Teorema da Convolução em Transformadas de Laplace

$$\left\{\begin{array}{l} F(s) = \mathcal{L}\{f(t)\} \\ G(s) = \mathcal{L}\{g(t)\} \\ \text{Existem para} \\ s > a \geq 0 \end{array}\right. \Rightarrow H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}$$

quando $s > a$

onde $h(t) = f * g$

Nota: Na prática vamos usar que $\mathcal{L}^{-1}\{F(s)G(s)\} = f * g$

__Prova__: Sejam $F(s) = \int_0^\infty e^{-s\lambda} f(\lambda)\, d\lambda$ e

$G(s) = \int_0^\infty e^{-s\beta} g(\beta)\, d\beta$ as transformadas de Laplace das funções $f(t)$ e $g(t)$, respetivamente.

$$F(s)G(s) = \underbrace{\int_0^\infty e^{-s\lambda} f(\lambda)\, d\lambda}_{\text{não depende de } \beta} \int_0^\infty e^{-s\beta} g(\beta)\, d\beta$$

$$F(s)G(s) = \int_0^\infty e^{-s\beta} g(\beta) \left[\int_0^\infty e^{-s\lambda} f(\lambda)\, d\lambda\right] d\beta \quad \leftarrow \text{Integral Iterada}$$

não depende de λ

$$F(s)G(s) = \int_0^\infty g(\beta) \left[\int_0^\infty \underbrace{e^{-s\beta} e^{-s\lambda}}_{e^a e^b = e^{a+b}} f(\lambda)\, d\lambda\right] d\beta$$

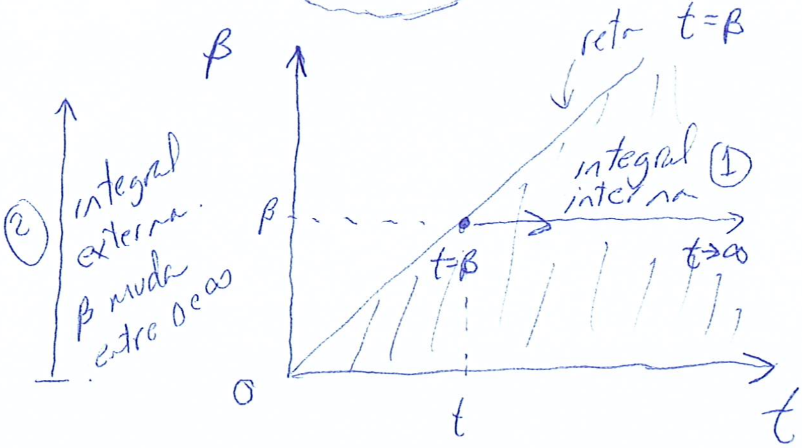
$$F(s)G(s) = \int_0^\infty g(\beta) \underbrace{\left[\int_0^\infty e^{-s(\lambda+\beta)} f(\lambda)\, d\lambda\right]}_{\text{troca de variáveis, } \beta \text{ fixo}} d\beta$$

$t = \lambda + \beta$, $\lambda = t - \beta$, $d\lambda = dt$ (pois β é cte.)

$\lambda = 0 \Rightarrow t = \beta$

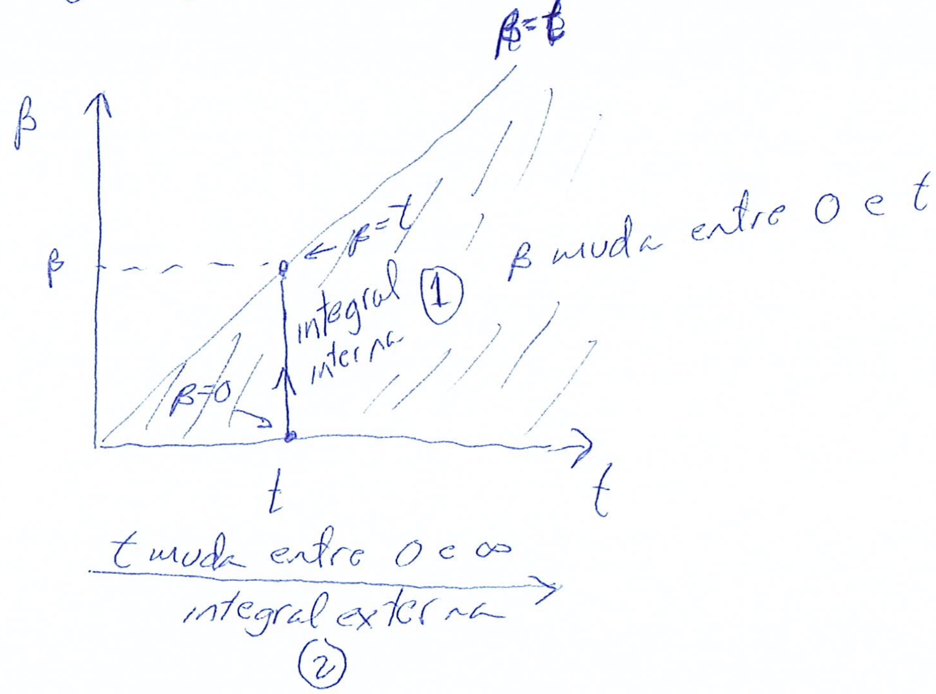
$\lambda \to \infty \Rightarrow t \to \infty$

$$F(s)G(s) = \int_0^\infty g(\beta)\left[\int_\beta^\infty e^{-st} f(t-\beta)\, dt\right] d\beta$$ integral iterada



t muda entre t=β e t→∞

$$F(s)G(s) = \int_0^\infty e^{-st}\left[\int_0^t g(\beta) f(t-\beta)\, d\beta\right] dt$$ integral iterada



Trocando novamente $\boxed{\beta = \tau}$

$$F(s)G(s) = \int_0^\infty e^{-st}\underbrace{\left[\int_0^t g(\tau) f(t-\tau)\, d\tau\right]}_{g(t) * f(t).} dt$$

$$F(s)G(s) = \int_0^\infty e^{-st} h(t)\, dt = \mathcal{L}\{h(t)\}$$

onde $h(t) = g * f$