Appendix A
Some Atomic Constants

Quantity	Symbol	Value in SI (cgs) units[a]
Speed of light in vacuum	c	2.99792458×10^{8} m/s (10^{10} cm/s)
Elementary charge	e	$1.6021765 \times 10^{-19}$ C (4.803242×10^{-10} esu)
Planck's constant	h	6.626069×10^{-34} J s ($\times 10^{-27}$ erg s)
	$\hbar$	$1.0545716 \times 10^{-34}$ J s ($\times 10^{-27}$ erg s)
Electron rest mass	m_e	9.109382×10^{-31} kg ($\times 10^{-28}$ g)
Boltzmann constant	k_B	1.380650×10^{-23} J/K ($\times 10^{-16}$ erg/K)
	k_B/hc	($0.6950356\,\mathrm{cm}^{-1}\,\mathrm{K}^{-1}$)
Rydberg constant	R_∞	$1.09737315685 \times 10^{7}\ \mathrm{m}^{-1}$ ($\times 10^{5}\ \mathrm{cm}^{-1}$)
	$R_\infty hc$	$2.179872 \times 10^{-18}\ \mathrm{J} = 13.605691\ \mathrm{eV}$
Fine-structure constant	α^{-1}	137.0359997
Bohr radius	a_0	$0.529177208 \times 10^{-10}$ m ($\times 10^{-8}$ cm)
Atomic mass unit	1 u = m_u	$1.6605388 \times 10^{-27}$ kg ($\times 10^{-24}$ g)
Proton rest mass	m_p	$1.6726216 \times 10^{-27}$ kg ($\times 10^{-24}$ gm)
	m_p/m_e	1836.152672
Electron g factor	g_e	-2.002319304362
Bohr magneton	μ_B	$9.274009 \times 10^{-24}\ \mathrm{J\,T}^{-1}$
	μ_B/hc	($4.668645 \times 10^{-5}\ \mathrm{cm}^{-1}\ \mathrm{gauss}^{-1}$)
Nuclear magneton	μ_N	$5.0507832 \times 10^{-27}\ \mathrm{J\,T}^{-1}$

[a]P.J. Mohr, B.N. Taylor and D.B. Newell, Rev. Mod. Phys. **80**, 633 (2008)

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Appendix B
Polynomials and Spherical Harmonics

The associated Laguerre polynomials are defined as

$$\mathrm{L}_{\lambda}^{\mu}(x) = \frac{1}{\lambda!} x^{-\mu} \mathrm{e}^{x} \frac{\mathrm{d}^{\lambda}}{\mathrm{d}x^{\lambda}} \left(x^{\lambda+\mu} \mathrm{e}^{-x} \right)$$

The Legendre polynomials are defined as

$$\mathrm{P}_{\ell} = \frac{1}{2^{\ell} \ell!} \frac{\mathrm{d}^{\ell}}{\mathrm{d}x^{\ell}} \left(x^{2} - 1 \right)^{\ell}; \qquad \mathrm{P}_{\ell}(1) = 1 \quad \text{for all} \quad \ell$$

The spherical harmonics are

$$\mathrm{Y}_{\ell}^{m}(\theta,\phi) = (-1)^{m} \mathrm{e}^{im\phi} \left[\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!} \right]^{1/2} \sin^{m}\theta \frac{\mathrm{d}^{m}}{\mathrm{d}x^{m}} \mathrm{P}_{\ell}(x) \quad \text{for} \quad x = \cos\theta$$

Orthonormality and completeness are given by

$$\int_{0}^{2\pi} \int_{0}^{\pi} \mathrm{Y}_{\ell}^{m}(\theta,\phi) \mathrm{Y}_{\ell'}^{*m}(\theta,\phi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = \delta_{\ell\ell'} \delta_{mm'} \tag{B.1}$$

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathrm{Y}_{\ell}^{*m}(\theta',\phi') \mathrm{Y}_{\ell}^{m}(\theta,\phi) = \delta(\phi-\phi') \delta(\cos\theta - \cos\theta') \tag{B.2}$$

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$$\mathrm{Y}_\ell^{*m}(\theta,\phi) = (-1)^m \mathrm{Y}_\ell^{-m}(\theta,\phi) \tag{B.3}$$

$$\mathrm{Y}_\ell^m(\pi-\theta,\phi+\pi) = (-1)^\ell \mathrm{Y}_\ell^m(\theta,\phi) \quad \text{Inversion: } \vec{r} \to -\vec{r} \tag{B.4}$$

$$\mathrm{Y}_\ell^0(\theta,\phi) = \left(\frac{2\ell+1}{4\pi}\right)^{1/2} \mathrm{P}_\ell(\cos\theta) \quad \text{No } \phi \text{ dependence} \tag{B.5}$$

$$\mathrm{Y}_\ell^m(0,\phi) = \mathrm{Y}_\ell^0(0)\delta_{m0} = \sqrt{\frac{2\ell+1}{4\pi}}\delta_{m0} \tag{B.6}$$

The expansion of $\frac{1}{r_{12}}$ occurs often in this text, and its derivation will be given here:

$$\frac{1}{|\vec{r}_1-\vec{r}_2|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{4\pi}{(2\ell+1)} \frac{r_<^\ell}{r_>^{\ell+1}} \mathrm{Y}_\ell^{*m}(\theta_2,\phi_2)\,\mathrm{Y}_\ell^m(\theta_1,\phi_1) \tag{B.7}$$

$\vec{r}_1$ and $\vec{r}_2$ are arbitrary vectors having the usual spherical coordinate angles θ_1,ϕ_1 and θ_2,ϕ_2, respectively. Let γ be the angle between these vectors, $\hat{r}_1\cdot\hat{r}_2 = \cos\gamma$, such that if the z-axis of a coordinate system were aligned with either $\vec{r}_1$ or $\vec{r}_2$, γ would play the role of θ for that coordinate frame. In that coordinate frame the role of ϕ is played by ω. The first step is to recall that

$$\nabla^2(|\vec{r}_1-\vec{r}_2|^{-1}) = 0 \quad \text{except at } \vec{r}_1 = \vec{r}_2$$

If $\vec{r}_2$ is chosen to lie along the z-axis, there is azimuthal symmetry with the solution

$$\frac{1}{|\vec{r}_1-\vec{r}_2|} = \sum_{\ell=0}^{\infty}[A_\ell\, r^\ell + B_\ell\, r^{-(\ell+1)}]\; P_\ell(\cos\theta)$$

This is the general solution to Laplace's equation in spherical coordinates with azimuthal symmetry. Since this solution is valid everywhere (except at $\vec{r}_1 = \vec{r}_2$), it must be valid for $\vec{r}_1$ on the z-axis. Then

$$\text{RHS} = \sum_{\ell=0}^{\infty}[A_\ell\, r^\ell + B_\ell\, r^{-(\ell+1)}]$$

$$\begin{aligned}
\text{LHS} = \frac{1}{(r_1-r_2)} &= \frac{1}{r_1}(1-r_2/r_1)^{-1} \qquad r_1 > r_2 \\
&= \frac{1}{r_1}\left[1+\frac{r_2}{r_1}+\left(\frac{r_2}{r_1}\right)^2+\left(\frac{r_2}{r_1}\right)^3+\cdots\right] \\
&= \frac{1}{r_1}\sum_{\ell=0}^{\infty}\left(\frac{r_2}{r_1}\right)^\ell = \sum_{\ell=0}^{\infty}\frac{{r_2}^\ell}{{r_1}^{\ell+1}}
\end{aligned}$$

which holds whenever $r_1 > r_2$. Whenever $r_2 > r_1$, one obtains

$$\mathrm{LHS} = \sum_{\ell=0}^{\infty} \frac{{r_1}^{\ell}}{{r_2}^{\ell+1}}$$

These two possibilities can be combined into the single expression:

$$\mathrm{LHS} = \sum_{\ell=0}^{\infty} \frac{{r_<}^{\ell}}{{r_>}^{\ell+1}}$$

where $r_<(r_>)$ is the lesser (greater) of r_1 and r_2. This is compatible with the right-hand side. For example, if $r_1 > r_2$ $B_\ell = r_2^\ell$ and $A_\ell = 0$ while if $r_2 > r_1$ $A_\ell = 1/r_2^{(\ell+1)}$ and $B_\ell = 0$. When r_1 is not along the z-axis, the solution would look like

$$\frac{1}{|\vec{r}_1 - \vec{r}_2|} = \sum_{\ell=0}^{\infty} \frac{{r_<}^{\ell}}{{r_>}^{\ell+1}} P_\ell(\cos\theta)$$

Finally if r_2 had not been along the z-axis, θ would have been γ yielding

$$\frac{1}{|\vec{r}_1 - \vec{r}_2|} = \sum_{\ell=0}^{\infty} \frac{{r_<}^{\ell}}{{r_>}^{\ell+1}} P_\ell(\cos\gamma). \tag{B.8}$$

It remains to show that $P_\ell(\cos\gamma)$ can be expanded in spherical harmonics. The expression is referred to as the spherical harmonic addition theorem:

$$P_\ell(\cos\gamma) = \left(\frac{4\pi}{2\ell+1}\right) \sum_{m=-\ell}^{\ell} \mathrm{Y}_\ell^{*m}(\theta_2, \phi_2)\mathrm{Y}_\ell^{m}(\theta_1, \phi_1) \tag{B.9}$$

First consider a function $g(\theta, \phi)$ which will at first be identified with the spherical harmonic having coordinates θ_1, ϕ_1. It will then be expanded in spherical harmonics using γ, ω coordinates. It's value at $\gamma = 0$ will prove to be important, but at that value for γ, it becomes equal to the spherical harmonic having coordinates θ_2, ϕ_2. Let's see how this unfolds:

$$g(\theta_1, \phi_1) \equiv \mathrm{Y}_\ell^m(\theta_1, \phi_1) \tag{B.10}$$

$$= \sum_{m'=-\ell}^{\ell} a_{\ell m'} \mathrm{Y}_\ell^{m'}(\gamma, \omega) \tag{B.11}$$

No summation over ℓ is needed as the spherical harmonics do not change ℓ value under a coordinate rotation:

$$g(\theta_1, \phi_1)\,|_{\gamma=0} = \sum_{m'=-\ell}^{\ell} a_{\ell m'} \left[\frac{(2\ell+1)}{4\pi}\right]^{1/2} \delta_{m'0} = a_{\ell 0} \left[\frac{(2\ell+1)}{4\pi}\right]^{1/2} \tag{B.12}$$

This follows from property (B.6) of spherical harmonics. Using (B.11), one can see that

$$\int g(\theta_1, \phi_1)\, \mathrm{Y}_\ell^{*0}(\gamma, \omega)\, \mathrm{d}\Omega_{\gamma,\omega} = a_{\ell 0}$$

But from (B.10), this means that

$$\int \mathrm{Y}_\ell^{m}(\theta_1, \phi_1)\, \mathrm{Y}_\ell^{*0}(\gamma, \omega)\, \mathrm{d}\Omega_{\gamma,\omega} = a_{\ell 0} \tag{B.13}$$

It is now possible to expand $P_\ell(\cos\gamma)$ itself in spherical harmonics:

$$P_\ell(\cos\gamma) = \sum_{m'=-\ell}^{\ell} b_{\ell m'}\, \mathrm{Y}_\ell^{m'}(\theta_1, \phi_1) \tag{B.14}$$

If one now multiplies both sides by Y_ℓ^{*m} and integrates over all space,

$$\int P_\ell(\cos\gamma)\mathrm{Y}_\ell^{*m}\, \mathrm{d}\Omega = \sum_{m'=-\ell}^{\ell} b_{\ell m'}\, \delta_{mm'} = b_{\ell m} \tag{B.15}$$

From Equation (B.5), it follows that

$$P_\ell(\cos\gamma) = \left[\frac{4\pi}{(2\ell+1)}\right]^{1/2} \mathrm{Y}_\ell^{0}(\gamma, \omega)$$

though in this expression ω is irrelevant. Inserting this into (B.15) yields

$$\left[\frac{4\pi}{(2\ell+1)}\right]^{1/2} \int \mathrm{Y}_\ell^{0}(\gamma, \omega)\mathrm{Y}_\ell^{*m}(\theta_1, \phi_1)\, \mathrm{d}\Omega = b_{\ell m}$$

But from (B.13), it follows that

$$b_{\ell m}^{*} = a_{\ell 0}\left[\frac{4\pi}{(2\ell+1)}\right]^{1/2}$$

Substituting the right-hand side of the above from (B.12) yields

$$b_{\ell m}^{*} = \frac{4\pi}{(2\ell+1)}\, g(\theta_1, \phi_1)\,|_{\gamma=0}$$

But as stated in the introduction of this derivation at $\gamma = 0$, one can write

$$g(\theta_1, \phi_1)\,|_{\gamma=0} = \mathrm{Y}_\ell^{m}(\theta_2, \phi_2)$$

from which it follows that

$$b_{\ell m}^{*} = \frac{4\pi}{(2\ell+1)}\mathrm{Y}_\ell^{m}(\theta_2, \phi_2).$$

Taking the complex conjugate and putting back into Eq. (B.14) yields the result

$$P_\ell(\cos\gamma) = \left(\frac{4\pi}{2\ell+1}\right) \sum_{m=-\ell}^{\ell} \mathrm{Y}_\ell^{*m}(\theta_2, \phi_2)\mathrm{Y}_\ell^m(\theta_1, \phi_1)$$

where the dummy index m' has been replaced by m everywhere. This completes the derivation of Equation (B.7).

Sometimes one sees spherical harmonics redefined to emphasize their relations to the Cartesian coordinates x, y, and z. Define

$$\mathrm{C}_\ell^m \equiv \left(\frac{2\ell+1}{4\pi}\right)^{1/2} \mathrm{Y}_\ell^m(\theta, \phi)$$

(eliminates some annoying constants.) Note that $\mathrm{C}_1^0 = \cos\theta = Z/r$:

$$\frac{\mathrm{C}_1^{-1} - \mathrm{C}_1^1}{\sqrt{2}} = \sin\theta\cos\phi = x/r$$

$$\frac{\mathrm{C}_1^{-1} + \mathrm{C}_1^1}{-\sqrt{2}i} = \sin\theta\sin\phi = y/r$$

These linear combinations have the spatial symmetries of x, y, and z. Wave functions using these combinations are labeled p_x, p_y, and p_z (p because $\ell = 1$).

Similarly for $\ell = 2$, one may write

$$\mathrm{C}_2^0 = \frac{3}{2}\cos^2\theta - \frac{1}{2} = \frac{1}{r^2}\left(z^2 - \frac{x^2+y^2}{2}\right)$$

$$\frac{\mathrm{C}_2^{-1} - \mathrm{C}_2^1}{\sqrt{2}} = \sqrt{3}\sin\theta\cos\theta\cos\phi = \sqrt{3}\frac{xz}{r^2}$$

$$\frac{\mathrm{C}_2^{-1} + \mathrm{C}_2^1}{-\sqrt{2}i} = \sqrt{3}\sin\theta\cos\theta\sin\phi = \sqrt{3}\frac{yz}{r^2}$$

$$\frac{\mathrm{C}_2^2 = \mathrm{C}_2^{-2}}{\sqrt{2}i} = \frac{\sqrt{3}}{2}\sin^2\theta\sin 2\phi = \sqrt{3}\frac{xy}{r^2}$$

$$\frac{\mathrm{C}_2^{-2} + \mathrm{C}_2^2}{\sqrt{2}} = \frac{\sqrt{3}}{2}\sin^2\theta\cos 2\phi = \frac{\sqrt{3}}{2}\frac{x^2-y^2}{r^2}$$

These linear combinations are sometimes labeled d_{xy}, $d_{x^2-y^2}$, etc. This labeling is not usually carried beyond $\ell = 2$. Such wave functions are often used for molecular orbital theory.

Appendix C
Some Tensor Background

A vector may be defined as any object which transforms like a coordinate point

$$A'_i = \lambda_{ij} A_j$$

A coordinate point transforms by coordinate rotation by

$$x'_i = \lambda_{ij} x_j$$

where $\lambda_{ij} \equiv \cos(x'_i, x'_j)$.

In n-dimensional space, an mth rank tensor is an object which transforms under coordinate rotations as

$$T'_{abcd\ldots} = \lambda_{ai}\lambda_{bj}\lambda_{ck}\lambda_{d\ell}\ldots T_{ijkl\ldots}$$

It has n^m components. Such a Cartesian tensor has a rank given by the number of indices. In three dimensions, an ℓth-rank tensor has 3^ℓ components.

A symmetric tensor is invariant to the interchange of any two indices. For an ℓth-rank tensor, this reduces the number of components from 3^ℓ to $(\ell+1)(\ell+2)/2$. (Can you show this?) For example, a 4th rank tensor is reduced from 81 to 15 components.

Now a second rank tensor is traceless whenever

$$\delta_{ij} T_{ij} = 0 \quad \text{or} \quad T_{11} + T_{22} + T_{33} = 0$$

The generalization of this is that

$$\delta_{mn} T_{ijk\ldots\ell} = 0$$

where m and n are *any* two indices. Such a tensor is said to be irreducible and has only $(2\ell + 1)$ independent components. So a 4th rank tensor which started with 81 components would have only 9.

Most tensors which describe physical phenomena are symmetric, and by being clever, one can usually make them irreducible.

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Consider, for example, an electrostatic multipole moment. You may recall that the quadrupole moment is defined as

$$Q_{ij} = \frac{1}{2}\int \rho(\vec{r}\,')\left(3x'_j x'_i - {r'}^2\delta_{ij}\right)\,\mathrm{d}\tau'$$

The $2^{\ell\text{th}}$ pole moment is defined as

$$Q_{ijk\ldots\ell} \equiv \frac{(-1)^\ell}{\ell!}\int \rho(\vec{r}\,'){r'}^{(2\ell+1)}\nabla'_i\nabla'_j\nabla'_k\ldots\nabla'_\ell\left(\frac{1}{r'}\right)\,\mathrm{d}\tau'.$$

Such a moment satisfies $\delta_{mn}Q_{ijk\ldots\ell} = 0$ and is symmetric.

Recall that for Y_ℓ^m m ranges from $-\ell$ to ℓ and takes on $(2\ell+1)$ values. In this way Y_ℓ^m can be used as a *basis* for irreducible tensors or spherical tensors. The spherical tensor analog of $Q_{ijk\ldots\ell}$ is

$$q_{lm} \equiv \int \mathrm{Y}_\ell^{*m}(\theta',\phi'){r'}^\ell\rho(\vec{r}\,')\,\mathrm{d}\tau'$$

Appendix D
Magnetic Dipole Interaction Energy

Recall that the definition of the magnetic dipole, $\vec{\mu}$, of a current distribution is

$$\vec{\mu} \equiv \frac{1}{2c} \int \vec{r'} \times \vec{J}(\vec{r})\, \mathrm{d}\tau'$$

But

$$\vec{J} = Nq\vec{v} = N\frac{q}{m}\vec{p}$$

where N is the number of particles (of mass m and charge q) per unit volume and $\vec{p}$ is the momentum. So

$$\vec{\mu} = \frac{Nq}{2cm} \int (\vec{r'} \times \vec{p'})\, \mathrm{d}\tau'$$

If there is but one particle in a volume V with charge $q = -e$ whose angular momentum is a constant of the motion, the dipole moment may be written as

$$\vec{\mu} = -\frac{e\vec{\ell}}{2cmV} \int \mathrm{d}\tau' = -\frac{e\vec{\ell}}{2mc} \tag{D.1}$$

The interaction energy (potential energy) of a magnetic dipole moment in an external magnetic field is what is desired. (The analogous result for an electric dipole in an external electric field is $-\vec{p}\cdot\vec{E}$.) Expand the magnetic field about some suitable origin:

$$B_i(\vec{r}) = B_i(0) + \vec{r} \cdot \vec{\nabla} B_i(0) + \cdots \tag{D.2}$$

Now the force on a current distribution in an external field is

$$\vec{F} = \frac{1}{c} \int \vec{J}(\vec{r'}) \times \vec{B}(\vec{r'})\, \mathrm{d}\tau' \tag{D.3}$$

(This is just an extension of the Lorentz law, $\vec{F} = (q/c)\vec{v} \times \vec{B}$.) Putting (D.2) into (D.3) gives

$$\vec{F} = \frac{-1}{c}\vec{B}(0) \times \int \vec{J}(\vec{r'})\, \mathrm{d}\tau' + \frac{1}{c} \int \vec{J}(\vec{r'}) \times [(\vec{r'} \cdot \vec{\nabla})\vec{B}(0)]\, \mathrm{d}\tau' + \cdots$$

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The first term is zero for steady-state localized currents. Next note that

$$\vec{J}(\vec{r'}) \times [(\vec{r'} \cdot \vec{\nabla})\vec{B}] = \vec{J}(\vec{r'}) \times \vec{\nabla}(\vec{r'} \cdot \vec{B})$$

This follows by the vector identity

$$\vec{\nabla}(\vec{r'} \cdot \vec{B}) = \vec{r'} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{r'}) + (\vec{r'} \cdot \vec{\nabla})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{r'}$$

However, $\vec{\nabla} \times \vec{B} = 0$ and ∇ do not operate on primed variables, so only the third term on the RHS is nonzero. Next note that

$$\vec{\nabla} \times (\vec{r'} \cdot \vec{B})\vec{J} = (\vec{r'} \cdot \vec{B})\vec{\nabla} \times \vec{J}(\vec{r'}) + \vec{\nabla}(\vec{r'} \cdot \vec{B}) \times \vec{J}(\vec{r'})$$

This is a vector identity and the first term of the RHS is zero because ∇ does not operate on $\vec{J}(\vec{r'})$. So

$$\vec{F} = -\frac{1}{c}\vec{\nabla} \times \int \vec{J}(\vec{r'})(\vec{r'} \cdot \vec{B}(0))\,\mathrm{d}\tau' \tag{D.4}$$

Now use the identity

$$\vec{B} \times (\vec{r'} \times \vec{J'}) = \vec{r'}(\vec{B} \cdot \vec{J'}) - \vec{J'}(\vec{r'} \cdot \vec{B})$$

to express the integral as

$$\int \vec{J}(\vec{r'})(\vec{r'} \cdot \vec{B})\,\mathrm{d}\tau' = \int \vec{r'}(\vec{B} \cdot \vec{J'})\,\mathrm{d}\tau' - \vec{B} \times \int (\vec{r'} \times \vec{J'})\,\mathrm{d}\tau' \tag{D.5}$$

On the LHS, there is

$$B_i \int J'_j x'_i\,\mathrm{d}\tau' = B_i \int \left[\nabla'_\ell (x'_j J'_\ell)\right] x'_i\,\mathrm{d}\tau'$$

(This is easy to get by working on the right to obtain the left.) Now integrate the RHS by parts:

$$\begin{aligned}
&= -B_i \int x'_j J'_\ell \nabla'_\ell x'_i\,\mathrm{d}\tau' \\
&= -B_i \int x'_j J'_i\,\mathrm{d}\tau' \\
&= -\int \vec{r'}(\vec{B} \cdot \vec{J'})\,\mathrm{d}\tau'
\end{aligned}$$

So the first term on the RHS of (D.5) is the negative of the LHS. (D.5) becomes

$$\int \vec{J}(\vec{r'})(\vec{r'} \cdot \vec{B})\,\mathrm{d}\tau' = -\frac{1}{2}\vec{B} \times \int (\vec{r'} \times \vec{J'})\,\mathrm{d}\tau'$$

Putting this into (D.4) gives

$$\vec{F} = \vec{\nabla} \times \left[\vec{B} \times \frac{1}{2c} \int (\vec{r'} \times \vec{J'})\, \mathrm{d}\tau' \right]$$
$$\text{or} \quad \vec{F} = \vec{\nabla} \times (\vec{B} \times \vec{\mu}) \tag{D.6}$$

Now use the vector identity

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A}) + (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B}$$
$$\text{and} \quad \vec{F} = (\vec{\mu} \cdot \vec{\nabla})\vec{B} = \vec{\nabla}(\vec{\mu} \cdot \vec{B}) \tag{D.7}$$

remembering that $\vec{\mu}$ is a constant vector and that $\operatorname{div} \vec{B} = \operatorname{curl} \vec{B} = 0$.

So if $\vec{F} = -\nabla W$ where W is the potential energy, it follows that

$$W = -\vec{\mu} \cdot \vec{B} \tag{D.8}$$

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