Eletromagnetismo

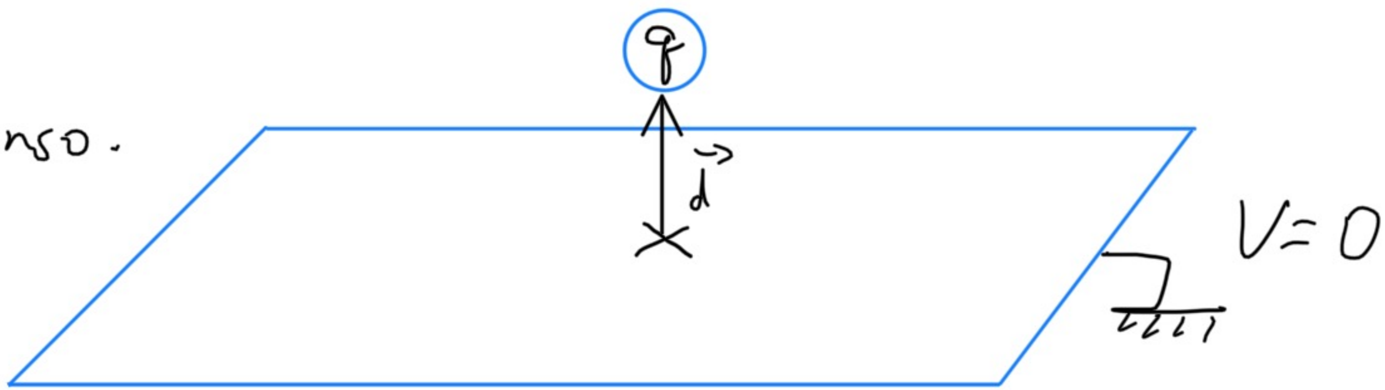
Equação de Laplace – mais soluções

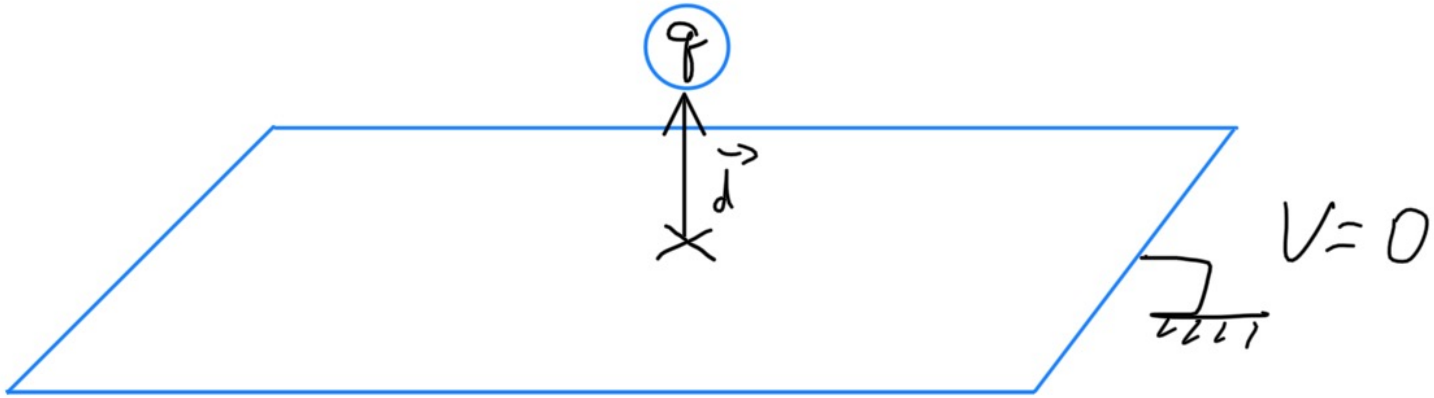
- Para além da solução da equação de Laplace por separação de variáveis, temos outras formas de resolver o problema.
- Com várias cargas e condições de contorno bem definidas, podemos usar equivalências para satisfazer as condições de contorno com distribuições *equivalentes*. Temos o Método das Imagens.
    - Situação semelhante reaparecerá em ótica! Holografia e difração.
- Com uma distribuição definida de cargas, podemos fazer expansões, decompondo a mesma em uma série de multipolos.
    - Esta técnica será útil para além do campo elétrico estático.

Aula 6: Método das imagens

Particularmente útil com cargas e condições de contorno definidas.

O exemplo básico: a carga q, e um plano condutor, muito extenso.



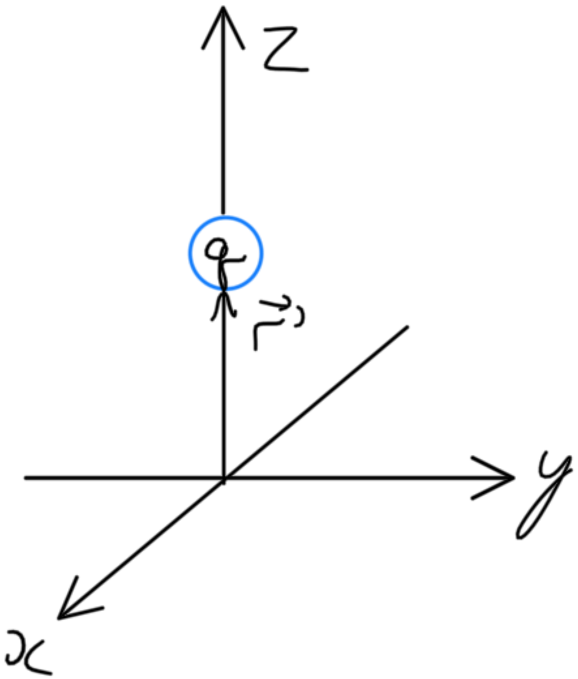


Evidente que o campo da carga induz uma redistribuição de cargas no condutor. O aterramento permite adicionar ou remover elétrons do condutor livremente.

O que temos de fixo é o potencial no plano!

As condições de contorno definem a função!

Podemos reescrever o problema, levando às mesmas condições de contorno



Potencial de uma carga q em

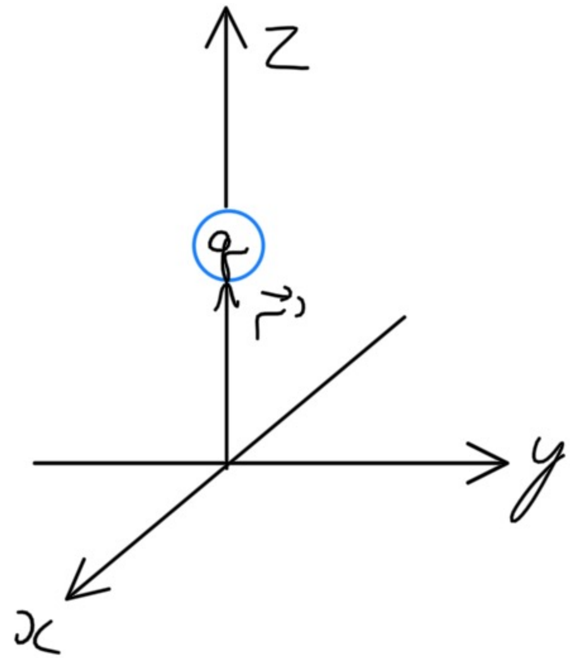
$$\vec{r}\,' = d\,\hat{k}$$

$$V_q = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r}\,'|} =$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}$$

Potencial de uma carga q em

$\vec{r}\,' = d\,\hat{k}$



$$V_q = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r}\,'|} =$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}$$

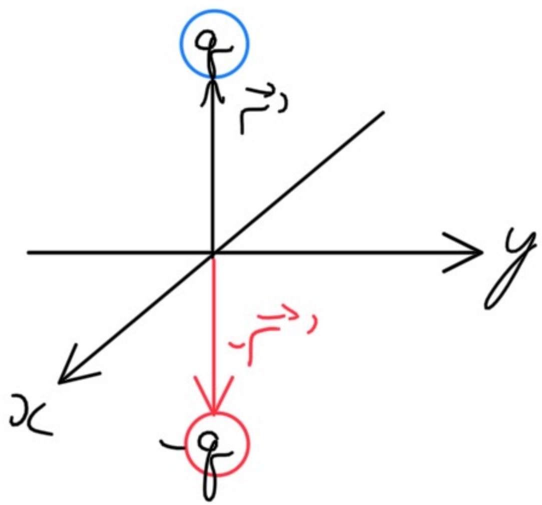
Cargas induzidas pelo plano $\rightarrow V_p = ?$

Potencial no plano $\rightarrow V(x,y,0) = 0 = V_q + V_p$

Dado ainda que $V(\vec{r}) \rightarrow 0$ se $r \rightarrow \infty$

Como é V_p? Pense em um espelho!

$$q \to q' = -q$$
$$\vec{r}' \to -\vec{r}'$$



$$V_{q'} = \frac{1}{4\pi\varepsilon_0} \frac{-q}{\sqrt{x^2 + y^2 + (z+d)^2}}$$

Note que $V(x, y, 0) = V_q + V_{q'} = 0$

$$\lim_{r \to \infty} V(r) = \lim_{r \to \infty} (V_q + V_{q'}) = 0$$

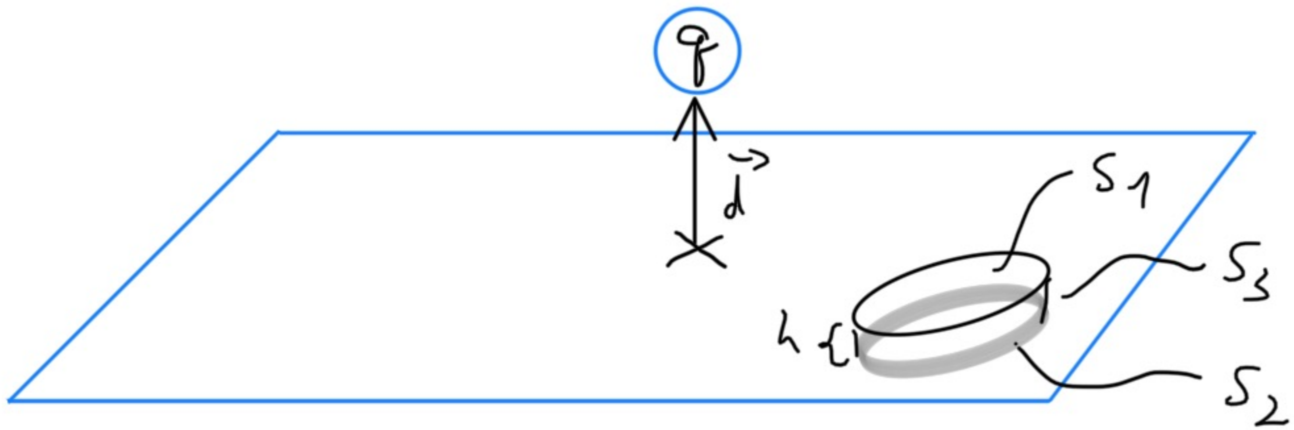
$\therefore$ $V_q + V_{q'}$ satisfazem a condição de contorno

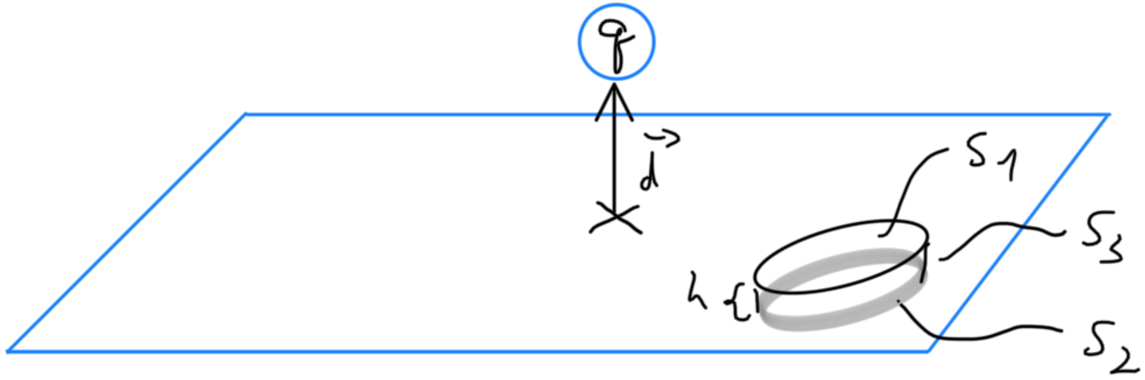
$\Rightarrow$ $V = V_q + V_{q'}$ é a solução única!

Não precisa existir $q' \rightarrow$

ela emula a distribuição de carga na superfície condutora

Com V, podemos calcular essa distribuição





Gauss $\int_{S_1} \vec{E}\cdot d\hat{a} + \int_{S_2} \vec{E}\cdot d\vec{a} + \int_{S_3} \vec{E}\cdot d\vec{a} = \int \sigma \cdot da / \varepsilon_0$

mas $\int_{S_2} \vec{E}\cdot d\vec{a} = 0$ (dentro do condutor)

$$\lim_{h\to 0} \int_{S_3} \vec{E}\cdot d\vec{a} = 0$$

$$\int_{S_1} \vec{E}\cdot d\vec{a} = \int_{S_1} -\frac{\partial}{\partial n} V \cdot \hat{n}\cdot d\vec{a} = \int \frac{\sigma}{\varepsilon_0}\, da$$

$$\Rightarrow \quad \sigma = -\varepsilon_0 \frac{\partial}{\partial n} V$$

$$\Rightarrow \qquad \sigma = -\varepsilon_0 \frac{\partial}{\partial n} V$$

$$\sigma = -\varepsilon_0 \frac{\partial}{\partial z} V \Big|_{z=0} = -\varepsilon_0 \frac{\partial}{\partial z} \left(V_q + V_{q'} \right) \Big|_{z=0}$$

$$\frac{\partial}{\partial z} \frac{1}{\left[x^2+y^2+(z-d)^2\right]^{1/2}} = -\frac{1}{2} \frac{1}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} \cdot 2(z-d)$$

$$\frac{\partial}{\partial z} \frac{1}{\left[x^2+y^2+(z+d)^2\right]^{1/2}} = -\frac{z+d}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}$$

$$\sigma = \frac{\varepsilon_0}{4\pi\varepsilon_0} \left[\frac{q(z-d)}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} + \frac{-q(z+d)}{\left[x^2+y^2+(z+d)^2\right]^{3/2}} \right] \Bigg|_{z=0}$$

$$\underline{\sigma(x,y) = -\frac{qd}{2\pi} \frac{1}{\left(x^2+y^2+d^2\right)^{3/2}}}$$

Força entre carga e placa

O campo gerado pela placa é igual ao gerado por uma carga-imagem

$$E_{q'}(\vec{r}') = \frac{1}{4\pi\varepsilon_0}\frac{(-q)}{|\vec{r}' - (-\vec{r}')|^3}\left[\vec{r}' - (-\vec{r}')\right]$$

$$= -\frac{q}{4\pi\varepsilon_0}\frac{2 \cdot d\,\hat{k}}{(2d)^3} = -\frac{q}{16\pi\varepsilon_0 d^2}\hat{k}$$

$$\vec{F}_q = q \cdot \vec{E} = -\frac{q^2}{16\pi\varepsilon_0 d^2}\hat{k}$$

Trabalho: Trazendo a carga do infinito

$$W = \int_{\infty}^{d} -\frac{q^2}{16\pi\varepsilon_0}\frac{1}{z^2}\hat{k}\cdot(\underbrace{\hat{k}\cdot dz}_{d\vec{l}}) = \frac{q^2}{16\pi\varepsilon_0}\int_{\infty}^{d}\frac{1}{z^2}dz$$

$$W = \frac{q^2}{16\pi\varepsilon_0}\left[-\frac{1}{z}\right]\Bigg|_{\infty}^{d} = -\frac{q^2}{16\pi\varepsilon_0 d}$$

Lembra do trabalho para um conjunto de N cargas?

$$W = \frac{1}{2}\sum_{\substack{i,j \\ i\neq j}} q_i V(r_i)$$

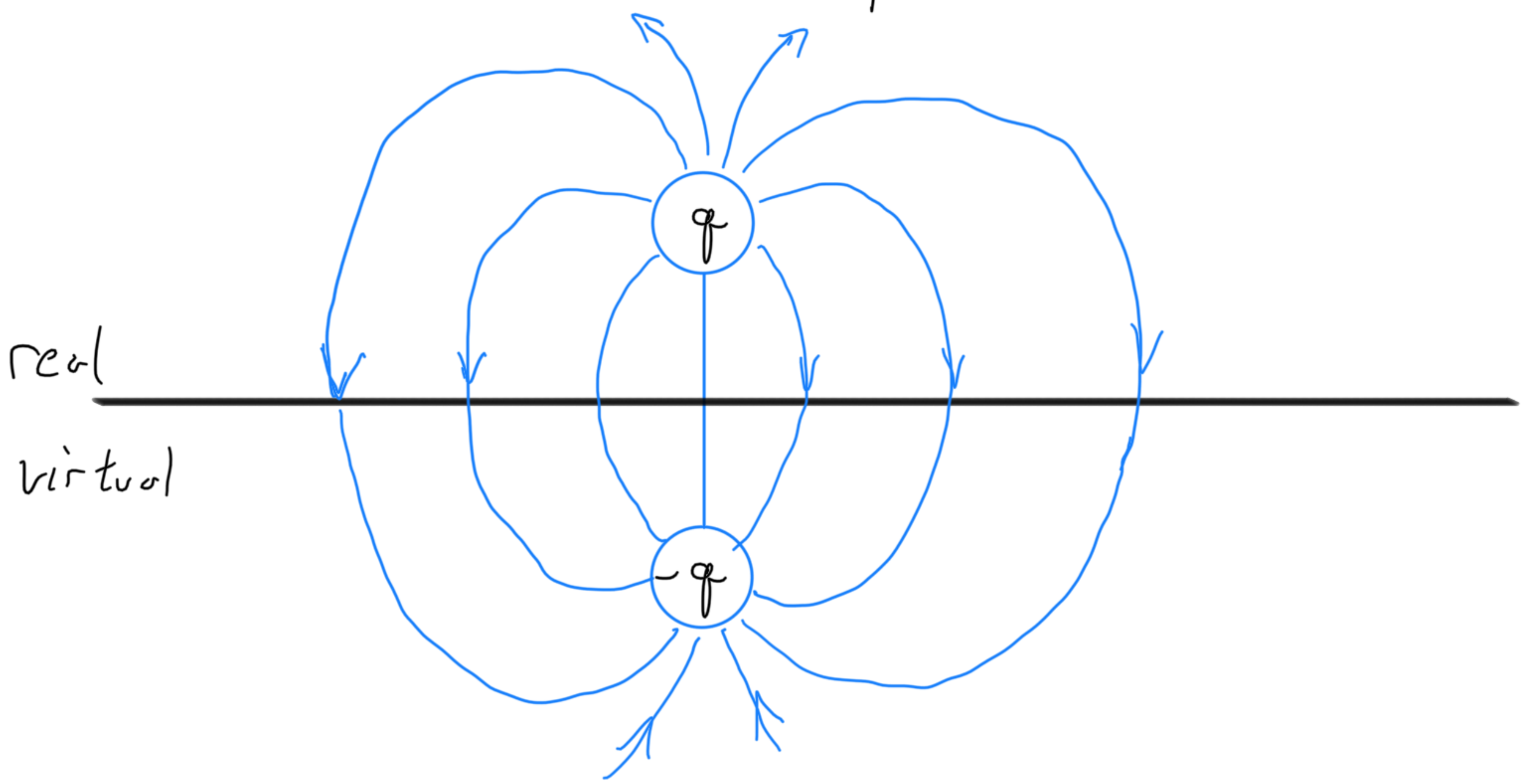
Com a carga e a carga-imagem $\rightarrow$ $V = \frac{-q}{4\pi\varepsilon_0 \cdot 2d}$

$$W = \frac{1}{2}\left[q_1 V(r_1) + q_2 V(r_2)\right] =$$

$$= \frac{q^2}{8\pi\varepsilon_0 d}$$ $\rightarrow$ Por quê o fator 2?

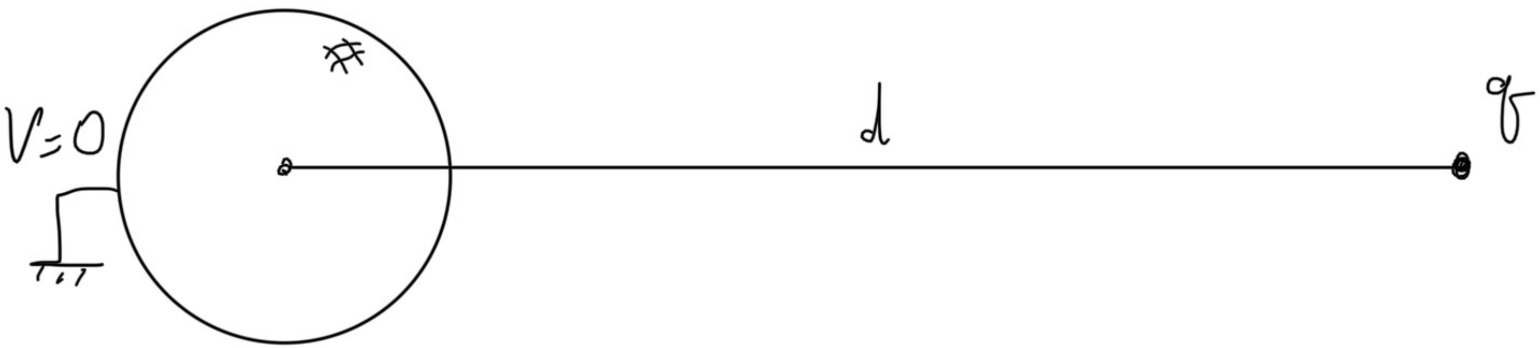
→ Não existe a carga imagem

→ $W = \frac{1}{2}\int \varepsilon_0 E^2 dV$, mas só temos metade do espaço!

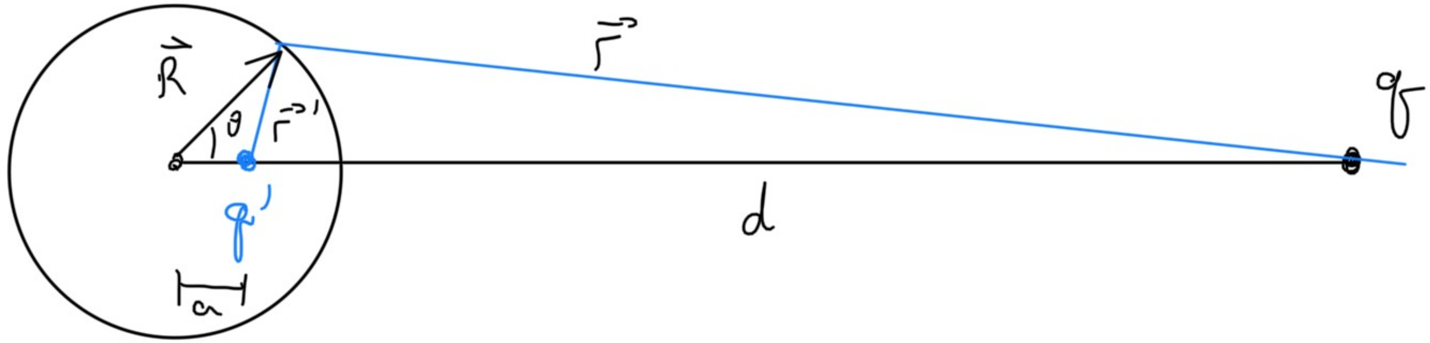


Um belo exemplo de <u>dipolo elétrico</u>

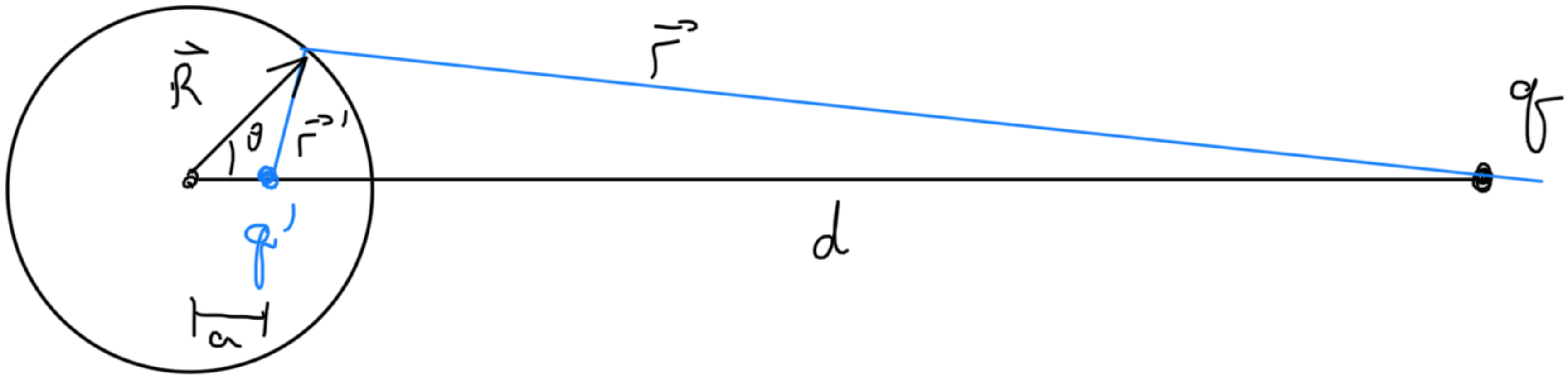
Uma carga e uma esfera aterrada



Surpreendente, mas dá para substituir a esfera por uma única carga!



$$V(\vec{R}) = V_q + V_{q'} = 0$$



$$V(\vec{R}) = V_q + V_{q'} = 0$$

$$r^2 = (R \,\mathrm{sen}\, \theta)^2 + (d - R\cos\theta)^2 = R^2 + d^2 - 2Rd\cos\theta$$

$$r'^2 = (R\,\mathrm{sen}\,\theta)^2 + (R\cos\theta - a)^2 = R^2 + a^2 - 2Ra\cos\theta$$

$$\frac{q}{4\pi\varepsilon_0 r} = \frac{-q'}{4\pi\varepsilon_0 r'} \qquad q \cdot q' < 0 \text{ (evidente)}$$

$$r'^2 = \left(\frac{q'}{q}\right)^2 r^2$$

$$R^2 + a^2 - 2Ra\cos\theta = \left(\frac{q'}{q}\right)^2 R^2 + \left(\frac{q'}{q}\right)^2 d^2 - 2Rd\left(\frac{q'}{q}\right)^2 \cos\theta$$

Para ser independente de θ, $\quad a = d\left(\frac{q'}{q}\right)^2 \Rightarrow \left(\frac{q'}{q}\right)^2 = \frac{a}{d}$

$$\Rightarrow R^2 + a^2 = \frac{a}{d}\left(R^2 + d^2\right) \Rightarrow a^2 - a\left(\frac{R^2+d^2}{d}\right) + R^2 = 0$$

$$\Rightarrow R^2 + a^2 = \frac{a}{d}(R^2 + d^2) \Rightarrow a^2 - a\left(\frac{R^2+d^2}{d}\right) + R^2 = 0$$

$$\Delta = \frac{(R^2+d^2)^2}{d^2} - 4R^2 = \frac{R^4 + d^4 + 2R^2d^2 - 4R^2d^2}{d^2} = \frac{R^4 - 2R^2d^2 + d^4}{d^2} = \left(\frac{R^2-d^2}{d}\right)^2$$

$$a = \frac{R^2+d^2}{2d} \pm \frac{1}{2}\,\frac{R^2-d^2}{d} \left\{ \quad a = R^2/d \quad ; \quad \underset{\substack{\downarrow \\ \text{solução de uma} \\ \text{carga } q' \text{ sobre } q\text{!}}}{a = d} \right.$$

$$\left(\frac{q'}{q}\right)^2 = \frac{a}{d} = \frac{R^2}{d}\cdot\frac{1}{d} = \left(\frac{R}{d}\right)^2 \Rightarrow q' = -q\frac{R}{d}$$

Temos a posição: $a = R^2/d = R \cdot \frac{R}{d} \leq R$

e a carga: $q' = -q \cdot \frac{R}{d}$; $|q'| \leq |q|$

Surpreendente simetria!

O potencial no espaço, para $r \geq R$, temos

$$V(r,\theta,\varphi) = \frac{1}{4\pi\varepsilon_0} q \left[\frac{1}{\sqrt{r^2 + d^2 - 2rd\cos\theta}} - \frac{R}{d\sqrt{r^2 + \frac{R^4}{d^2} - 2\frac{R^2}{d} r \cos\theta}} \right]$$

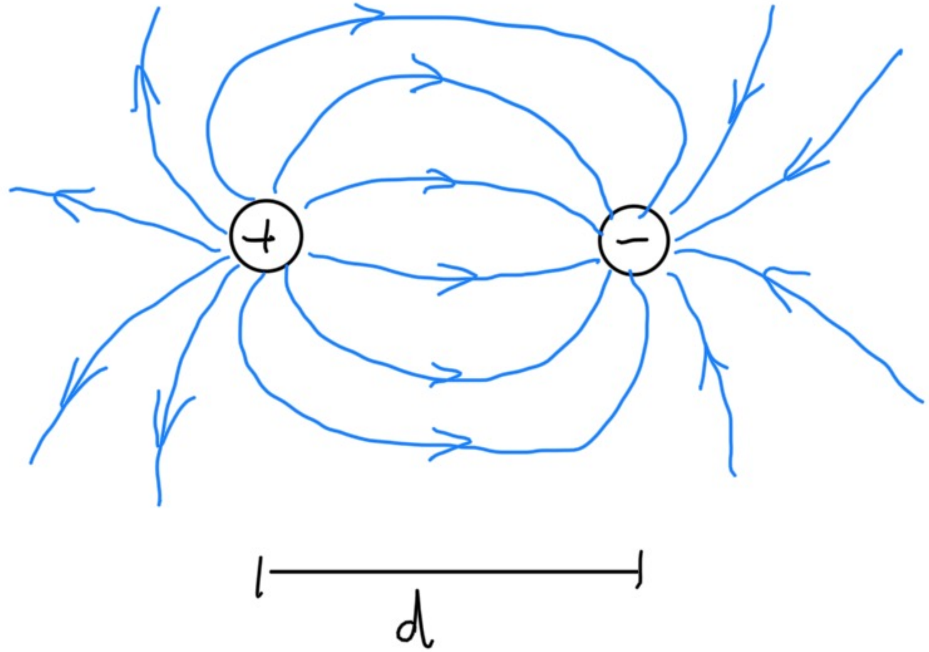
$$= \frac{1}{4\pi\varepsilon_0} q \left[\frac{1}{\sqrt{r^2 - 2rd\cos\theta + d^2}} - \frac{1}{\sqrt{R^2 - 2rd\cos\theta + d^2\left(\frac{r}{R}\right)^2}} \right]$$

Daqui podemos calcular tanto $\vec{E}(\vec{r}) = -\nabla V(\vec{r})$

quanto a carga na superficie da esfera: $\sigma = -\varepsilon_0 \frac{\partial}{\partial r} V \Big|_{r=R}$

Aula 6: Expansão em Multipolos

Dipolo Elétrico



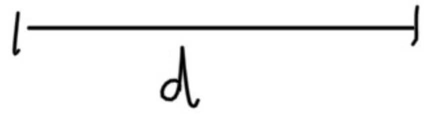
Como fonte de campo:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q(\vec{r}-\vec{r}_+)}{|\vec{r}-\vec{r}_+|^3}$$

$$- \frac{1}{4\pi\varepsilon_0} \frac{Q(\vec{r}-\vec{r}_-)}{|\vec{r}-\vec{r}_-|^3}$$

Definindo $\vec{d} = \vec{r}_+ - \vec{r}_-$ $\qquad \vec{r}_0 = \dfrac{\vec{r}_+ + \vec{r}_-}{2}$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q(\vec{r}-\vec{r}_+)}{|\vec{r}-\vec{r}_+|^3}$$



$$-\frac{1}{4\pi\varepsilon_0}\frac{Q(\vec{r}-\vec{r}_-)}{|\vec{r}-\vec{r}_-|^3}$$

Definindo $\vec{d} = \vec{r}_+ - \vec{r}_-$ $\qquad$ $\vec{r}_0 = \dfrac{\vec{r}_+ + \vec{r}_-}{2}$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} Q\left[\frac{\vec{r}-\vec{r}_0-\vec{d}/2}{|\vec{r}-\vec{r}_0-\vec{d}/2|^3} - \frac{\vec{r}-\vec{r}_0+\vec{d}/2}{|\vec{r}-\vec{r}_0+\vec{d}/2|^3}\right]$$

Módulo: $|\vec{r} \pm \vec{d}/2| = \sqrt{r^2 + \dfrac{d^2}{4} \pm \vec{r}\cdot\vec{d}}$

Para $r' = |\vec{r}'| >> d$; $|\vec{r}' \pm \vec{d}/2| \simeq \sqrt{r'^2 \pm \vec{r}' \cdot \vec{d}}$

$$= r' \sqrt{1 \pm \frac{\vec{r}' \cdot \vec{d}}{r'^2}}$$

Expandindo: $f(\alpha) = \dfrac{1}{(1+\alpha)^{3/2}} = f(0) + \alpha \dfrac{d}{d\alpha} f(\alpha)\Big|_{\alpha=0} + \dfrac{\alpha^2}{2!} \dfrac{d^2}{d\alpha^2} f(\alpha)\Big|_{\alpha=0} + \cdots$

$$= 1 - \alpha \cdot \frac{3}{2} + \frac{\alpha^2}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} + \cdots$$

Usando $\vec{r}' = \vec{r} - \vec{r}_0$, e $\alpha = \vec{r}' \cdot \vec{d} / r'^2$

$$\vec{E}(\vec{r}') = \frac{1}{4\pi\varepsilon_0} Q \left[\frac{\vec{r}' - \vec{d}/2}{|\vec{r}' - \vec{d}/2|^3} - \frac{\vec{r}' + \vec{d}/2}{|\vec{r}' + \vec{d}/2|^3} \right]$$

Usando $\vec{r}\,' = \vec{r} - \vec{r}_0$, e $\alpha = \vec{r}\,' \cdot \vec{d} / r'^2$

$$\vec{E}(\vec{r}\,') = \frac{1}{4\pi\varepsilon_0} Q \left[\frac{\vec{r}\,' - \vec{d}/2}{|\vec{r}\,' - \vec{d}/2|^3} - \frac{\vec{r}\,' + \vec{d}/2}{|\vec{r}\,' + \vec{d}/2|^3} \right]$$

$$\simeq \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r'^3} \left[\frac{\vec{r}\,'}{(1-\alpha)^{3/2}} - \frac{\vec{d}}{2} \frac{1}{(1-\alpha)^{3/2}} - \frac{\vec{r}\,'}{(1+\alpha)^{3/2}} - \frac{\vec{d}}{2} \frac{1}{(1+\alpha)^{3/2}} \right]$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r'^3} \left\{ \vec{r}\,' \left[\frac{1}{(1-\alpha)^{3/2}} - \frac{1}{(1+\alpha)^{3/2}} \right] - \frac{\vec{d}}{2} \left[\frac{1}{(1-\alpha)^{3/2}} + \frac{1}{(1+\alpha)^{3/2}} \right] \right\}$$

$$\simeq \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r'^3}\left[\frac{\vec{r}\,'}{(1-\alpha)^{3/2}} - \frac{\vec{d}}{2}\,\frac{1}{(1-\alpha)^{3/2}} - \frac{\vec{r}\,'}{(1+\alpha)^{3/2}} - \frac{\vec{d}}{2}\,\frac{1}{(1+\alpha)^{3/2}}\right]$$

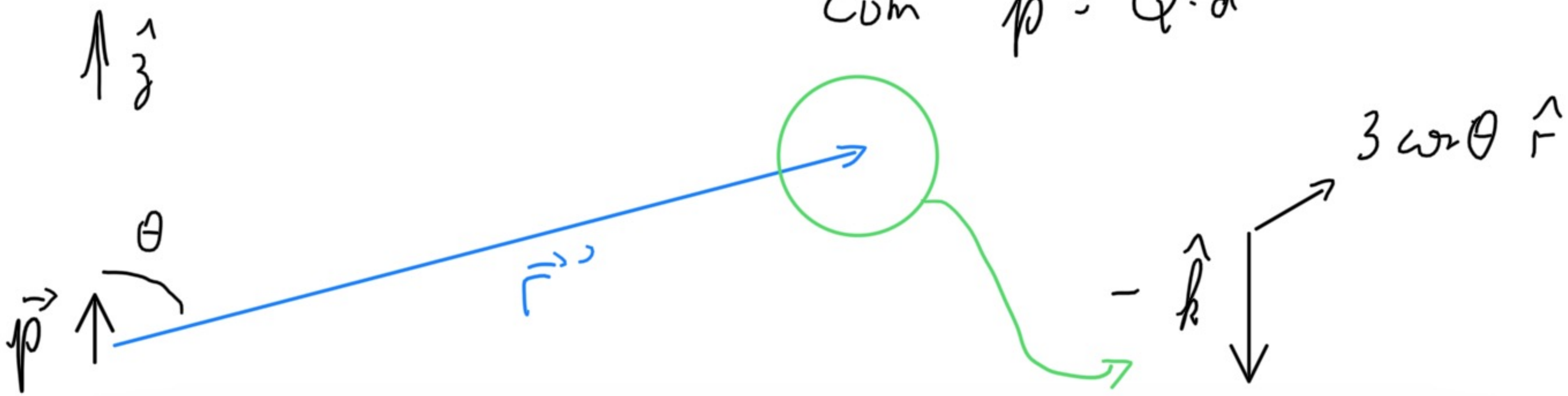
$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r'^3}\left\{\vec{r}\,'\underbrace{\left[\frac{1}{(1-\alpha)^{3/2}} - \frac{1}{(1+\alpha)^{3/2}}\right]}_{1+\frac{3}{2}\alpha+\frac{15}{18}\alpha^2 - \left(1-\frac{3}{2}\alpha+\frac{15}{18}\alpha^2\right)} - \frac{\vec{d}}{2}\underbrace{\left[\frac{1}{(1-\alpha)^{3/2}} + \frac{1}{(1+\alpha)^{3/2}}\right]}_{1-\frac{3}{2}\alpha+\frac{15}{18}\alpha^2 + 1+\frac{3}{2}\alpha+\frac{15}{18}\alpha^2}\right\}$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{Q}{r'^3}\left\{3\frac{\vec{r}\,'\,(\vec{r}\,'\cdot\vec{d})}{r'^2} - \frac{\vec{d}}{2}\cdot\left[2 + \frac{15}{8}\,\underbrace{\frac{(\vec{r}\,'\cdot\vec{d})^2}{r'^4}}_{<<\,\vec{d}}\right]\right\}$$

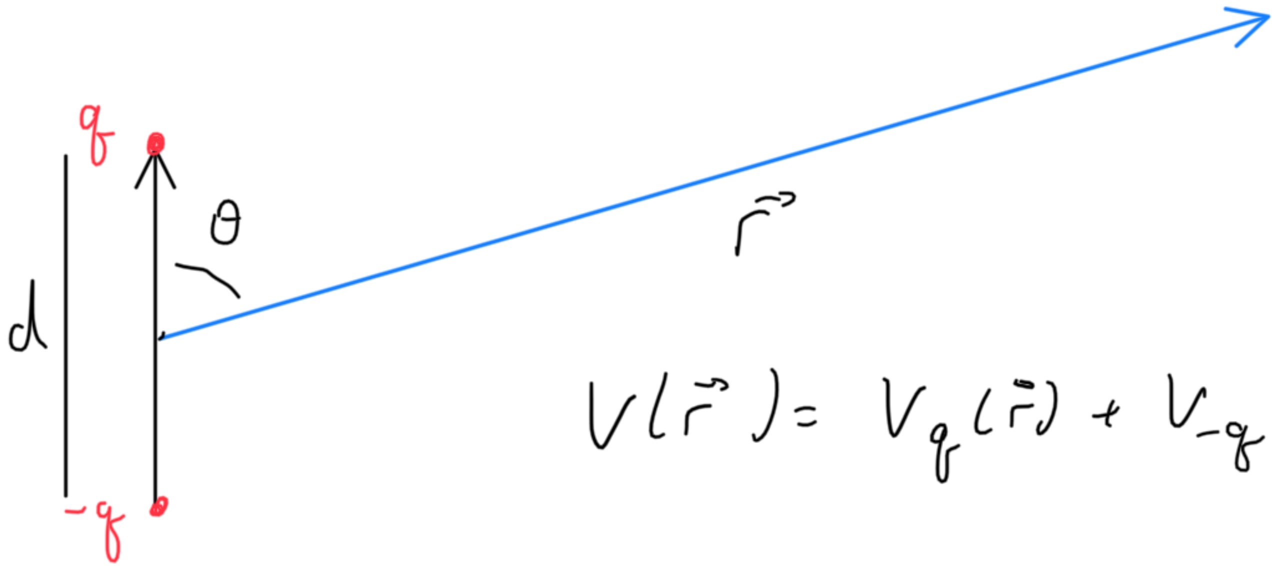
$$= \frac{1}{4\pi\varepsilon_0}\,\frac{Q}{r'^3}\left\{3\,\frac{\vec r\,'\,(\vec r\,'\cdot\vec d)}{r'^2} - \frac{\vec d}{2}\cdot\left[2+\frac{15}{8}\,\underbrace{\frac{(\vec r\,'\cdot\vec d)^2}{r'^4}}_{\ll \vec d}\right]\right\}$$

$$\vec E(\vec r\,') \doteq \frac{1}{4\pi\varepsilon_0}\,\frac{1}{r'^3}\left[3\,\frac{(\vec r\,'\cdot\vec p)\cdot\vec r\,'}{r'^2} - \vec p\right]$$

com $\vec p = Q\cdot\vec d$



Potencial elétrico de um dipolo



$$V(\vec{r}) = V_q(\vec{r}) + V_{-q}(\vec{r})$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{r}-\vec{d}/2|} - \frac{1}{|\vec{r}+\vec{d}/2|}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{r}-\vec{d}/2|} - \frac{1}{|\vec{r}+\vec{d}/2|}\right]$$

$$(\vec{r}-\vec{d}/2)^2 = r^2 \operatorname{sen}^2\theta + \left(r\cos\theta - \frac{d}{2}\right)^2 = r^2\operatorname{sen}^2\theta + r^2\cos\theta - dr\cos\theta + d^2/4$$

$$= r^2 + d^2/4 - dr\cos\theta$$

$$(\vec{r}+\vec{d}/2)^2 = r^2 + d^2/4 + dr\cos\theta$$

$$r >> d \qquad (\vec{r}-\vec{d}/2)^2 \simeq r^2\left(1 - \frac{d}{r}\cos\theta\right)$$

$$(\vec{r}+\vec{d}/2) \simeq r^2\left(1 + \frac{d}{r}\cos\theta\right)$$

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r}\left[\frac{1}{\sqrt{1-\frac{d}{r}\cos\theta}} - \frac{1}{\sqrt{1+\frac{d}{r}\cos\theta}}\right]$$

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r}\left[\frac{1}{\sqrt{1-\frac{d}{r}\cos\theta}} - \frac{1}{\sqrt{1+\frac{d}{r}\cos\theta}}\right]$$

$$\frac{1}{\sqrt{1+\alpha}} \simeq 1 - \frac{1}{2}\alpha$$

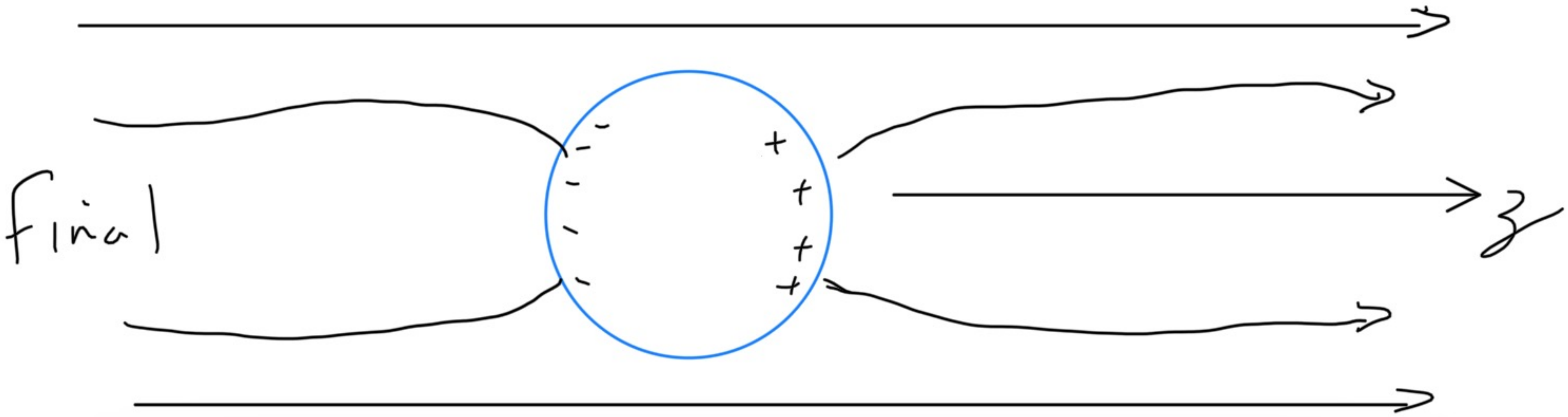
$$\Rightarrow \quad V(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r} \cdot \frac{d\cos\theta}{r} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}$$

com $\vec{p} = q\cdot\vec{d}$

Lembra o que vimos na aula passada:

esfera metálica em campo elétrico

Aplicando: Uma esfera metálica é colocada em um campo inicialmente uniforme. O campo redistribui as cargas, de forma que o campo dentro da esfera se anule. Qual o potencial fora da esfera?



A esfera é equipotencial

longe da esfera $\quad V = -E_0 z + C$

Adotando $\quad V = 0 \quad p/ \ r \leq R$

$$V \to -E_0 \, r \cos\theta \qquad r >> R$$

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

$$V(R,\theta) = 0 \Rightarrow \qquad \sum_{l=0}^{\infty} \left(A_l R^l + \frac{B_l}{R^{l+1}} \right) P_l(\cos\theta) = 0$$

$$A_l R^l = -\frac{B_l}{R^{l+1}} \Rightarrow \quad B_l = -A_l R^{2l+1}$$

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta)$$

$$r >> R \therefore V(r,\theta) \cong \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r \cos\theta$$

Somente $P_1(\cos\theta) = \cos\theta$ é conveniente

$$\Rightarrow A_1 = -E_0 \, , \quad A_l = 0 \quad l \neq 1$$

$$\therefore V(r,\theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos\theta$$

$$= \underbrace{-E_0 r \cos\theta}_{\text{potencial externo}} + \underbrace{E_0 \frac{R^3}{r^2} \cos\theta}_{\text{potencial da esfera}}$$

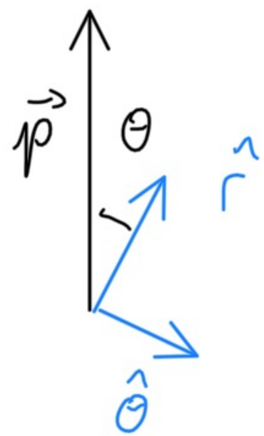
$$\nabla \cdot V(\vec{r}) = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\,\frac{\partial}{\partial \theta} + \hat{\varphi}\frac{1}{r\,\mathrm{sen}\,\theta}\,\frac{\partial}{\partial \varphi}\right) \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$= -\hat{r}\cdot\frac{p\cos\theta}{2\pi\varepsilon_0 r^3} - \hat{\theta}\,\frac{p}{4\pi\varepsilon_0 r^3}\,\mathrm{sen}\,\theta$$

$$\vec{E}(\vec{r}) = -\nabla\cdot V = \frac{1}{4\pi\varepsilon_0 r^3}\left[2(\vec{p}\cdot\vec{r})\cdot\frac{\vec{r}}{r^2} + p\,\mathrm{sen}\,\theta\;\hat{\theta} + \vec{p} - \vec{p}\right]$$

$$\vec{p} = p\cos\theta\;\hat{r} - p\,\mathrm{sen}\,\theta\;\hat{\theta}$$

$$= (\vec{p}\cdot\vec{r})\cdot\frac{\vec{r}}{r^2} - p\,\mathrm{sen}\,\theta\;\hat{\theta}$$

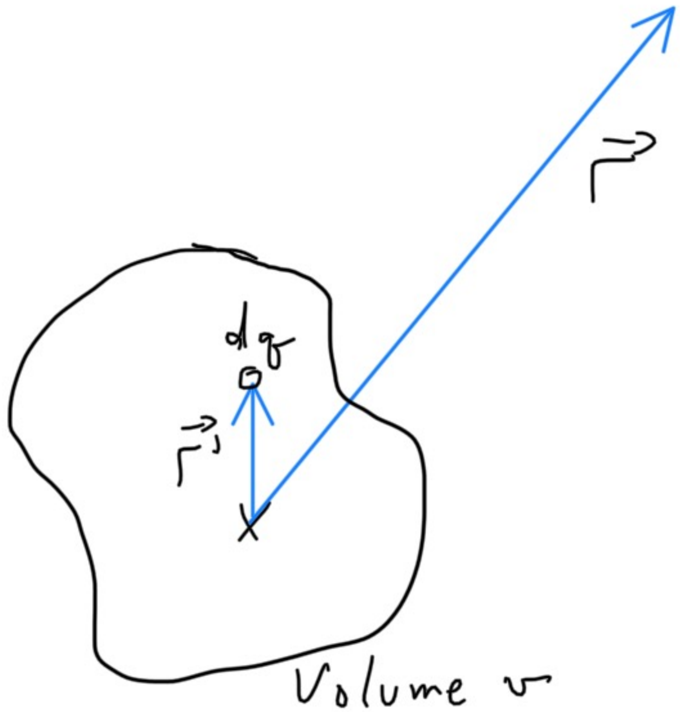


$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{r^3}\left[3\,\frac{(\vec{r}\cdot\vec{p})\cdot\vec{r}}{r^2} - \vec{p}\right]$$

Para uma distribuição de cargas $\rho(\vec{r}')$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{1}{|\vec{r}-\vec{r}'|} dq$$

$$= \frac{1}{4\pi\varepsilon_0} \int_v \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} dv$$



O termo $\frac{1}{|\vec{r}-\vec{r}'|}$ pode ser expandido

em uma série envolvendo os polinômios de Legendre

Considerando $r > r'$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{4\pi}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \left(\frac{r'}{r}\right)^{\ell} Y^{*}_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi)$$

com $Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}}\, P_{\ell m}(\cos\theta) e^{im\varphi}$

Harmônicos esféricos: $Y_{\ell,-m}(\theta,\varphi) = (-1)^m Y^{*}_{\ell,m}(\theta,\varphi)$

ℓ, m	$Y_{\ell,m}(\theta,\phi)$
0,0	$\frac{1}{\sqrt{4\pi}}$
1,0	$\sqrt{\frac{3}{4\pi}}\cos\theta$
1,1	$-\sqrt{\frac{3}{8\pi}}\,\mathrm{sen}\,\theta e^{i\phi}$
1,-1	$\sqrt{\frac{3}{8\pi}}\,\mathrm{sen}\,\theta e^{-i\phi}$
2,0	$\frac{1}{2}\sqrt{\frac{5}{4\pi}}(3\cos^2\theta - 1)$

ℓ, m	$Y_{\ell,m}(\theta,\phi)$
2,1	$-\sqrt{\frac{15}{8\pi}}\cos\theta\,\mathrm{sen}\,\theta e^{i\phi}$
2,-1	$\sqrt{\frac{15}{8\pi}}\cos\theta\,\mathrm{sen}\,\theta e^{-i\phi}$
2,2	$\frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\mathrm{sen}^2\,\theta e^{2i\phi}$
2,-2	$\frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\mathrm{sen}^2\,\theta e^{-2i\phi}$
ℓ, m	$\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_{\ell,m}(\cos\theta)e^{im\phi}$

Colocando na expansão:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}\,')\frac{4\pi}{r}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left(\frac{r'}{r}\right)^{l} Y^{*}_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi)\, dv$$

$$= \frac{1}{\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[\int \rho(\vec{r}\,')\, r'^{\,l}\, Y^{*}_{lm}(\theta',\varphi')\, dv\right]\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

$$V(\vec{r}) = \frac{1}{\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\, q_{lm}\,\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

$$q_{lm} = \int \rho(\vec{r}\,')\, r'^{\,l}\, Y^{*}_{lm}(\theta',\varphi')\, dV \rightarrow \text{momento de multipolo}$$

$$V(\vec{r}) = \frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} q_{lm} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

$$q_{lm} = \int \rho(\vec{r}\,') \, r'^{\,l} \, Y^*_{lm}(\theta',\varphi') \, dV \Rightarrow \text{momento de multipolo}$$

$l = 0 \Rightarrow$ monopolo

$$q_{00} = \int \rho(\vec{r}\,') \, Y_{00}(\theta',\varphi') \, dv = \frac{1}{\sqrt{4\pi}} \int \rho(r) \, dv = \frac{q}{\sqrt{4\pi}}$$

$$Y_{00}(\theta',\varphi') = \frac{1}{\sqrt{4\pi}}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}$$

$l = 1 \rightarrow$ dipolo

$$q_{10} = \int \rho(\vec{r}\,') \, r' \, Y^*_{10}(\theta', \varphi') \, dv = \int \rho(\vec{r}\,') \, r' \sqrt{\frac{3}{4\pi}} \cos\theta \, dv$$

$$= \sqrt{\frac{3}{4\pi}} \int \rho(\vec{r}\,') \underbrace{r' \cos\theta}_{z} \, dv$$

$$q_{10} = \sqrt{\frac{3}{4\pi}} \, p_z$$

$$V(\vec{r}) = \frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}}$$

$$V(\vec{r}) = \frac{1}{\varepsilon_0} \frac{1}{3} \sqrt{\frac{3}{4\pi}} \, p_z \cdot \sqrt{\frac{3}{4\pi}} \cos\theta \cdot \frac{1}{r^3}$$

$$= \frac{p_z \cdot \cos\theta}{4\pi \varepsilon_0 r^2}$$

Temos ainda contribuição para $l=1$, $m=1$ e $m=-1$

$$q_{1,1} = -\sqrt{\frac{3}{8\pi}}\,(p_x - i\,p_y) \quad ; \quad q_{1,-1} = \sqrt{\frac{3}{8\pi}}\,(p_x + i\,p_y)$$

O potencial total do termo dipolar envolve as 3 componentes de $\vec{p}$

$$V = \frac{1}{4\pi\varepsilon_0}\,\frac{\vec{p}\cdot\vec{r}}{r^3}$$

E $\vec{p} = \int \vec{r}\,'\rho(\vec{r}\,')\,dV$

$l=2 \rightarrow$ quadrupolo ; $l=n \rightarrow 2n$-polo ; etc.