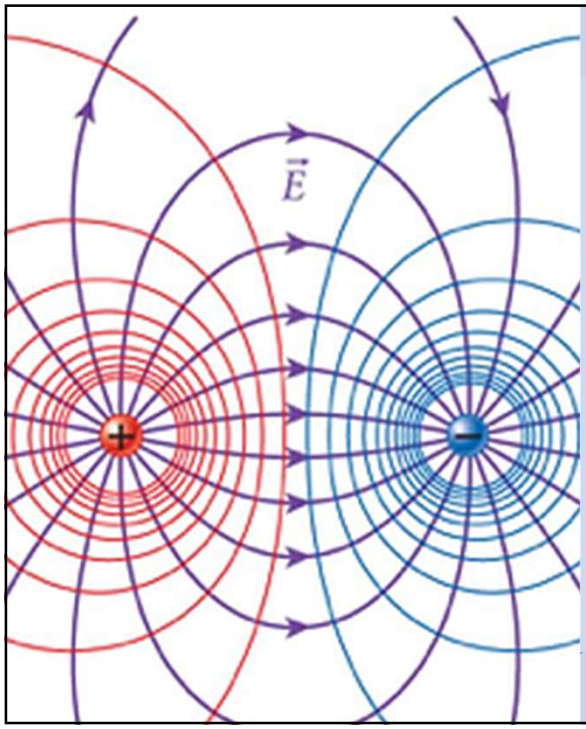

Potenciais 4

Eletromagnetismo II (4302304)

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Equações de Jefimenko

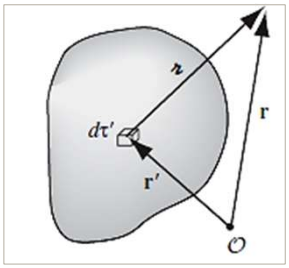
Recordando

- Equações de *Jefimenko* para os campos elétrico e magnético:

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\mathbf{r}',t_r)}{\mathcal{r}^2}\hat{\boldsymbol{\mathcal{r}}} + \frac{\dot{\rho}(\mathbf{r}',t_r)}{c\mathcal{r}}\hat{\boldsymbol{\mathcal{r}}} - \frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c^2\mathcal{r}}\right]d\tau'.$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\left[\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathcal{r}^2} + \frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c\mathcal{r}}\right]\times\hat{\boldsymbol{\mathcal{r}}}\,d\tau'.$$

$V(\mathbf{r},t)$, $\mathbf{A}(\mathbf{r},t)$



- Essas equações são as soluções fundamentais (causais) para as eqs. de Maxwell, publicadas nessa forma em 1966 por **Oleg Jefimenko** .

Potenciais de Liénard-Wiechert

Recordando

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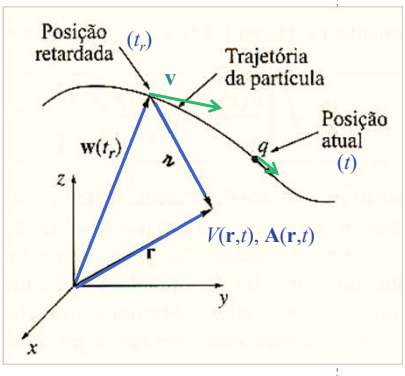
- **P**otenciais $V(\mathbf{r},t)$ e $\mathbf{A}(\mathbf{r},t)$, no calibre de Lorentz, para uma carga elétrica *puntual* em movimento numa trajetória $\mathrm{w}(t_r)$.
- Potencial escalar elétrico e potencial vetor magnético:

$$V(\mathrm{r},t)=\frac{1}{4\pi\epsilon_0}\frac{qc}{(\varkappa c-\boldsymbol{\varkappa}\cdot\mathrm{v})},$$

$$\boldsymbol{\varkappa}=\mathrm{r}-\mathrm{w}(t_r).$$

$$\mathbf{A}(\mathrm{r},t)=\frac{\mu_0}{4\pi}\frac{qc\mathrm{v}}{(\varkappa c-\boldsymbol{\varkappa}\cdot\mathrm{v})}=\frac{\mathrm{v}}{c^2}V(\mathrm{r},t).$$

$$t_r\equiv t-\frac{\varkappa}{c}.$$



... **v** é a velocidade da carga no tempo retardado, e $\boldsymbol{\varkappa}$ é o vetor da posição retardada até o ponto **r** do campo.

- No caso de movimento com vel. constante:...

$$V(\mathrm{r},t)=\frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{(c^2t-\mathrm{r}\cdot\mathrm{v})^2+(c^2-v^2)(r^2-c^2t^2)}},$$

$$\mathbf{A}(\mathrm{r},t)=\frac{\mu_0}{4\pi}\frac{qc\mathrm{v}}{\sqrt{(c^2t-\mathrm{r}\cdot\mathrm{v})^2+(c^2-v^2)(r^2-c^2t^2)}}.$$

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Potenciais de Liénard-Wiechert

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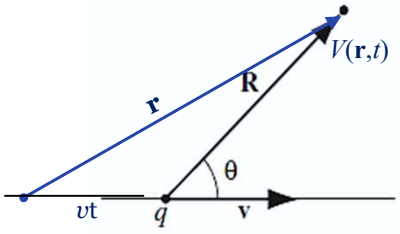
Exercício 6: *(proposto)*

- Mostre que o potencial escalar de uma carga pontual em movimento com velocidade constante pode ser escrito de forma mais simplificada como:

$$V(\mathrm{r},t)=\frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{1-v^2\sin^2\theta/c^2}},$$



onde $\mathbf{R}\equiv\mathbf{r}-\mathbf{v}t$ é o vetor desde a posição *presente* (!) da partícula até o ponto **r** do campo, e θ é o ângulo entre **R** e **v**. Para velocidades não relativísticas ($v^2 << c^2$), então

$$\therefore\quad V(\mathrm{r},t)\approx\frac{1}{4\pi\epsilon_0}\frac{q}{R}.$$

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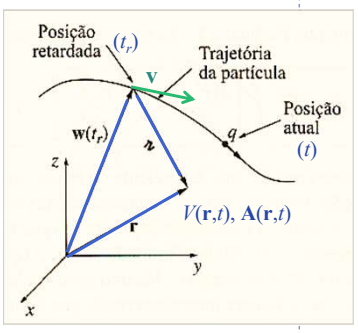
Os campos de uma carga pontual em movimento

- Vamos partir dos potenciais de Liénard-Wiechert para cargas pontuais em movimento e das equações para $\boldsymbol{E}$ e $\boldsymbol{B}$:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$



- Vamos, também, levar em conta as relações vistas:

$$|\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r), \qquad \mathcal{r} = \mathbf{r} - \mathbf{w}(t_r). \qquad t_r \equiv t - \frac{\mathcal{r}}{c}.$$

- Comecemos com o *gradiente* de V:

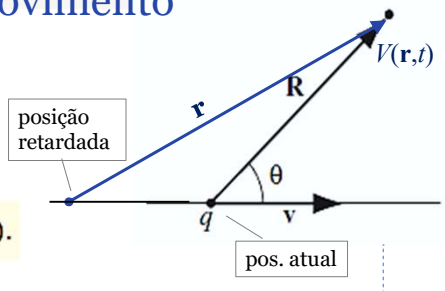
$$\therefore \quad \nabla V = \frac{qc}{4\pi\epsilon_0} \frac{-1}{(\mathcal{r}c - \boldsymbol{\mathcal{r}} \cdot \mathbf{v})^2} \nabla(\mathcal{r}c - \boldsymbol{\mathcal{r}} \cdot \mathbf{v}).$$

P/ o 1º. termo da direita, desde que $\mathcal{r} = c(t - t_r)$:

$$\nabla \mathcal{r} = -c \nabla t_r.$$

Os campos de uma carga pontual em movimento

$$\nabla V = \frac{qc}{4\pi\epsilon_0} \frac{-1}{(\mathcal{r}c - \boldsymbol{\mathcal{r}} \cdot \mathbf{v})^2} \nabla(\mathcal{r}c - \boldsymbol{\mathcal{r}} \cdot \mathbf{v}).$$



- Ainda do 2º. termo acima, vamos usar a regra (4):

$$\nabla(\boldsymbol{\mathcal{r}} \cdot \mathbf{v}) = (\boldsymbol{\mathcal{r}} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\boldsymbol{\mathcal{r}} + \boldsymbol{\mathcal{r}} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \boldsymbol{\mathcal{r}}).$$

- Vamos avaliar os termos à direita, um por um :

$$\therefore \mathbf{1)} \quad (\boldsymbol{\mathcal{r}} \cdot \nabla)\mathbf{v} = \left(\mathcal{r}_x \frac{\partial}{\partial x} + \mathcal{r}_y \frac{\partial}{\partial y} + \mathcal{r}_z \frac{\partial}{\partial z}\right) \mathbf{v}(t_r)$$

$$= \mathcal{r}_x \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial x} + \mathcal{r}_y \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial y} + \mathcal{r}_z \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial z} = \mathbf{a}(\boldsymbol{\mathcal{r}} \cdot \nabla t_r),$$

onde $\mathbf{a} \equiv \dot{\mathbf{v}}$ é a aceleração da carga no instante t_r.

$$t_r \equiv t - \frac{\mathcal{r}}{c}. \qquad |\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r),$$

Os campos de uma carga pontual em movimento

$$\nabla(\boldsymbol{\imath}\cdot\mathbf{v}) = (\boldsymbol{\imath}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\boldsymbol{\imath} + \boldsymbol{\imath}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\boldsymbol{\imath}).$$

$\therefore$ **1)** $(\boldsymbol{\imath}\cdot\nabla)\mathbf{v} = \mathbf{a}(\boldsymbol{\imath}\cdot\nabla t_r)$,

- Quanto ao 2º. Termo:

$\therefore$ **2)** $(\mathbf{v}\cdot\nabla)\boldsymbol{\imath} = (\mathbf{v}\cdot\nabla)\mathbf{r} - (\mathbf{v}\cdot\nabla)\mathbf{w}$,

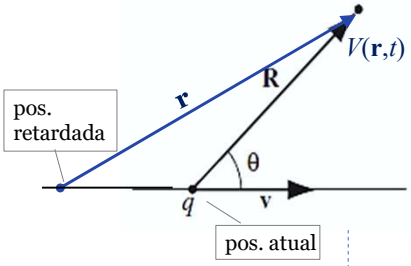
$$\rightarrow\quad (\mathbf{v}\cdot\nabla)\mathbf{r} = \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}})$$

$$= v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}} + v_z\,\hat{\mathbf{z}} = \mathbf{v},$$

$\rightarrow$ $(\mathbf{v}\cdot\nabla)\mathbf{w} = \mathbf{v}(\mathbf{v}\cdot\nabla t_r)$ *sendo* $\mathbf{w}(t) = \mathbf{v}t$. *pois:*

$$(\mathbf{v}\cdot\nabla)\mathbf{w} = \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)\mathbf{w}(t) =$$

$$= v_x\frac{\partial \mathrm{w}}{\partial \mathrm{t_r}}\frac{\partial t_r}{\partial x} + v_y\frac{\partial \mathrm{w}}{\partial \mathrm{t_r}}\frac{\partial \mathrm{t_r}}{\partial \mathrm{y}} + v_z\frac{\partial \mathrm{w}}{\partial \mathrm{t_r}}\frac{\partial \mathrm{t_r}}{\partial \mathrm{x}} = \mathbf{v}(\mathbf{v}\cdot\nabla t_r)$$



Os campos de uma carga pontual em movimento

$$\nabla(\boldsymbol{\imath}\cdot\mathbf{v}) = (\boldsymbol{\imath}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\boldsymbol{\imath} + \boldsymbol{\imath}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\boldsymbol{\imath}).$$

$\therefore$ **1)** $(\boldsymbol{\imath}\cdot\nabla)\mathbf{v} = \mathbf{a}(\boldsymbol{\imath}\cdot\nabla t_r)$,

$\therefore$ **2)** $(\mathbf{v}\cdot\nabla)\boldsymbol{\imath} = \mathbf{v} - \mathbf{v}(\mathbf{v}\cdot\nabla t_r)$

- Quanto ao 3º. Termo:

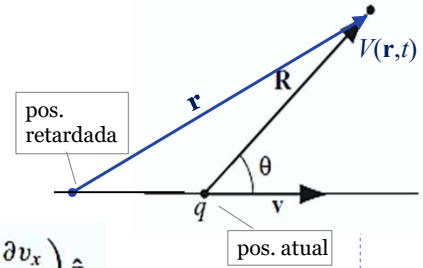
$\therefore$ **3)** $$\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$= \left(\frac{dv_z}{dt_r}\frac{\partial t_r}{\partial y} - \frac{dv_y}{dt_r}\frac{\partial t_r}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{dv_x}{dt_r}\frac{\partial t_r}{\partial z} - \frac{dv_z}{dt_r}\frac{\partial t_r}{\partial x}\right)\hat{\mathbf{y}}$$

$$+ \left(\frac{dv_y}{dt_r}\frac{\partial t_r}{\partial x} - \frac{dv_x}{dt_r}\frac{\partial t_r}{\partial y}\right)\hat{\mathbf{z}} \quad = -\mathbf{a}\times\nabla t_r.$$

- Em relação ao 4º. Termo:

$\therefore$ **4)** $\nabla\times\boldsymbol{\imath} = \nabla\times\mathbf{r} - \nabla\times\mathbf{w}$, teremos $\nabla\times\mathbf{r} = 0$, e $\nabla\times\mathbf{w} = -\mathbf{v}\times\nabla t_r$.



Os campos de uma carga pontual em movimento

- Substituindo os resultados (1) a (4):

$$\nabla(\boldsymbol{\imath}\cdot\mathbf{v}) = (\boldsymbol{\imath}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\boldsymbol{\imath} + \boldsymbol{\imath}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\boldsymbol{\imath}).$$

$$\therefore\quad \nabla(\boldsymbol{\imath}\cdot\mathbf{v}) = \mathbf{a}(\boldsymbol{\imath}\cdot\nabla t_r) + \mathbf{v} - \mathbf{v}(\mathbf{v}\cdot\nabla t_r) - \boldsymbol{\imath}\times(\mathbf{a}\times\nabla t_r) + \mathbf{v}\times(\mathbf{v}\times\nabla t_r)$$

- Usando a regra do produto vetorial triplo nos dois últimos termos (*conferir*):

$$(2)\quad \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$\therefore\quad \nabla(\boldsymbol{\imath}\cdot\mathbf{v}) = \mathbf{v} + (\boldsymbol{\imath}\cdot\mathbf{a} - v^2)\nabla t_r.$$

- Assim, o potencial escalar fica, até agora:

$$\nabla V = \frac{qc}{4\pi\epsilon_0}\frac{1}{(\imath c - \boldsymbol{\imath}\cdot\mathbf{v})^2}\left[\mathbf{v} + (c^2 - v^2 + \boldsymbol{\imath}\cdot\mathbf{a})\nabla t_r\right].$$

Para completar a dedução, precisamos achar ∇t_r ...

Os campos de uma carga pontual em movimento

- Como já vimos: $-c\nabla t_r = \nabla\imath = \nabla\sqrt{\boldsymbol{\imath}\cdot\boldsymbol{\imath}} = \frac{1}{2\sqrt{\boldsymbol{\imath}\cdot\boldsymbol{\imath}}}\nabla(\boldsymbol{\imath}\cdot\boldsymbol{\imath})$
- Usando a regra (4): $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

$$\therefore\quad -c\nabla t_r = \frac{1}{\imath}\left[(\boldsymbol{\imath}\cdot\nabla)\boldsymbol{\imath} + \boldsymbol{\imath}\times(\nabla\times\boldsymbol{\imath})\right].$$

- Mas $(\boldsymbol{\imath}\cdot\nabla)\boldsymbol{\imath} = \boldsymbol{\imath} - \mathbf{v}(\boldsymbol{\imath}\cdot\nabla t_r)$, (assim como **(2)** anterior)

e $\nabla\times\boldsymbol{\imath} = (\mathbf{v}\times\nabla t_r).$, já que $\boldsymbol{\imath} = \mathbf{r} - \mathbf{w}(t_r).$

- Então: $-c\nabla t_r = \frac{1}{\imath}\left[\boldsymbol{\imath} - \mathbf{v}(\boldsymbol{\imath}\cdot\nabla t_r) + \boldsymbol{\imath}\times(\mathbf{v}\times\nabla t_r)\right] = \frac{1}{\imath}\left[\boldsymbol{\imath} - (\boldsymbol{\imath}\cdot\mathbf{v})\nabla t_r\right],$

E assim: $\therefore\quad \nabla t_r = \frac{-\boldsymbol{\imath}}{\imath c - \boldsymbol{\imath}\cdot\mathbf{v}}.$

Vamos juntar tudo em ∇V ...

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Os campos de uma carga pontual em movimento

$$\nabla t_r = \frac{-\mathscr{r}}{\mathscr{r}c - \mathscr{r}\cdot\mathbf{v}}. \quad \rightarrow \quad \nabla V = \frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}c - \mathscr{r}\cdot\mathbf{v})^2}\left[\mathbf{v} + (c^2 - v^2 + \mathscr{r}\cdot\mathbf{a})\nabla t_r\right].$$

- Assim: $$\nabla V = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \mathscr{r}\cdot\mathbf{v})^3}\left[(\mathscr{r}c - \mathscr{r}\cdot\mathbf{v})\mathbf{v} - (c^2 - v^2 + \mathscr{r}\cdot\mathbf{a})\mathscr{r}\right].$$
- Por outro lado, um cálculo similar fornece:

$$\therefore \quad \frac{\partial \mathbf{A}}{\partial t} = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \mathscr{r}\cdot\mathbf{v})^3}\left[(\mathscr{r}c - \mathscr{r}\cdot\mathbf{v})(-\mathbf{v} + \mathscr{r}\mathbf{a}/c) + \frac{\mathscr{r}}{c}(c^2 - v^2 + \mathscr{r}\cdot\mathbf{a})\mathbf{v}\right].$$

- Combinando os resultados e introduzindo o vetor $\mathbf{u} \equiv c\hat{\mathscr{r}} - \mathbf{v}$:

$$\therefore \quad \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}. \quad \rightarrow \quad \mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\mathscr{r}\cdot\mathbf{u})^3}\left[(c^2 - v^2)\mathbf{u} + \mathscr{r}\times(\mathbf{u}\times\mathbf{a})\right].$$

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Os campos de uma carga pontual em movimento

- Vamos agora determinar o campo **B**:

$$\therefore \quad \nabla\times\mathbf{A} = \frac{1}{c^2}\nabla\times(V\mathbf{v}) = \frac{1}{c^2}[V(\nabla\times\mathbf{v}) - \mathbf{v}\times(\nabla V)]$$

- Usando resultados já calculados:

$$\rightarrow \quad \mathbf{B}(\mathbf{r}, t) = \nabla\times\mathbf{A} = -\frac{1}{c}\frac{q}{4\pi\epsilon_0}\frac{1}{(\mathbf{u}\cdot\mathscr{r})^3}\mathscr{r}\times\left[(c^2 - v^2)\mathbf{v} + (\mathscr{r}\cdot\mathbf{a})\mathbf{v} + (\mathscr{r}\cdot\mathbf{u})\mathbf{a}\right]$$

- O termo entre colchetes é semelhante àquele da expressão de **E**, desenvolvendo o produto triplo, a menos da troca de **u** por **v**. Porém, $\mathscr{r}\times\mathbf{u} = -\mathscr{r}\times\mathbf{v}$. Então, podemos escrever que:

$$\therefore \quad \mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\hat{\mathscr{r}}\times\mathbf{E}(\mathbf{r}, t). \quad \text{e} \quad \mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\mathscr{r}\cdot\mathbf{u})^3}\left[(c^2 - v^2)\mathbf{u} + \mathscr{r}\times(\mathbf{u}\times\mathbf{a})\right].$$

B é perpendicular a **E** e a $\mathscr{r}$ ...

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Os campos de uma carga pontual em movimento

- O campo **E** deduzido pode ser separado em duas componentes:

$\therefore$ $$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{\mathscr{r}}{(\mathscr{r}\cdot\boldsymbol{u})^3}(c^2-v^2)\mathbf{u} + \frac{q}{4\pi\varepsilon_0}\frac{\mathscr{r}}{(\mathscr{r}\cdot\boldsymbol{u})^3}\mathscr{r}\times(\mathbf{u}\times\mathbf{a})$$

Campo de Coulomb generalizado ou Campo de velocidade

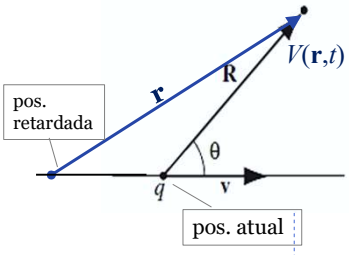
Campo de radiação ou Campo de aceleração

- $\therefore$ $\mathbf{E} = \boldsymbol{E}_v + \boldsymbol{E}_a$
- p/ $\mathbf{r} \to \infty$, $\boldsymbol{E}_v \to {}^1/_{\mathscr{r}^2}$ e $\boldsymbol{E}_a \to {}^1/_{\mathscr{r}}$ (domina a grandes distâncias)
- $\therefore$ A mesma terminologia se aplica ao campo B: $\mathbf{B} = \boldsymbol{B}_v + \boldsymbol{B}_a$

Os campos de uma carga pontual em movimento

Exercício 7:

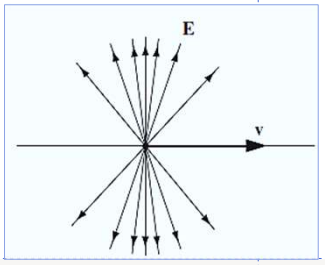
- Encontre os campos **E** e **B** para uma carga se movendo com velocidade constante ($a = 0$).



$\therefore$ $$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{(c^2-v^2)\mathscr{r}}{(\mathscr{r}\cdot\mathbf{u})^3}\mathbf{u}.$$, que considerando $\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$:

(v. ex. 6)

$\therefore$ $$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1-v^2/c^2}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2}.$$

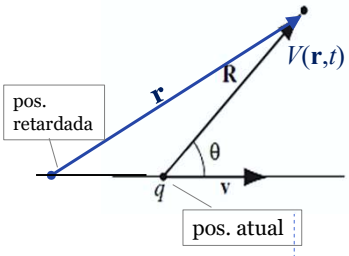


- P/ $\theta = 0, \pi$ → **E** é atenuado pelo fator $(1 - v^2/c^2)$
- P/ $\theta = \pi/2$ → **E** é aumentado pelo fator $1/\sqrt{1 - v^2/c^2}$.

Os campos de uma carga pontual em movimento

Exercício 7:

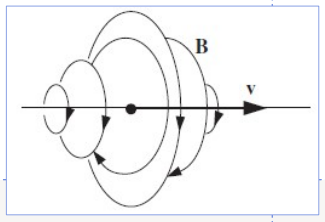
- Encontre os campos **E** e **B** para uma carga se movendo com velocidade constante ($a = 0$).



Para o campo B:

$$\mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\imath}} \times \mathbf{E}) = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E}).$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2 \sin^2\theta/c^2)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2}.$$

- O campo **B** segue distribuição angular semelhante a **E**...
- Quando $v^2 \ll c^2$, então **E** e **B** se reduzem a :

$$\therefore \quad \mathbf{E}(\mathbf{r}, t) \approx \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \hat{\mathbf{R}}; \qquad \mathbf{B}(\mathbf{r}, t) \approx \frac{\mu_0}{4\pi} \frac{q}{R^2} (\mathbf{v} \times \hat{\mathbf{R}}).$$