Propriedades da Integral de Convolução

Def.: $h(t) = \underbrace{f(t) * g(t)}_{\text{convolução das funções } f \text{ e } g.} = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$

Propriedades

(I) $f * g = g * f$ (comutatividade)

(II) $f * (g_1 + g_2) = f * g_1 + f * g_2$ (distributividade do "produto" em relação a soma.)

(III) $(f * g) * h = f * (g * h)$ (associatividade)

(IV) $f * 0 = 0 * f = 0$ ~~(elemento neutro do "produto")~~

0 é a função $f(t) = 0, \forall t \geq 0$.

Vamos provar a propriedade (I).

$$f(t) * g(t) = f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Faremos uma troca de variáveis: $u = t - \tau$ (t fixo)

Logo, $\tau = t - u$, $du = -d\tau$ (t é constante, fixo)

$\tau = 0 \Rightarrow u = t$, $\tau = t \Rightarrow u = 0$

$$f * g = \int_t^0 f(t-u)\, g(u)\, (-du)$$

$$f * g = \int_0^t g(u)\, f(t-u)\, du$$

Mudando novamente a variável de integração

$u = \tau$

$$f \times g = \int_0^t g(\tau) f(t-\tau)\, d\tau = g * f = g(t) * f(t) = h(t)$$

Logo $f * g = g * f$

<u>Ex.:</u> Calcule $e^t * \operatorname{sen}(t)$.

<u>Sol.:</u> $f(t) = e^t$ e $g(t) = \operatorname{sen}(t)$

$$f(t) * g(t) = \int_0^t e^{\tau} \operatorname{sen}(t-\tau)\, d\tau$$

$\operatorname{sen}(x+y) = \operatorname{sen}(x)\cos(y) + \operatorname{sen}(y)\cos(x)$.

Leva a calcular duas integrais

$$\underbrace{f(t) * g(t) = g(t) * f(t)}_{\text{a convolução comuta}} = \int_0^t \operatorname{sen}(\tau)\, e^{t-\tau}\, d\tau$$

$$= e^t \underbrace{\int_0^t e^{-\tau} \operatorname{sen}(\tau)\, d\tau}_{\text{uma única integral} = I} \qquad (1)$$

$$\overbrace{\int_0^t e^{-\tau} \operatorname{sen}(\tau)\, d\tau}^{I} = -e^{-\tau} \operatorname{sen}(\tau)\Big|_0^t - \int_0^t -e^{-\tau} \cos(\tau)\, d\tau$$

$u = \operatorname{sen}(\tau) \rightarrow du = \cos(\tau)\, d\tau$

$dv = e^{-\tau} d\tau \rightarrow v = -e^{-\tau}$

$$\underbrace{\int_0^t e^{-\tau} \operatorname{sen}(\tau)\, d\tau}_{I} = -e^{-t} \operatorname{sen}(t) + \int_0^t e^{-\tau} \cos(\tau)\, d\tau \qquad (2)$$

(3

$$\int_0^t e^{-\tau}\cos(\tau)\,d\tau = -e^{-\tau}\cos(\tau)\Big|_0^t - \int_0 (-e^{-\tau})(-\operatorname{sen}(\tau)\,d\tau)$$

$$u=\cos(\tau) \rightarrow du = -\operatorname{sen}(\tau)\,d\tau$$

$$dv = e^{-\tau}d\tau \rightarrow v = -e^{-\tau}$$

$$\int_0^t e^{-\tau}\cos(\tau)\,d\tau = -e^{-t}\cos(t) + 1 - \underbrace{\int_0^t e^{-\tau}\operatorname{sen}(\tau)\,d\tau}_{I} \quad (3)$$

Chamando $I = \int_0^t e^{-\tau}\operatorname{sen}(\tau)\,d\tau$ e colocando (3) em (2)

temos

$$I = -e^{-t}\operatorname{sen}(t) - e^{-t}\cos(t) + 1 - I$$

$$2I = -e^{-t}[\operatorname{sen}(t) + \cos(t)] + 1$$

$$I = \frac{-e^{-t}[\operatorname{sen}(t) + \cos(t)] + 1}{2}$$

Logo, voltando em (1) temos

$$f(t)*g(t) = g(t)*f(t) = e^t\left[\frac{-e^{-t}[\operatorname{sen}(t) + \cos(t)] + 1}{2}\right]$$

$$e^t * \operatorname{sen}(t) = \frac{1}{2}\left[-\operatorname{sen}(t) - \cos(t) + e^t\right]$$

$\blacksquare$