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## 16.a Motivation

We saw that QED can be "derived" from the free model

$$\mathcal{L}_0 = \bar{\Psi}(i\not{\partial}-m)\Psi \qquad (1)$$

by requiring the ~~local~~ global invariance $\Psi\to\Psi' = e^{i\alpha}\Psi$ (constant $\alpha$) to be promoted to local invariance

$$\Psi \longrightarrow e^{-iqe\Lambda(x)}\Psi = \Psi' \qquad (2)$$

$$\bar{\Psi} \longrightarrow \bar{\Psi}' = e^{iqe\Lambda(x)}\bar{\Psi} .$$

This is accomplished by:

i) Changing

$$\partial_\mu \longrightarrow D_\mu \equiv \partial_\mu + iqeA_\mu \qquad (3)$$

such that

$$D'_\mu\Psi' = e^{-iqe\Lambda} D_\mu\Psi \qquad (4)$$

with

$$A'_\mu = A_\mu + \partial_\mu\Lambda$$

ii) Adding a kinetic term for the new gauge field $A_\mu$

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (5)$$

invariant under (4)

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{1}{iqe}[D_\mu, D_\nu] \qquad (6)$$

This procedure leads to

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\not{D}-m)\Psi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\not{\partial}-m)\Psi - qe\bar{\Psi}\gamma^\mu\Psi A_\mu \qquad (7)$$

Idea: A Lie group is a group G whose elements can be expressed in terms of N continuous parameters ($\theta^a$). A group element continuously connected to the identity can be written as

$$U = \exp[-i\,\theta^a T^a] \qquad (8)$$

where $T^a$ are the group generators.

The algebra of the $T^a$'s is called a Lie algebra. Due to the group properties

$$[T^a, T^b] = i f^{abc} T^c \qquad (9)$$

→ depend on the group multiplication law

$f^{abc}$ are the structure constants. If $f^{abc}=0$ the group is said to be abelian, otherwise it is ~~an~~ non-abelian group.

The generators $T^a$ obey the Jacobi identity

$$[T^a,[T^b,T^c]] + [T^b,[T^c,T^a]] + [T^c,[T^a,T^b]] = 0 \qquad (10)$$

that is just a ~~property~~ property of commutators! (9+10) leads to

$$f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0 \qquad (11)$$

Representation: A representation is an association of an operator in a vector space to group elements such that

$$a \in G \longrightarrow M(a) : V \to V$$

with ~~that~~ $ab = c \Rightarrow M(a)\,M(b) = M(c)$ (12)

The dimension of a representation is the one of V. $d(R)$

Of course, we can work with the matrices $M_{ij} = \langle i | M(a) | j \rangle$

A representation is said to be reducible if it can be written ~~as~~ in the block-diagonal form

$$M(a)_{ij} = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

If the representation can not be brought to this form it is called irreducib[le]

Fact: Any representation of a compact Lie group is equivalent to a representation by unitary operators.

The generators of the group are not unique, since a linear combination of the generators is also a generator. In fact $\{T^a\}$ form a vector space!
We can ~~now~~ choose the generators such that

for generator $T^a \longrightarrow T^a_R$ ↳ $T^a$ representation,

$$\mathrm{Tr}[T^a_R T^b_R] = C(R)\,\delta^{ab} \qquad (13)$$

Hint to the proof: $\mathrm{Tr}[T^a_R T^b_R]$ is a symmetric matrix in $ab$!

[C]asimir operator: To classify the representations in a basis-independent way one use the ~~Ca~~ Casimir operator(s) that commutes with all generators. One [e]xample is

$$C_2 = \sum_a T^a T^a \qquad (14)$$

[S]chur's lemma implies that $T^a T^\alpha$ is proportional to the unit:

$$T^a_R T^a_R = C_2(R)\,\mathbb{1} \qquad (15)$$

Note that $\sum_{ab} (13)\Big|_{a=b} \Rightarrow \quad d(R)\,C_2(R) = C(R)\,d(G) \quad (16)$
↳ number of generators of the group

The fundamental representation is the one with smallest $d(R)$. Usually we choose to have $C(\text{fundamental}) = \frac{1}{2} \qquad (17)$

In the <u>adjoint representation</u>, we have

$$[T^b_{Adj}]_{ac} = i f^{abc} \qquad (18)$$

Exercise: Show that this is in fact a representation!

Notice that $d(\mathrm{adj}) = d(G)$!

For each irreducible representation $R$ we can define a <u>complex conjugate representation</u> $\bar{R}$ by

$$T^a_{\bar{R}} = -(T^a_R)^* \underset{\substack{\uparrow \\ T^a \text{ is hermitian}}}{=} -(T^a)^T \qquad (19)$$

Exercise: Show that $T^a_{\bar{R}}$ satisfies (19).

A <u>representation</u> is said to be <u>real</u> if there is an unitary transformation $U$ such that

$$T^a_{\bar{R}} = U T^a_R U^{-1}$$

i.e., $T^a_{\bar{R}}$ and $T^a_R$ are equivalent. For example, the adjoint representation is real

$$(T^b_{\overline{adj}})_{ac} = -i f^{cba} \underset{f \text{ is anti-symmetric}}{=} i f^{abc} = [T^b_{ADJ}]_{ac}$$

## A few classical groups:

<u>$SO(N)$</u>: For $u$ and $v$ $N$-dimensional vector, we define the linear transformation

$$u \to u' = Ou \qquad v \to v' = Ov$$

such that $\langle u|v\rangle = \langle u'|v'\rangle \Rightarrow O^+ = O^{-1} \qquad (20)$

Further, we require that

$$\det U = 1 \qquad (21)$$

to remove a freedom in the choice of $U$. Notice that

$U = e^{i\theta^a T^a}$ with $T^a$ hermitian. Moreover $(21) \Rightarrow Tr[T^a] = 0$.

The dimension of $SU(N)$ is $d(SU(N)) = N^2 - 1$. Notice that the fundamental representation is used to define the group and that the casimir of this representation is

$$C_F \equiv C_2(\text{fund}) = \frac{1}{\underbrace{d(\text{fund})}_{N}} \underbrace{C(\text{fund})}_{\frac{1}{2} \text{ by convention}} \overbrace{d(SU(N))}^{N^2-1} = \frac{N^2-1}{2N} \qquad (22)$$

On the other hand, ~~for the adjoint representation~~, we can define the generators such that

$$\sum_{cd} f^{acd} f^{bcd} = N \delta^{ab} \qquad (23)$$

With this convention,

$$\cancel{\phantom{xx}}\; C(\text{adjoint}) = N \qquad (24)$$

and

$$C_A = C_2(\text{adjoint}) = \frac{1}{\cancel{N^2-1}} N \cancel{(N^2-1)} = N$$

For $SU(2)$ $f^{abc} = \epsilon^{abc}$ and the fundamental representation is

$$T^a = \frac{\sigma^a}{2} \qquad (25)$$

where $\sigma^a$ are the Pauli matrices.

SO(N): Given the N-dimensional ~~vecto~~ real vector $u$ and $\theta$ [16.6]

we perform a linear transformation $O$

$$u' = O u \qquad \theta' = O u$$

such that $\langle u' | \theta' \rangle = \langle u | \theta \rangle \Rightarrow O = O^T \qquad (26)$

furthermore, we also impose that $\det O = 1$, (27)

to exclude reflections. In this case

$$d(SO(N)) = \frac{N(N-1)}{2} \qquad (28)$$

To learn more: see Georgi, "Lie Algebras in Particle Physics", chapters 1 and 2.

Srednicki, "Quantum Field Theory", chapter 70.

## 16.C Non-Abelian Gauge Symmetry: Classical aspects [C.N. Yang & R. Mills, Phys. Rev. 96 (1954) 91]

Let us consider a set of spinorial fields $\psi_j$ and a group $G$. A global transformation of the fields $\psi_j$ is

$$\psi_j \longrightarrow \psi'_j = \left(e^{-i\theta^a T^a_R}\right)_{jk} \psi_j \qquad (29)$$

where $R$ is an unitary representation of $G$ and the parameters $\theta^a$ are constant. This transformation is a symmetry of the $\mathcal{L}$

$$\mathcal{L} = \sum_j \bar{\psi}_j \,(i\partial\!\!\!/ - m)\,\psi_j \qquad (30)$$

where we assume that all fields in the multiplet have the same mass. Usually the $\sum_j$ is ommited, and in general we do not write even the $j$! In this case we understand that $\psi$ stands for the multiplet!!

In analogy to QED we want to promote the global ~~local~~ symmetry (29) to a local one:

$$\psi_j \longrightarrow \psi'_j = \left( e^{-i\theta^a(x) T^a_R} \right)_{jk} \psi_k \qquad (30)$$

For that we need to trade the derivative $\partial_\mu$ by a covariant one, in analogy to QED

$$\partial_\mu \psi \longrightarrow D_\mu \psi = \left[ \partial_\mu + i g A^a_\mu(x) T^a_R \right] \psi \qquad (31)$$

such that

$$[D_\mu \psi]' = D'_\mu \psi' = \underbrace{e^{-i\theta^a(x) T^a_R}}_{\substack{\parallel \\ U(x)}} D_\mu \psi \qquad (32)$$

so,

$$D'_\mu \psi' = \left( \partial_\mu + i g T^a_R A'^a_\mu \right) U \psi = U \left( \partial_\mu + i g U^{-1} T^a_R A'^a_\mu U + U^{-1} \partial_\mu U \right) \psi$$

$$= U \left( \partial_\mu + i g T^a_R A^a_\mu \right) \psi$$

$$\Rightarrow T^a_R A'^a_\mu(x) = U(x) T^a_R A^a_\mu(x) U^{-1}(x) + \frac{i}{g} (\partial_\mu U) U^{-1} \qquad (33)$$

~~The terms~~ For an infinitesimal transformation $U = 1 - i\theta^b T^b_R$ we have

$$T^a_R A'^a_\mu(x) = T^a_R A^a_\mu + \frac{1}{g} \partial_\mu \theta^b T^b_R + i \underbrace{\left[ A^a_\mu T^a_R , \theta^b T^b_R \right]}_{} \qquad (34)$$

ut $[A^b_\mu T^b_R, \theta^c T^b_R] = i f^{bca} T^a_R A^b_\mu \theta^c$ [16.8]

$$= i f^{abc} T^a_R A^b_\mu \theta^c$$

(34) $\Rightarrow$ $A'^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu \theta^a - f^{abc} A^b_\mu \theta^c$ (35)

lotice that for a global transformation

$$A'^a_\mu T^a = U T^a_R A^a_\mu U^{-1} \quad (36)$$

n the top of that, $A^a_\mu$ is on the adjoint representation of G!

Now, we need the analog of $F_{\mu\nu}$ in (6) to construct an invariant kinetic term for $A^a_\mu$:

$$[D_\mu, D_\nu]\Psi = [\partial_\mu + igA^a_\mu T^a_R, \partial_\nu + igA^b_\nu T^b_R]\Psi$$

$$= \underbrace{ig\{(\partial_\mu A^a_\nu T^a_R - \partial_\nu A^a_\mu T^a_R) + ig[A^a_\mu T^a_R, A^b_\nu T^b_R]\}}\Psi \quad (37)$$

$$\equiv G_{\mu\nu}\Psi$$

lotice that

$$\left.\begin{aligned} [D'_\mu, D'_\nu]\Psi' &= U[D_\mu, D_\nu]\Psi = U G_{\mu\nu}\Psi \\ &\parallel \\ G'_{\mu\nu}\Psi' &= G'_{\mu\nu} U\Psi \end{aligned}\right\} \Rightarrow G'_{\mu\nu} = U G_{\mu\nu} U^{-1} \quad (38)$$

Defining $G_{\mu\nu} = ig G^a_{\mu\nu} T^a_R \overset{(37)}{\Longrightarrow} G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$ (39)

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(38) implies that $Tr[G_{\mu\nu}G^{\mu\nu}]$ is gauge invariant. So, we can write the gauge-invariant Lagrangian density

$$\mathcal{L} = \frac{1}{2g^2} Tr[G_{\mu\nu}G^{\mu\nu}] + \bar{\Psi}(i\not{D}-m)\Psi \qquad (40)$$

where

$$\frac{1}{2g^2}Tr[G_{\mu\nu}G^{\mu\nu}] = \frac{1}{2g^2}(-g^2)\underbrace{Tr[T_R^a T_R^b]}_{\text{imposing that } = \frac{1}{2}\delta^{ab}}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{adc} A_\mu^d A_\nu^c) \times$$
$$\times(\partial^\mu A^{\nu b} - \partial^\nu A^{\mu b} - g f^{bef} A^{e\mu} A^{e\nu})$$

$$= -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + g f^{abc} A_\mu^b A_\nu^c \partial^\mu A^{\nu a}$$
$$- \frac{1}{4} g^2 f^{abc} f^{aed} A_\mu^b A_\nu^c A^{\mu e} A^{\nu d} \qquad (41)$$

The EOM of the fields in (40) are:

$$\partial_\mu G^{a\mu\nu} - g f^{abc} A_\mu^b G^{c\mu\nu} = g \bar{\Psi}\gamma^\nu T^a \psi \qquad (42) \checkmark$$

and

$$(i\not{\partial} - m)\psi = g \not{A}^a T^a \Psi \qquad (43)$$

Exercise: Obtain (42) and (43).

We can see from (42) that not only the fermions generate gauge fields but also the gauge field themselves, ie, they carry charge!

To obtain the conserved charge, we note that (40) is invariant under global transformations (29) and (36) that in the infinitesimal form are:

$$\psi \to \psi'_j = \left(1_{jk} - i\,\theta^a T^a_{jk}\right)\Psi_k \qquad (43)$$

and

$$A^a_\mu \to A'^a_\mu = A^a_\mu - f^{abc} A^b_\mu \theta^c \qquad (44)$$

Now, using the Noether's theorem, the conserved current is

$$J^a_\mu = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_n)} \frac{\delta \phi_n}{\delta \theta^a}$$ with $\phi^n = \psi$ and $A^{\mu a}$ $\Longrightarrow$

$$\Rightarrow J^a_\mu = \bar{\Psi}\gamma_\mu T^a \Psi + f^{abc} A^b_\nu G^{c\,\nu}{}_{\mu} \qquad (45)$$

with $$\partial^\mu J^a_\mu = 0 \quad (46)$$

Remarks: i) (45) also shows that the fields $A^a_\mu$ are charged.

ii) The conserved current <u>is not</u> gauge invariant. So, the conserved charges $Q^a = \int d^3x\, J^a_0$ depend on the choice of gauge and <u>it is not</u> a physical observable! ⊗ Notice that, in QED, the conserved charge is gauge invariant and an observable!

⊗ this is due to $A^c_\mu$ having charge. (Weinberg-Witten theorem)

## 16.d Origin of the gauge fields

Consider two points x and y. In QED we have the freedom

$$\left.\begin{array}{l}\phi(x) \to e^{i\alpha(x)}\phi(x)\\ \phi(y) \to e^{i\alpha(y)}\phi(y)\end{array}\right\} \Rightarrow \left| e^{i\alpha(x)}\phi(x) - e^{i\alpha(y)}\phi(y)\right|$$ depends on the choice of the local phases!

=> the derivative $\partial_\mu \phi$ also has this dependence! So, we need to know how to compare fields in different points. The simplest way is to consider a scalar, called Wilson line, such that under phase transformations

$$W(x,y) \longrightarrow e^{i\alpha(x)}\, W(x,y)\, e^{-i\alpha(y)} \qquad (47)$$

"$W(x,x) = 1$"

With this: $\underbrace{W(x,y)\phi(y)}$ $- \phi(x) \Longrightarrow e^{i\alpha(x)}\left(W(x,y)\phi(y) - \phi(x)\right)$

↳ "kind of parallel transport".

This allow us to define the covariant derivative

$$D_\mu \phi(x) = \lim_{\delta x^\mu \to 0} \frac{W(x, x+\delta x)\phi(x+\delta x) - \phi(x)}{\delta x^\mu} \qquad (48)$$

that satisfies: $D_\mu \phi(x) \longrightarrow e^{i\alpha}\partial_\mu \phi$ as desired!

For small $\delta x$: $W(x, x+\delta x) = 1 + ie\,\delta x^\mu A_\mu(x) + \mathcal{O}(\delta x^2)$ (49)

Now (48+49) $\Rightarrow$ $D_\mu \phi = (\partial_\mu + ieA_\mu)\phi$ (50)

Moreover, (47) $\Rightarrow$ $1 + ie\,\delta x^\mu A'_\mu(x) = (1 + i\alpha(x))(1 + ie\,\delta x^\mu A_\mu)(1 - i\alpha(x+\delta x))$

$$\Rightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha \qquad (51)$$

From (49) we can write $W_P(x,y) = \exp\left\{ie\int_y^x dz^\mu A_\mu(z)\right\}$ (52)

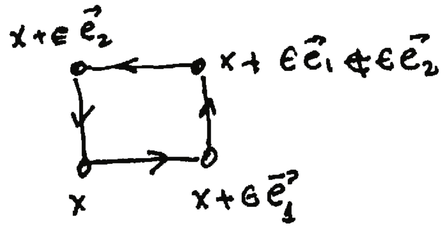
where P is a path connecting y to x and $z^\mu(\lambda)$ $0 \le \lambda \le 1$ with $z(0) = y$ and $z(1) = x$

under a gauge transformation (51) 

$$W_p(x,y) \longrightarrow \exp\left\{ -ie\int_y^x dz^\mu A_\mu(z) - ie\int_y^x dz^\mu \partial_\mu\alpha(x)\left(-\frac{1}{e}\right)\right\}$$

$$= e^{i\alpha(x)}\, W_p(x,y)\, e^{-i\alpha(y)} \quad \text{as it should be!}$$

To understand the meaning of $F_{\mu\nu}$, let's evaluate $W_p(x,x)$ going around the loop



where $\epsilon \ll 1$.

(51)

$$W(x, x+\epsilon\vec{e}_2)\, W(x+\epsilon\vec{e}_2, x+\epsilon\vec{e}_1+\epsilon\vec{e}_2)\, W(x+\epsilon\vec{e}_1+\epsilon\vec{e}_2, x+\epsilon\vec{e}_1)\, W(x+\epsilon\vec{e}_1, x)$$

$$= \left(1+ie\,\epsilon\vec{e}_2\cdot A_\mu(x+\tfrac{\epsilon}{2}\vec{e}_2)\right)\left(1+ie\,\epsilon\vec{e}_1\cdot A_\mu(x+\epsilon\vec{e}_2+\tfrac{\epsilon}{2}\vec{e}_1)\right)\otimes$$

$$\otimes\left(1-ie\,\epsilon\vec{e}_2\cdot A_\mu(x+\tfrac{\epsilon}{2}\vec{e}_2+\epsilon\vec{e}_1)\right)\left(1-ie\,\epsilon\vec{e}_1\cdot A_\mu(x+\epsilon\tfrac{\vec{e}_1}{2})\right)$$

$$= 1+ie\epsilon\left[A_2(x+\tfrac{\epsilon}{2}\vec{e}_2) - A_2(x+\tfrac{\epsilon}{2}\vec{e}_2+\epsilon\vec{e}_1)\right.$$

$$\left. + A_1(x+\epsilon\vec{e}_2+\tfrac{\epsilon}{2}\vec{e}_1) - A_1(x+\tfrac{\epsilon\vec{e}_1}{2})\right]$$

$$= 1+ie\epsilon\left[-\epsilon\,\partial_1 A_2 + \epsilon\,\partial_2 A_1\right] = 1-ie\epsilon^2\underbrace{\left[\partial_1 A_2-\partial_2 A_1\right]}_{F_{12}}$$

In the case of non-abelian transformations like (30) the situation is similar. However, the wilson line is

$$W_P(x,y) = P\left\{\exp\left(-i g \int_y^x A^a_\mu(t)\, T^a dt^\mu\right)\right\} \quad (53)$$

where the path-ordering operator P has to be used since the group generators do not commute!

## References for 16.c and 16.d:

- Pokorski, section 1.3. We used his conventions.
- Schwartz, sections 25.1 to 25.3
- Peskin, chapter 15.

## 16.e QCD

The strong interactions are described by the Quantum Chromo dynamics (QCD), which is a non-abelian gauge theory based on the $SU(3)_c$ group. The subscript $\underline{c}$ is just to ~~indicate the~~ indicate the charge of the group!

Each quark comes in "three colors" (nothing to do with light!) that we denote as R(ed), B(lue) and G(reen). The quarks transform according the fundamental representation of $SU(3)_c$. For a quark of flavor f

$$q_f = \begin{pmatrix} q_R \\ q_G \\ q_B \end{pmatrix} \text{ transforms as } q_f \to q'_f = e^{-i\theta^a \frac{\lambda^a}{2}} q_f$$

where $T^a_f = \frac{\lambda^a}{2}$ are the representation of the generators of SU(3) in the fundamental representation. Choosing the $\lambda^a$ to be the Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \quad \lambda^4 = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ 1 & & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & & -i \\ & 0 & \\ i & & 0 \end{pmatrix}$$

$$\lambda^6 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix} \text{ and } \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

Notice that $\mathrm{Tr}[T^a_f T^b_f] = \frac{1}{2}\delta^{ab}$

The QCD Lagrangian is

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_f \bar{q}_f (i\not{D} - m_f) q_f \qquad (54)$$

with ~~the~~ $G^a_{\mu\nu}$ given by (39) and $D_\mu$ by (31). The gauge bosons $A^a_\mu$ are called ~~the~~ gluons and there are 8 of them, since $d(SU(3)) = 3^2-1$

In order to explore the consequences of non-abelian theories we must quantize them!

# 16.F Quantization: Faddeev-Popov determinant



The problem in the quantization of non-abelian gauge theories, as in QED, can be traced to a constraint:

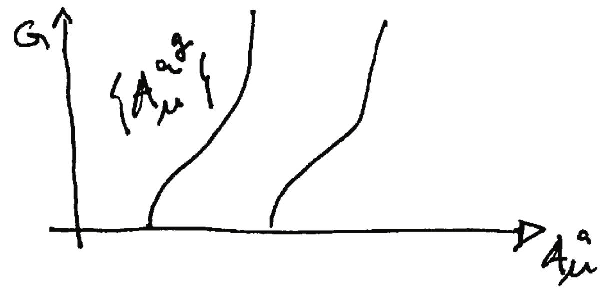
$$\Pi^a_\alpha \equiv \frac{\partial \mathcal{L}}{\partial \partial^0 A^\alpha} \overset{(A1)}{=} \partial_\alpha A^a_0 - \partial_0 A^a_\alpha - g f^{abc} A^b_\alpha A^c_0 \equiv G^a_{\alpha 0}$$

so $\Pi^a_0 \equiv 0$. This leads to necessity of using the cumbersome method of Dirac to quantize constrained systems.

In the path integral approach

$$\int DA^a_\mu \; e^{i\int d^4x\, \mathcal{L}}$$

the measure is invariant under an arbitrary g transformation g of the gauge group. However, there are infinite gauge ~~transformation~~ configurations that are physically equivalent since they are connected by gauge transformations. We divide the configuration space $\{A^a_\mu\}$ into equivalent classes $\{A^{ag}_\mu\}$, called orbits.




So the integral is proportional to the infinite volume of the gauge group!

When we try to do perturbation theory, the quadratic form in the gauge fields has zero eigenvalues, as we had in QED

$$\int d^4x \, -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 \longrightarrow \text{zero eigenvalue!}$$

$$= \int d^4x \, \frac{1}{2} A_\mu^a \left( g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu \right) \delta_{ab} A_\nu^b$$

So, we will use the same procedure we employ in QED.

Recuerdo:

If the integral

$$Z[A] = \int_{-\infty}^{+\infty} \prod_{j=1}^{N} e^{-x_k A_{kn} x_n} = \frac{\pi^{N/2}}{\sqrt{\det A}} \qquad (52)$$

is divergent due to $L$ zero eigenvalues of $A$, we first diagonalise $A$ changing the variables to $y_k$ ($y = Rx$). Then, we impose $L$ conditions to fix the variables associated to flat directions

$$f_k(\bar{y}) = 0 .$$

Since

$$1 = \int_{-\infty}^{+\infty} \prod_{k=1}^{L} d\bar{y}_k \; \delta(f_k(\bar{y})) \det\left(\frac{\partial f_k}{\partial \bar{y}_i}\right)$$

$$Z[A] = \int_{-\infty}^{+\infty} \prod_{j=1}^{N} dx_j \; e^{-x_n A_{nm} x_m} \times 1 = \underbrace{\int_{-\infty}^{+\infty} \prod_{k=1}^{L} d\bar{y}_k}_{\text{just an infinite normalization constant}} \int \prod_{j=1}^{N} dx_j \; e^{-x_n A_{nm} x_m} \delta(f_k(\bar{y})) \left(\frac{\partial f_k}{\partial \bar{y}_i}\right)$$

$$\Rightarrow \quad Z[A] = \bar{N} \int \prod_{j=1}^{N} dx_j \; e^{-x_n A_{nm} x_m} \delta(f_k(\bar{y})) \det\left(\frac{\partial f_k}{\partial \bar{y}_i}\right) \qquad (53)$$

| Schwartz | Peskin | Pokorski |
| --- | --- | --- |
| $= e^{i\alpha^a T^a}\psi$ | $\psi' = e^{i\alpha_i T^i}\psi$ | $\psi' = e^{-i\theta^a T^a}\psi$ |
| $D_\mu = \partial_\mu - ig\, T^a A^a_\mu$ | $D_\mu = \partial_\mu - igT^a A^a_\mu$ | $D_\mu = \partial_\mu + igT^a A^a_\mu$ |
| $A'^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + f^{abc}\alpha^b A^c_\mu$ | $A'^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu\alpha^a + f^{abc}A^b_\mu\alpha^c$ | $A'^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu\theta^a - f^{abc}A^b_\mu\theta^c$ |
| $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$ | $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$ | $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu$ |
| $=$ | $\partial^\mu F^a_{\mu\nu} + gf^{abc}A^{\mu b}F^c_{\mu\nu} = -g\bar\psi\gamma_\nu T^a\psi$ ✓ | $\partial_\mu G^{a\mu\nu} - gf^{abc}A^b_\mu G^{c\mu\nu} = g\bar\psi T^a\gamma^\nu\psi$ ✓ |

Schawrtz ≡ Peskin $\qquad$ Pokorski

$\alpha^a \longrightarrow -\theta^a$

$g \longrightarrow -g$

Srednick $\longrightarrow$ Pokorski

$g \longrightarrow -g$

$gT^a \longrightarrow \theta^a$

Faddeev-Popov procedure : | 16.11 |

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Let $Dg$ be the invariant measure on the group $G$

$$Dg = \prod_x dg(x) \quad \text{with} \quad D(gg') = Dg$$

We choose functionals $F^a$ to fix the gauge through

$$F^a[A_\mu^{bg}] = 0$$

Remember that, $\delta(f(x))$ represents $\prod_x \delta(f(x))$ at each space-time point.

So, we write

$$1 = \int Dg \, \delta\left(F^a[A_\mu^{ag}]\right) \det\left(\frac{\delta F^b[A_\mu^{ag}]}{\delta g}\right) \qquad (54)$$

Faddeev-Popov determinant

Now

$$\int DA_\mu^a \, e^{iS[A_\mu^a]} \otimes 1 = \int Dg \int DA_\mu^a \; e^{iS[A_\mu^a]} \; \delta\left(F^b[A_\mu^{ag}]\right) \det\left(\frac{\delta F^b[A_\mu^{ag}]}{\delta g}\right) \quad (5$$

Once again $\int Dg$ is just a normalization constant that we absorb into the definition of $DA_\mu^a$! Notice that this whole procedure is gauge invariant!

Now, let's trade the $\delta$ function in (55) by a more manageable gauge condition of the functional. For that, we consider gauge $a$ of the form

$$F^b[A_\mu^a] - \omega^b(x) \qquad (56)$$

where $\omega^b(x)$ are arbitrary functions of $x$. The Faddeev-Popov determin oes not depend on $\omega^b$:

$$\det\left[\frac{\delta}{\delta g}\left(F^b[A^a_\mu]-\cancel{\omega^b}\right)\right]=\det\left(\frac{\delta F^b}{\delta g}\right)$$

$0 \rightarrow$ it's just a function!

Now, we substitute (56) into (55) and perform the integration

$$\int D\omega^b \; e^{-\frac{i}{2\alpha}\int d^4x(\omega^b)^2} \qquad (57)$$

$$\Rightarrow\left(\int Dg\right)\int DA^a_\mu \; e^{iS[A^a_\mu]}\det\left(\frac{\delta F^b}{\delta g}\right)e^{-\frac{i}{2\alpha}\int d^4x\,(F^b)^2} \qquad (57)$$

$\Rightarrow$ this amounts to replace the original Lagrangian density $\mathcal{L}$ by

$$\mathcal{L}\rightarrow\mathcal{L}-\frac{1}{2\alpha}\left(F^b[A^a_\mu]\right)^2=\mathcal{L}_{eff} \qquad (58)$$

Finally, we write the generating functional as

$$Z[J^a_\mu]=N\int DA^a_\mu \det\left(\frac{\delta F^b}{\delta g}\right)e^{i\int d^4x\left\{\mathcal{L}-\frac{1}{2\alpha}(F^b)^2+J^a_\mu A^{a\mu}\right\}} \qquad (59)$$

<u>Example</u>: Let's consider the Lorenz gauge

$$F^a=\partial_\mu A^{a\mu}=0 \qquad (60)$$

under an infinitesimal gauge transformation (35)

$$\delta A^{b}_{\mu}(z) = \frac{1}{g}\partial^{(z)}_{\mu}\theta^{b}(z) - f^{bce}A^{c}_{\mu}(z)\,\theta^{e}(z)\,.$$

$$= \frac{1}{g} D^{be}_{\mu}\theta^{e}$$ → adjoint representation!

ow,

$$\frac{\delta F^{a}(x)}{\delta\theta^{d}(y)} = \int d^4z \;\frac{\delta F^{a}(x)}{\delta A^{b}_{\mu}(z)}\;\frac{\delta A^{b}_{\mu}(z)}{\delta\theta^{d}(y)} \qquad (61)$$

$$= \int d^4z \left\{ \partial^{(x)}_{\mu}\delta^{a}_{b}\,\delta(x-z)\left[\frac{1}{g}\partial^{\mu}_{(z)}\delta^{b}_{d}\,\delta(z-y) - f^{bcd}A^{c\mu}(z)\,\delta(z-y)\right]\right\}$$

$$= \int d^4z \left\{ \partial^{(x)}_{\mu}\delta^{a}_{b}\,\delta(x-z)\left[f^{bdc}A^{c\mu}(z) + \frac{1}{g}\partial^{\mu}_{(z)}\delta^{b}_{d}\right]\delta(z-y)\right\}$$

$$= \left[\frac{1}{g}\partial^{2}_{(x)}\delta^{a}_{d} + f^{adc}A^{c\mu}\partial^{(x)}_{\mu}\right]\delta(x-y) \qquad (62)$$

here we used that $\partial_\mu \hat{A}^\mu = 0$. Notice that the Faddeev-Popov determinant depends on ...

~~...~~ In order to perform the functional integral
n (59) it is convenient to introduce some auxiliary
omplex fields $\eta$ that are Grassmann variables, which are called
hosts!

$$\det\frac{\delta F^{a}(x)}{\delta\theta^{d}(y)} = N'\int D\eta\, D\eta^{*}\exp\left[i\int d^4x\, d^4y\;\eta^{*a}(x)\,\frac{\delta F^{a}(x)}{\delta\theta^{d}(y)}\,\eta^{d}(y)\right] \qquad (63)$$

For QED, $f^{abc} \equiv 0$ and the Lagrangian density is $\boxed{16.20}$

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\not{\partial} - m)\Psi - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 - \eta^* \Box \eta \qquad (64)$$

where we used (58), (60), (62), and (63).

## 16.G BRST invariance

In (64), the gauge invariance is broken, so what prevents the appearance of divergences in operators (Green's functions) that do not respect gauge invariance? This would spoil the renormalization process! In QED this does not happen since the gauge fixing term modifies just the photon propagator and the ~~propagators~~ photon couples to a conserved current!

In non-abelian gauge theories the gauge boson do not couple to a conserved current! However, there is a global symmetry, called BRST, discovered by Becchi, Rouet and Stora (76) and Tyutin (75). For simplicity, we will consider only the Lorenz gauge (63). First, let's introduce an auxiliary field (i.e. without derivatives) $B^a$:

$$e^{\int d^4x \, -\frac{1}{2\alpha}(\partial_\mu A^{a\mu})^2} = \int \mathcal{D}B^a \; e^{i\int d^4x \left(\frac{\alpha}{2} B^a B^a - B^a \partial_\mu A^{a\mu}\right)} \qquad (65)$$

(65) is true up to a normalization constant. So, the effective Lagrangian in (59) + (63) + adding a matter field is

$$\mathcal{L} = -\frac{1}{4}(G^a_{\mu\nu})^2 + \bar{\Psi}(i\not{D}-m)\Psi + \frac{\alpha}{2}B^aB^a - B^a\partial_\mu A^{a\mu} + \eta^{*a}\partial^\mu D^{ab}_\mu \eta^b \qquad (66)$$

with $D^{ab}_\mu \eta^b \equiv (\delta^{ab}\partial_\mu - gA^c_\mu f^{acb})\eta^b$. We have rescaled $\eta^a \to \eta^a/g$

This is invariant under

$$\delta A^a_\mu = \epsilon D^{ab}_\mu \eta^b \qquad (67.a)$$

$$\delta\Psi = -igT^a\epsilon\eta^a\Psi \qquad (67.b)$$

$$\delta\eta^a = \frac{1}{2}g\epsilon f^{abc}\eta^b\eta^c \qquad (67.c)$$

$$\delta\eta^{*a} = \epsilon B^a \qquad (67.d)$$

$$\delta B^a = 0 \qquad (67.e)$$

where $\epsilon$ is an infinitesimal anticommuting parameter.

Notice that (67.a) and (67.b) are just a gauge transformation with $\theta^a(x) = g\epsilon\eta^a(x)$, thus, the first two terms in (66) are automatically invariant. The third term in (66) is invariant due to (67.e). We are left with:

$$\delta(-B^a\partial_\mu A^{a\mu} + \eta^{*a}\partial^\mu D^{ab}_\mu\eta^b) = -B^a\cancel{\partial_\mu}\epsilon D^{ab}_\mu\eta^b + \epsilon B^a\cancel{\partial^\mu}D^{ab}_\mu\eta^b + \eta^{*a}\partial^\mu\delta(D^{ab}_\mu\eta^b)$$

$$\cancel{+\eta^{*a}\partial^\mu(-gf^{acb}\epsilon D^{cd}_\mu\eta^d\eta^b)}$$

$$\cancel{+\eta^{*a}D^{ab}_\mu\frac{1}{2}g\epsilon f^{bde}\eta^d\eta^e}$$

Focusing on

$$\delta(D^{ab}_\mu \eta^b) = D^{ab}_\mu \delta\eta^b - g f^{acb}(\delta A^c_\mu)\eta^b$$

$$= \partial_\mu\left(\tfrac{1}{2} g\epsilon \cancel{f^{abc}} \eta^b\eta^c\right) - g A^c_\mu f^{acb}\left(\tfrac{1}{2} g\epsilon f^{bde}\eta^d\eta^e\right)$$

$$\cancel{- g f^{acb}(\partial_\mu \eta^c)\eta^b} - g f^{acb}(-g f^{cde}) A^d_\mu \eta_e \eta_b$$

$$= -\tfrac{1}{2} g^2\epsilon f^{acb} f^{bde}\left(A^c_\mu \eta^d\eta^e \overset{c\to d,\ d\to e}{+} A^d_\mu \eta_e\eta_c \overset{c\to e,\ d\to e}{+} A^e_\mu \eta_c \eta_d\right)$$

$$= -\tfrac{1}{2} g^2\epsilon A^c_\mu \eta^d\eta^e \underbrace{\Big(\underset{[T^c,[T^d,T^e]]}{f^{acb} f^{bde}} + \underset{[T^e,[T^c,T^d]]}{f^{aeb} f^{bcd}} + \underset{[T^d,[T^e,T^c]]}{f^{adb} f^{bec}}\Big)}_{=0 \text{ by Jacobi identity}}$$

$\Rightarrow$ (66) is invariant under (67)!

**Exercise:** Writing $\delta\phi = \epsilon Q \phi$ where $\phi = A^a_\mu, \psi, \eta, \eta^*$ with the transformations given by (67), show that

$$Q^2 = 0$$

---

**Fact:** Since BRST is a continuous transf. it generates relations similar to the Ward identities. However, they are easier to use than the gauge symmetry Ward identities.

## 16.9 Feynman Rules

Let's consider

$$Z[J^a_\mu, \eta^a, \bar\eta^a] = \int DA_\mu\, D\bar\Psi\, D\Psi\, D\bar{c}\, Dc \;\; e^{i\left\{S_{eff} + \int d^4_x \, J^a_\mu A^{a\mu} + \bar\eta^a \psi^a + \bar\psi^a \eta^a\right\}} \quad (68)$$

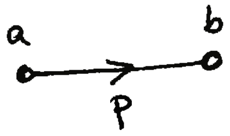
with $J^a_\mu$, $\bar\eta^a$ and $\eta^a$ being sources, i.e., we renamed the ghosts to be $\bar c, c$; In addition, we use the Lorenz gauge (60): $+\frac{1}{2\alpha} A^a_\mu \delta^{ab} \partial^\mu \partial^\nu$

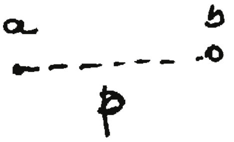
$$S_{eff} = \int d^4_x \Big\{ \frac{1}{2} A^a_\mu \delta^{ab}(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A^b_\nu + \bar\Psi^a (i\not\partial - m)\delta^{ab}\psi^b + \bar{c}^a \delta^{ab} \partial^2 c^b$$

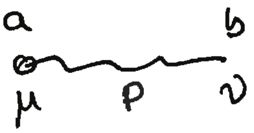
$$+ g f^{abc} \partial_\mu A^a_\nu A^{\mu b} A^{\nu c} - \frac{1}{4} g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e}$$

$$- g \bar\Psi \gamma_\mu A^{\mu a} T^a \psi + g f^{abc} A^c_\mu \eta^{*a} \partial^\mu \eta^b \Big\} \quad (69)$$

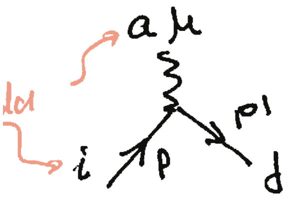
The first line of (69) define the propagators while the others are the interactions. The free propagators are, in momentum space,

fermion  $\delta^{ab} \dfrac{i}{\not p - m}$

ghost  $\delta^{ab} \dfrac{-i}{p^2}$ (70)

gauge boson  $\delta^{ab} \dfrac{-i}{p^2}\left[g^{\mu\nu} - (1-\alpha)\dfrac{p^\mu p^\nu}{p^2}\right]$

The interaction gauge boso to fermions leads to



$$-ig\, T^a_{ji} \gamma^\mu A^a_\mu \quad (71)$$

The last term of (69) is an interaction between ghosts and gauge field: $\boxed{16.24}$

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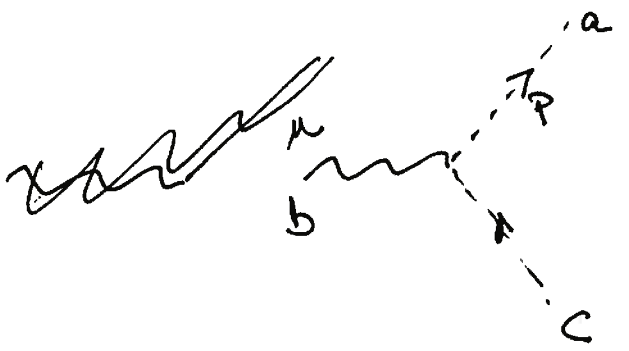



$$-g f^{abc} p_\mu \qquad (72)$$

Remember that in the momentum space $\partial_\mu \longrightarrow -i p_\mu$ where $p_\mu$ is the incoming momentum! Now we are left with the second line of (69) that contains the interactions between the gauge bosons.

Triple vertex: We evaluate in lowest order

$$G^{(3)abc}_{\mu\nu\lambda}(x_1 \dots x_3) = \langle 0 | T A^a_\mu(x_1) A^b_\nu(x_2) A^c_\lambda(x_3) | 0 \rangle$$

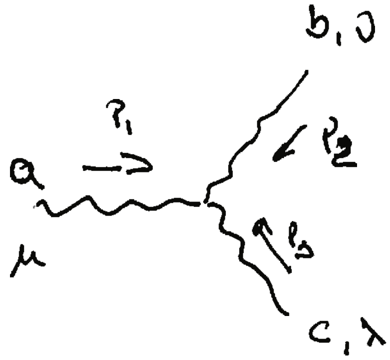
$$= \frac{1}{i^3} \frac{\delta^3}{\delta J^a_\mu(x_1)\, \delta J^b_\nu(x_2)\, \delta J^c_\lambda(x_3)} Z \Bigg|_{J=\eta=\bar\eta=0} \qquad (73)$$

However, to lowest order

$$Z = i\int d^4y\, g f^{def}\, \partial_\rho \frac{1}{i^3} \frac{\delta^3}{\delta J^d_\kappa(y)\, \delta J^{(e)}_\rho(y)\, \delta_\kappa^{f}(y)} Z_0 \qquad (74)$$

$Z_0$ → free $Z$, i.e., ~~neglecting~~ discarding interaction.

Now, we pair the $\frac{\delta}{\delta J}$ in (73) with the ones in (74)




$$-g\, f^{abc}\Big[ g^{\mu\nu}(p_1-p_2)^\lambda + g^{\lambda\nu}(p_2-p_3)^\mu + g^{\mu\lambda}(p_3-p_1)^\nu \Big] \qquad (75)$$

[Notice this Sostanziale sono statistico!]

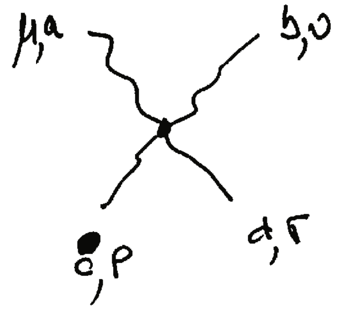
Example of 1 pairing:

global sign $(i)^6 \overset{\partial}{(-i)}\, i = -1.$ $\Big\|$ $\exists$ also $(-1)^3$ from $Z_0 = \exp\{-\frac{1}{2}\int J^\mu(x)\, D^F_{\mu\nu}(x-y)\, J^\nu(y)\}$

$$c \leftrightarrow f,\quad \lambda \leftrightarrow \kappa \quad \xrightarrow[\text{choices}]{2} \quad \begin{cases} \begin{bmatrix} a & d \\ \mu & \kappa \end{bmatrix} \begin{bmatrix} b & e \\ \nu & \rho \end{bmatrix} \Rightarrow g f^{abe}\, p_1^\nu\, g^{\mu\lambda} \\[2ex] \begin{bmatrix} a & e \\ \mu & \rho \end{bmatrix} \begin{bmatrix} b & d \\ \nu & \kappa \end{bmatrix} \Rightarrow g \underbrace{f^{bac}}_{-f^{abc}}\, p_2^\mu\, g^{\nu\lambda} \end{cases}$$

<u>Exercise</u>: Do the other pairings and ~~...~~ verify (75).

<u>Analogously</u>, we have



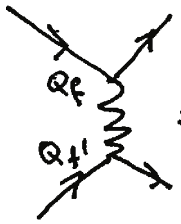

$$-ig^2\Big[ f^{abe} f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\Big] \qquad (76)$$

## 16.4 Attractive/Repulsive Potentials in QCD



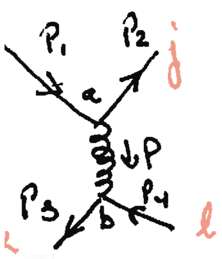
**Goal:** to study how QCD differs from QED

In QED, we can derive the potential between charges from




$$\Rightarrow \mathcal{M}_{QED} = Q_f Q_{f'} \mathcal{M}_0 \Rightarrow V(r) = -Q_f Q_{f'} \frac{e^2}{4\pi r} \qquad (77)$$

In QCD, quarks are in the fundamental representation of SU(3)




$$q\bar{q}' \to q\bar{q}'$$

$$i\mathcal{M}_{QED} = \bar{u}(p_2)(-ig_s T^a_{ji})\gamma^\mu u(p_1)\,\delta^{ab}\,\frac{-ig_{\mu\nu}}{p^2}\,\bar{v}(p_3)\gamma^\nu(-ig_s T^b_{ke})v(p_4)$$

$$= g_s^2\, T^a_{ji}\, T^a_{ke}\, \mathcal{M}_0$$

…, the potential between $q$ and $\bar{q}$ is given by (77) with $e^2 Q_f Q_{f'} \leftrightarrow g_s^2\, T^a_{ji} T^a_{ke}$

Let's call $(R, G, B) = (1, 2, 3)$. For instance,

$$i=1 \quad k=2 \Rightarrow T^a_{j1} T^a_{2\ell} = \begin{pmatrix} 0 & -1/6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\frac{1}{6}\delta^{j1}\delta^{2\ell} \Rightarrow \text{repulsive potential}$$

$R\ \bar{G}$ final state: $R\bar{G}$

$$i=1 \quad k=1 \Rightarrow T^a_{j1} T^a_{1\ell} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \Rightarrow \text{attractive potential}$$

$R\ \bar{R}$ final state: $R\bar{R}, G\bar{G}, B\bar{B}$

from group theory, $3 \otimes \bar{3} = 1 \oplus 8$

the singlet state is $|1\rangle = \frac{1}{\sqrt{3}}\left( |R\bar{R}\rangle + |G\bar{G}\rangle + |B\bar{B}\rangle \right)$

$$\Rightarrow \langle 1 | T^a_{ij} T^a_{ke} | 1 \rangle = \frac{4}{3} \quad \Rightarrow \quad V^{\text{singlet}}_{QCD} = -\frac{4}{3} \frac{g_s^2}{4\pi r} \quad \text{(attractive)}$$

while for the octet states $\Rightarrow$ $V^{\text{octet}}_{QCD} = \frac{1}{6} \frac{g_s^2}{4\pi r}$ (repulsive)

This is compatible with the fact that mesons ($q\bar{q}$ states) are not colored.

## 16.I Counterterms

Our starting point is the bare Lagrangian density

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu A^a_{\nu B} - \partial_\nu A^a_{\mu B}\right)^2 + \bar{\Psi}_B (i\not{\partial} - m_B)\Psi_B - C^{*a}_B \partial^2 C^a_B$$

$$- g_B A^a_{\mu B} \bar{\Psi}_B \gamma^\mu T^a \Psi_B + g_B f^{abc} \partial^\mu A^{\nu a}_B A^b_{\mu B} A^c_{\nu B}$$

$$- \frac{1}{4} g_B^2 f^{abc} f^{aed} A^b_{\mu B} A^c_{\nu B} A^{\mu e}_B A^{\nu d}_B + g_B C^{*a}_B f^{abc} \partial^\mu A^b_{\mu B} C^c_B$$

$$- \frac{1}{2\alpha_B} \left(\partial_\mu A^{a\mu}_B\right)^2 \qquad (78)$$

As before we express ~~the~~ (66) in terms of renormalized fields, couplings and masses:

$$\mathcal{L} = -\frac{1}{4} Z_3 \left[ \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g \frac{Z_{1YM}}{Z_3} f^{abc} A_\mu^b A_\nu^c \right]^2$$

$$+ Z_2 \bar{\Psi} i \gamma_\mu \left[ \partial^\mu + i \tilde{g} \frac{Z_1}{Z_2} A^{a\mu} T^a \right] \psi - m Z_0 \bar{\Psi} \Psi$$

$$+ \tilde{Z}_2 c^{*a} \left[ \delta^{ab} \partial^2 + \tilde{\tilde{g}} \frac{\tilde{Z}_1}{\tilde{Z}_2} f^{abc} A_\mu^c \partial^\mu \right] c^b - \frac{1}{2\alpha} (\partial_\mu A^{a\mu})^2 \qquad (79)$$

where $g \frac{Z_{1YM}}{Z_3} = \tilde{g} \frac{Z_1}{Z_2} = \tilde{\tilde{g}} \frac{\tilde{Z}_1}{\tilde{Z}_2}$ (80)

that follows from requiring BRST invariance. In the MS scheme it turns out that

$$g = \tilde{g} = \tilde{\tilde{g}} \qquad (81)$$

The relation between renormalized and bare quantities is:

$$A^a_{\mu\,B} = Z_3^{1/2} A^a_\mu \qquad \psi_B = Z_2^{1/2} \psi \qquad \alpha_B = Z_3 \alpha$$

$$c^a_B = \tilde{Z}^{1/2} c^a \qquad c^{*a}_B = \tilde{Z}_2^{1/2} c^b \qquad m_B = m \frac{Z_0}{Z_2} \qquad (82)$$

and

$$g_B = g \frac{Z_{1YM}}{Z_3^{3/2}} = \tilde{g} \frac{Z_1}{Z_2 Z_3^{1/2}} = \tilde{\tilde{g}} \frac{\tilde{Z}_1}{\tilde{Z}_2 Z_3^{1/2}}$$

Notice that the last eq'n is the source of (80).

Finally, the Lagrangian written ~~with~~ using the counterterms is, taking $g = \tilde{g} = \tilde{\tilde{g}}$,

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \bar\Psi(i\not{D}-m)\Psi - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2$$

$$+\, c^{*a}\left(\delta^{ab}\partial^2 + g f^{abc} A^{c\mu}\partial_\mu\right)c^b$$

$$+(Z_2-1)\bar\Psi i\not\partial\Psi - (Z_1-1)g\bar\Psi\gamma^\mu A^a_\mu T^a\Psi - m(Z_0-1)\bar\Psi\Psi$$

$$+(\tilde Z_2-1)c^{*a}\delta^{ab}\partial^2 c^b + (\tilde Z_1-1)g\, c^{*a} f^{abc} A^\mu_c \partial_\mu \eta^b \tag{83}$$

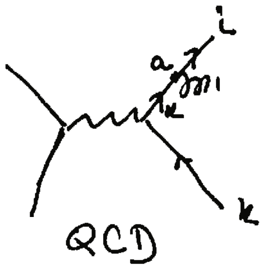
$$-\frac{1}{4}(Z_3-1)(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \frac{1}{2}(Z_{1YM}-1)g(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)c^{abc}A^{b\mu}A^{c\nu}$$

$$-\frac{1}{4}g^2\left(\frac{Z_{1YM}^2}{Z_3^{-1}}-1\right)f^{abc}A^b_\mu A^c_\nu f^{ade}A^{d\mu}A^{e\nu}$$

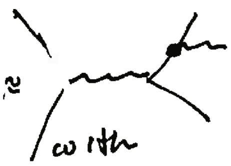
## 16.J $e^+e^-\to$ hadrons at NLO

Notice that it's like QED with $e\to g_s T^a$!

Let's obtain $e^+e^-\to q\bar q(g)$ at order $\alpha_{em}^2\alpha_s$ from the previous study of $e^+e^-\to\mu^+\mu^-(\gamma)$ that we did before.




$= T^a_{ik}\cdot$



with $e\to g_s$ $\times Q_q$

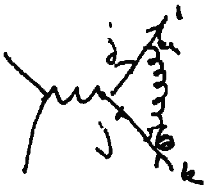
$\Rightarrow MM^\dagger \sim T^a_{ik}T^a_{ki}$ no sum

Now summing over color of quarks and gluons $\Rightarrow \mathrm{Tr}[T^aT^a] = 3\,\frac{4}{3} = C$

summing

$$\Rightarrow \sigma_R = 3Q_q^2 \overset{e^+e^-\to\mu^+\mu^-}{\sigma_0}\left(\frac{\alpha_s}{\pi}\right)C_F\left(\frac{\tilde\mu^2}{Q^2}\right)^{4-d}\left(\frac{1}{\epsilon^2}+\frac{13}{12\epsilon}-\frac{5\pi^2}{24}+\frac{259}{144}+O(\epsilon)\right) \tag{84}$$

On the other hand


$\Rightarrow T^a_{ij} T^a_{jk}$

$\alpha \to \alpha_s$

$\times Q_q$

$\Rightarrow$

$$\sigma_V = 3Q_q^2 \, \sigma_0 \left(\frac{4\alpha_s}{\pi}\right) \left(\frac{\tilde{\mu}^2}{Q^2}\right)^{4-d} C_F \left( -\frac{1}{\varepsilon^2} - \frac{13}{12\varepsilon} + \frac{5\pi^2}{24} - \frac{29}{18} + \mathcal{O}(\varepsilon) \right) \quad (85)$$

$$\Rightarrow \sigma_{NLO} = \sigma_0 + \sigma_R + \sigma_V = 3Q_q^2 \sigma_0 \left( 1 + \frac{3\alpha_s}{4\pi} C_F \right) \quad (86)$$

## 16.K One-loop renormalization

Goal: to gather the information to evaluate the $\beta$ function. To do so

$$\beta(\alpha) = \mu \frac{d}{d\mu} \alpha(\mu^2) \qquad \text{with} \qquad \alpha = \frac{g^2}{4\pi}$$

to do so we need $Z_\alpha$: $\alpha_B = \mu^{+\varepsilon} Z_\alpha \alpha_R$

where

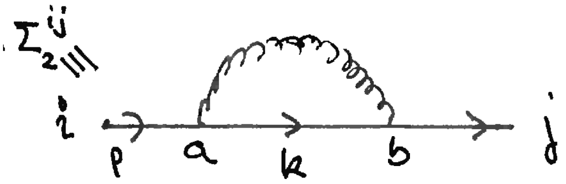
$$Z_\alpha \overset{(82)}{=} \frac{Z_{1YM}^2}{Z_3^2} \overset{*}{=} \frac{Z_1^2}{Z_2^2 Z_3} = \frac{\tilde{Z}_1^2}{\tilde{Z}_2^2 Z_3} \quad (87)$$

We are going to use $Z_\alpha = \dfrac{Z_1^2}{Z_2^2 Z_3}$

This is a long calculation that we will highlight the main points. The details are for a problem set.

## Fermion two-point function:

The one-loop contribution is




compared to the QED analog contribution the difference is

$$\sum_{abk} T^b_{jk} T^a_{ki} \delta^{ab} = \sum_a (T^a T^a)_{ji} = C_F \delta_{ij} \qquad (88)$$

where $C_F = \frac{N^2-1}{2N}$ since we assume the fermions are in the fundamental representation. Using the QED result for the loop

$$\Sigma_2^{ij} = \delta^{ij} \Big\{ -\frac{g^2}{8\pi^2} C_F \int_0^1 dx\, (2m - x\not{p}) \Big[ \frac{2}{\varepsilon} + \ln \frac{4\pi e^{-\gamma_E} \mu^2}{(1-x)(m^2 - p^2 x)} \Big]$$

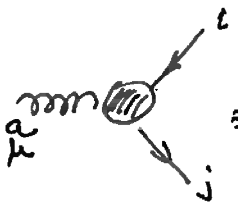
$$+ \underbrace{(Z_2 - 1)}_{\delta_2} \not{p} - \big( (Z_2-1) + \underbrace{(Z_0 - 1)}_{\delta_m} \big) m \Big\}$$

$$= \delta^{ij} \Big\{ \frac{g^2}{16\pi^2} C_F \Big( \frac{2\not{p} - 8m}{\varepsilon} \Big) + \text{finite} + \delta_2 \not{p} - (\delta_m + \delta_2) m \Big\} \qquad (89)$$

15

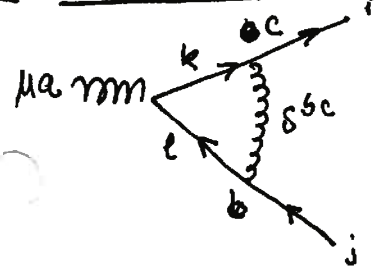
$$\Rightarrow \begin{cases} \delta_2 = \frac{1}{\varepsilon} \frac{g^2}{16\pi^2} (-2C_F) \\ \delta_m = \frac{1}{\varepsilon} \frac{g^2}{16\pi^2} (-6C_F) \end{cases} \qquad (90)$$

## Three-point function $A\bar{\Psi}\Psi$:




$= -ig\, \Gamma^{a\mu}_{ij}$ (91) receives ~~many~~ two contributions:

irst contribution




$$\equiv \sum_{\substack{bc\\kl}} \delta^{bc}\, T^{c}_{ik}\, T^{a}_{kl}\, T^{b}_{lj} \times \text{QED}(e\to g)$$

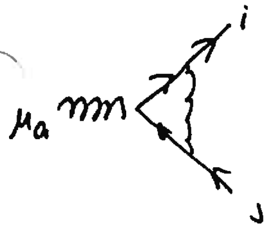
$$\underbrace{\qquad\qquad}_{(T^bT^aT^b)_{ij}}$$

However, $T^bT^aT^b = T^bT^bT^a + T^b[T^a,T^b]$

$$= C_F T^a + i f^{abc} T^b T^c = C_F T^a + \frac{i}{2} f^{abc}[T^b,T^c]$$

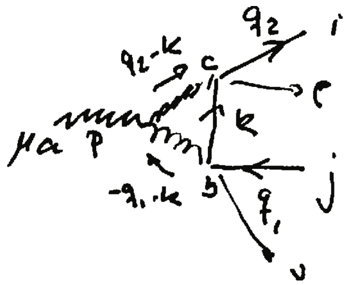
$$= C_F T^a - \frac{1}{2} f^{abc} f^{bcd} T^d \overset{(24)}{=} \left(C_F - \frac{1}{2} C_A\right) T^a$$

Now using the QED result:




$$= -ig\left(C_F - \frac{C_A}{2}\right) T^a_{ij}\, \gamma^\mu \left(\frac{g^2}{16\pi^2}\right)\left(\frac{2}{\epsilon} + \text{finite}\right) \qquad (91)$$

Second Contribution




$$= -ig\, \underbrace{f^{abc}\, (T^cT^b)_{ij}}_{\text{group factors}}\, \Gamma^\mu_{(2b)}$$

with

$$f^{abc}\, T^cT^b = \frac{1}{2} f^{abc}[T^c,T^b] = -\frac{i}{2} f^{abc} f^{dbc} T^d = -i\,\frac{C_A}{2} T^a$$

For the loop factor (with $m=0$)

$$(-ig)\,\Gamma^\mu_{2B} = -g(-ig)^2 \int \frac{d^4k}{(2\pi)^4}\, \gamma^\rho \frac{i\not{k}}{k^2} \gamma^\nu \frac{-i}{(q_1+k)^2}\, \frac{-i}{(q_2-k)^2}$$

$$\times\, g^{\mu\nu}(2q_1+q_2+k)^\rho + g^{\nu\rho}(-q_1+q_2-2k)^\mu + g^{\rho\mu}(k-2q_2-q_1)^\nu \Big)$$

To extract the UV divergence it's easy to set $p \equiv 0 = q_1 = q_2$

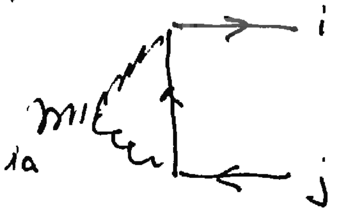
$$\Gamma^{\mu}_{2b}(0) = g^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\rho \not{k} \gamma^\sigma}{k^6} \left( g^{\mu\sigma} k^\rho - 2 g^{\sigma\rho} k^\mu + g^{\rho\mu} k^\sigma \right)$$

$$= g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^6} \left[ 2k^2 \gamma^\mu - 2 \gamma^\rho \not{k} \gamma^\rho k^\mu \right]$$

dimensional regularization ↓

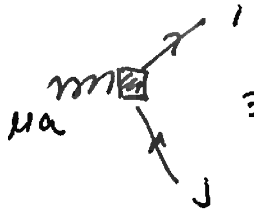
$$= i \gamma^\mu \frac{g^2}{8\pi^2} \left( \frac{3}{\varepsilon} + \text{finite} \right)$$

so



$$= -i g C_A T^a_{ij} \gamma^\mu \left( \frac{g^2}{16\pi^2} \right) \left( \frac{3}{\varepsilon} + \text{finite} \right) \qquad (92)$$

To renormalize (91) and (92) we use the counterterm



$$\equiv -i g T^a_{ij} \gamma^\mu \underbrace{\delta_1}_{(Z_1 - 1)} \qquad (93)$$

Using MS, we obtain that $\delta_1 = (Z_1 - 1) = \frac{1}{\varepsilon} \left( \frac{g^2}{16\pi^2} \right) \left[ -2C_F - 2C_A \right] \quad (94)$

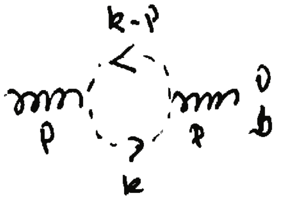
Vacuum polarization  receives many contributions

In QED

 $\equiv 0 \implies$  $= i(q^2 g^{\mu\nu} - q^\mu q^\nu)\, \Pi(q^2) \qquad (95)$

$$-ig^2\delta^{ab}C_A\,\frac{\mu^{4-d}}{(4\pi)^{d/2}}\,g^{\mu\nu}\int_0^1 dx\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}(d-1)\,x\left[-\frac{d}{2}\Gamma\left(1-\frac{d}{2}\right)\Delta+(1-x)^2p^2\Gamma\left(2-\frac{d}{2}\right)\right]\quad(9$$

Ghost bubble:




$$=ig^2\,\frac{\mu^{4-d}}{(4\pi)^{d/2}}\,\delta^{ab}C_A\int_0^1 dx\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\left\{g^{\mu\nu}\left[\frac{1}{2}\Gamma\left(1-\frac{d}{2}\right)\Delta\right]\right.$$

$$\left.+\,p^\mu p^\nu\left[x(1-x)\Gamma\left(2-\frac{d}{2}\right)\right]\right\}\qquad(99)$$

Complete vacuum polarization: Now, we add (97)-(98)-(99) to obtain

$$i\delta^{ab}C_A g^2\,\frac{\mu^{4-d}}{(4\pi)^{d/2}}\int_0^1 dx\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\left\{g^{\mu\nu}\Delta\left[\overbrace{\frac{3-3d}{d}+(d-1)+\frac{1}{d}}^{\frac{1}{d}(d-2)^2 \text{ to cancel the } d=2 \text{ pole.}}\right]\frac{d}{2}\,\Gamma\left(1-\frac{d}{2}\right)\right.$$

$$+\,p^\mu p^\nu\left[-3(x^2-x+1)+\frac{d}{2}(1-2x)^2+x(1-x)\right]\Gamma\left(2-\frac{d}{2}\right)$$

$$\left.+\,g^{\mu\nu}p^2\left[\left(x^2-x+\frac{5}{2}\right)-(1-x)^2(d-1)\right]\Gamma\left(2-\frac{d}{2}\right)\right\}\qquad(100)$$

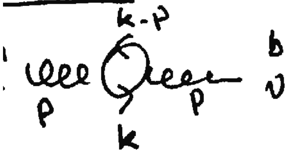
notice $\Gamma\left(1-\frac{d}{2}\right)(d-2)=-2\Gamma\left(2-\frac{d}{2}\right)$.

Going to $d=4-\varepsilon$ dimensions:

$$i\,C_A\delta^{ab}\,\frac{g^2}{16\pi^2}\left(g^{\mu\nu}p^2-p^\mu p^\nu\right)\left[\frac{10}{3\varepsilon}+\text{finite}\right]\qquad(101)$$

(95) still holds for non abelian gauge theories, however, not after we sum all contributions!

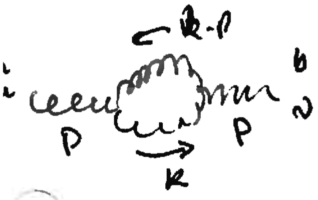
Fermion bubble:



in this case (95) holds since its result is

$$\propto \underbrace{Tr[T^a T^b]}_{T_F(=\frac{1}{2})} \times QED(e\to g) \Rightarrow i(p^2 g^{\mu\nu} - p^\mu p^\nu)\delta^{ab}\, T_F\left(\frac{g^2}{16\pi^2}\right)\left[-\frac{8}{3}\frac{1}{\varepsilon} + \text{finite}\right] \quad (96)$$

3-Point gauge boson bubble:



In the exercise you will get ~~that~~

$$-\frac{g^2}{2}\frac{\mu^{4-d}}{(4\pi)^{d/2}}\delta^{ab} C_A \int_0^1 dx \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \times \Big\{ g^{\mu\nu}\, 3(d-1)\,\Gamma\left(1-\frac{d}{2}\right)\Delta$$
$$+ p^\mu p^\nu\left[6(x^2-x+1) - d(1-2x)^2\right]\Gamma\left(2-\frac{d}{2}\right)$$
$$+ g^{\mu\nu} q^2\, \Gamma\left(2-\frac{d}{2}\right)(-2x^2+2x-5)\Big\} \quad (97)$$

where $\Delta = x(x-1)p^2$

Four-point gauge boson bubble

$$\sim \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2} = 0$$

however, we write it as follows ~~to see cancels parts~~ to see how the pole at $d=2$ cancels the pole in this position in the other ~~position~~ diagrams ~~of (97)~~! Again, from the problem set we have ~~that~~

Now, using (96) and (101) we obtain that

$$\delta_3 = Z_3 - 1 = \frac{1}{\varepsilon}\left(\frac{g^2}{16\pi^2}\right)\left[\frac{10}{3}C_A - \frac{8}{3}T_F\, n_f\right] \qquad (102)$$

where $n_f$ is the number of fermions running in the loop!