 = 10 \\ \text{final} \to 3\times 3+1 = 10 \end{cases}$ OK ✓

Goal: to study if it is possible SSB if the tree level vacuum has vanishing vev.

Warm up: Let's consider a massless $\lambda\varphi^4$ that is massless and study the breaking of $\varphi \leftrightarrow -\varphi$ symmetry:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{\lambda}{4!}\varphi^4 + \underbrace{\frac{A}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{B}{2}\varphi^2 - \frac{C}{4!}\varphi^4}_{\text{counterterms}} \qquad (38)$$

As we've seen before the 1-loop effective potential is

$$V_{eff}^{1\text{-loop}} = \frac{\lambda}{4!}\varphi^4 + \frac{B}{2}\varphi^2 + \frac{C}{4!}\varphi^4 + \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left(1 + \frac{\lambda\varphi^2}{2k^2}\right) \qquad (39)$$

Going into euclidean space and cutting off the integral at $k_E^2 = \Lambda^2$ leads to

$$V_{eff}^{1\text{-loop}} = \frac{\lambda}{4!}\varphi^4 + \frac{B}{2}\varphi^2 + \frac{C}{4!}\varphi^4 + \frac{\lambda\Lambda^2}{64\pi^2}\varphi^2 + \frac{\lambda^2\varphi^2}{256\pi^2}\left[\ln\left(\frac{\lambda\varphi^2}{2\Lambda^2}\right) - \frac{1}{2}\right]$$

Imposing that

$$\left.\frac{d^2V_{eff}}{d\varphi^2}\right|_{\varphi=0} = 0 \quad\Rightarrow\quad B = -\frac{\lambda\Lambda^2}{32\pi^2}$$

$$\left.\frac{d^4V_{eff}}{d\varphi^4}\right|_{\varphi=M} = \lambda(M)$$

$$V_{eff}^{1\text{-loop}} = \frac{\lambda(M)}{4!}\varphi^4 + \frac{\lambda^2(M)\varphi^4}{256\pi^2}\left(\ln\frac{\varphi^2}{M^2} - \frac{25}{6}\right. \qquad (40)$$

$M \neq 0$ due to IR divergences

Looking for the minimum of (40) $\Rightarrow$

$$\frac{\lambda}{6}\varphi^3 + \frac{\lambda^2\varphi^3}{64\pi^2}\left(\ln\frac{\varphi^2}{M^2} - \frac{25}{6}\right) + \frac{\lambda^2\varphi^3}{128\pi^2} = 0$$

$$\Rightarrow \begin{cases} \langle\varphi\rangle = 0 & \text{(maximum)} \\ \lambda \ln\frac{\langle\varphi\rangle}{M} = -\frac{32}{3}\pi^2 + \mathcal{O}(\lambda) \quad (41) & \text{(minimum)} \end{cases}$$

However, (41) is not reliable since higher order corrections lead to higher powers of $\lambda \ln\frac{\varphi^2}{M^2}$ in V_{eff}!

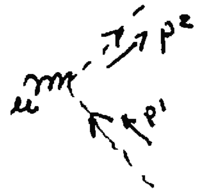
Scalar Eletrodynamics: (Coleman-Weinberg)

Let's consider

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu\phi)^\dagger(D^\mu\varphi) - \frac{\lambda}{4!}\varphi^4 \qquad (42)$$

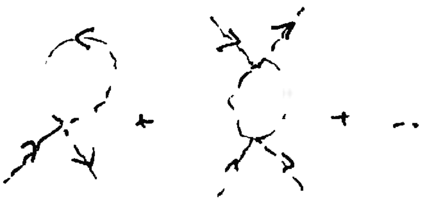
with $D_\mu\varphi = (\partial_\mu + igA_\mu)\varphi$

Feynman rules: we choose to work in the Landau gauge ($\xi=0$ in (17.12))

 $= \frac{-i}{k^2}\left[g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right]$  $\equiv ig(-p^1_\mu - p^2_\mu)$

 $2ig^2 g_{\mu\nu}$  $= -i\lambda$

One-loop contributions to V_{eff}:



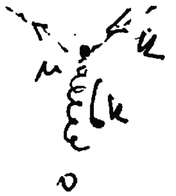
I

II

III

The diagrams in class (II) vanish identically for ~~the~~ V_{eff} since $P^{\mu}_{ext} = 0$:

in our vertex



$$\equiv k^{\mu} \times \left(g^{\mu\nu} - \frac{k^{\mu}k^{\nu}}{k^2} \right) \equiv 0 \, !$$

Notice

$$-i\lambda \cdot \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2}$$

$\Rightarrow$ are proportion[al]

$$i 2g^2 \cdot \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2} \underbrace{\left(g^{\mu\nu} - \frac{k^{\mu}k^{\nu}}{k^2} \right) g_{\mu\nu}}_{3}$$

$$(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \left(\frac{i}{k^2} \right)^2$$

proportional

$$(i2g^2)^2 \int \frac{d^4k}{(2\pi)^4} \left(\frac{-i}{k^2} \right)^2 \underbrace{\left(g^{\mu\nu} - \frac{k^{\mu}k^{\nu}}{k^2} \right) \left(g^{\nu\mu} - \frac{k^{\mu}k^{\nu}}{k^2} \right)}_{g^{\alpha\beta}\left(g_{\alpha\beta} - \frac{k_{\alpha}k_{\beta}}{k^2} \right) \equiv 3}$$

Following the same procedure that we did before with the renormalizati[on] conditions

$$\left. \frac{d^2 V_{eff}}{d\varphi \, d\varphi^*} \right|_{\varphi=\varphi^*=0} = 0$$

and

$$\left. \frac{d^4 V_{eff}}{d\varphi^2 \, d\varphi^{*2}} \right|_{\varphi=\varphi^*=M} = \lambda_R$$

we obtain

$$V_{eff} = \frac{\lambda_R \varphi_c^4}{4} + \frac{3g^4}{16\pi^2}\varphi_c^4\left(\ln\frac{2\varphi_c^2}{M^2} - \frac{25}{6}\right) \quad \text{with } \varphi_c^2 = \varphi_c^* \varphi_c$$

If we choose the vev of φ_c to be $\frac{M}{\sqrt{2}}$; then

$$\left.\frac{dV_{eff}}{d\varphi_c}\right|_{\varphi_c = \frac{M}{\sqrt{2}}} = 0 \Rightarrow \lambda_R = \frac{g^4 \, 11}{4\pi^2}$$

So finally we have

$$V_{eff} = \frac{3g^4}{16\pi^2}\varphi_c^4\left(\ln\left(\frac{\varphi_c^2}{\langle\varphi_0\rangle^2}\right) - \frac{1}{2}\right)$$

Notice that i) $\lambda_R \ll g^2$

ii) We have dimensional transmutation since the original La has no mass scale. This break leads to

$$m_A^2 = g^2 \langle\varphi\rangle^2$$

and

$$m_S = \frac{3e^4}{8\pi^2}\langle\varphi\rangle^2$$

$$\xrightarrow{\text{prediction}} \frac{m_S^2}{m_V^2} = \frac{3e^2}{8\pi^2}$$

References for sections 17.5 ~~and~~, 6 and 7

- Pokorski sections 11.1 and 11.2
- E. Weinberg's thesis whose link is available in ~~the~~ moodle
- Schwartz 28.3 and 28.4.