# Vibração Harmônica Forçada: 1 Grau de Liberdade

**Sistemas Dinâmicos II**

Profs. Ettore de Barros, Larissa Driemeier e Marcílio Alves

# Baseado no Livro:


If a force $F(t)$ acts on a viscously damped spring-mass system as shown in Fig. 3.1, the equation of motion can be obtained using Newton's second law:

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{3.1}$$

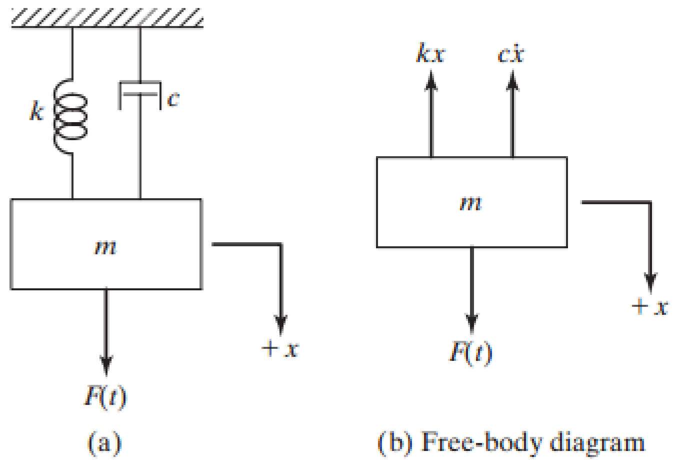


**FIGURE 3.1** A spring-mass-damper system.

Since this equation is nonhomogeneous, its general solution $x(t)$ is given by the sum of the homogeneous solution, $x_h(t)$, and the particular solution, $x_p(t)$. The homogeneous solution, which is the solution of the homogeneous equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{3.2}$$

represents the free vibration of the system and was discussed in Chapter 2. As seen in Section 2.6.2, this free vibration dies out with time under each of the three possible conditions of damping (underdamping, critical damping, and overdamping) and under all possible initial conditions. Thus the general solution of Eq. (3.1) eventually reduces to the particular solution $x_p(t)$, which represents the steady-state vibration. The steady-state motion is present as long as the forcing function is present. The variations of homogeneous, particular, and general solutions with time for a typical case are shown in Fig. 3.2. It can be seen that $x_h(t)$ dies out and $x(t)$ becomes $x_p(t)$ after some time ($\tau$ in Fig. 3.2). The part of the motion that dies out due to damping (the free-vibration part) is called *transient*. The rate at which the transient motion decays depends on the values of the system parameters $k$, $c$, and $m$. In this chapter, except in Section 3.3, we ignore the transient motion and derive only the particular solution of Eq. (3.1), which represents the steady-state response, under harmonic forcing functions.

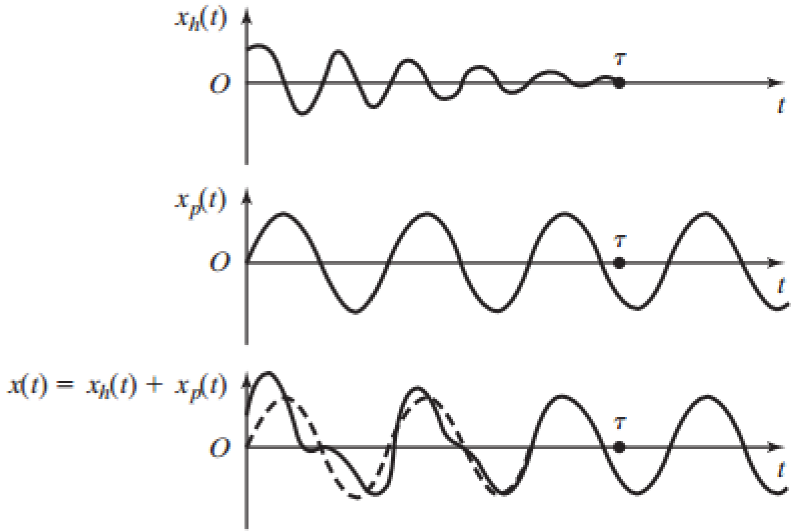


**FIGURE 3.2** Homogenous, particular, and general solutions of Eq. (3.1) for an underdamped case.

# Resposta de um sistema sem amortecimento e de 1GL à uma excitação harmônica

Before studying the response of a damped system, we consider an undamped system subjected to a harmonic force, for the sake of simplicity. If a force $F(t) = F_0 \cos \omega t$ acts on the mass $m$ of an undamped system, the equation of motion, Eq. (3.1), reduces to

$$m\ddot{x} + kx = F_0 \cos \omega t \tag{3.3}$$

The homogeneous solution of this equation is given by

$$x_h(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t \tag{3.4}$$

where $\omega_n = (k/m)^{1/2}$ is the natural frequency of the system. Because the exciting force $F(t)$ is harmonic, the particular solution $x_p(t)$ is also harmonic and has the same frequency $\omega$. Thus we assume a solution in the form

$$x_p(t) = X \cos \omega t \tag{3.5}$$

where $X$ is an constant that denotes the maximum amplitude of $x_p(t)$. By substituting Eq. (3.5) into Eq. (3.3) and solving for $X$, we obtain

$$X = \frac{F_0}{k - m\omega^2} = \frac{\delta_{st}}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2} \tag{3.6}$$

where $\delta_{st} = F_0/k$ denotes the deflection of the mass under a force $F_0$ and is sometimes called *static deflection* because $F_0$ is a constant (static) force. Thus the total solution of Eq. (3.3) becomes

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t \tag{3.7}$$

Using the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, we find that

$$C_1 = x_0 - \frac{F_0}{k - m\omega^2}, \qquad C_2 = \frac{\dot{x}_0}{\omega_n} \tag{3.8}$$

and hence

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right) \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n}\right) \sin \omega_n t + \left(\frac{F_0}{k - m\omega^2}\right) \cos \omega t \tag{3.9}$$

The maximum amplitude $X$ in Eq. (3.6) can be expressed as

$$\frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \tag{3.10}$$

The quantity $X/\delta_{st}$ represents the ratio of the dynamic to the static amplitude of motion and is called the *magnification factor, amplification factor*, or *amplitude ratio*. The variation of the amplitude ratio, $X/\delta_{st}$, with the frequency ratio $r = \omega/\omega_n$ (Eq. 3.10) is shown in Fig. 3.3. From this figure, the response of the system can be identified to be of three types.

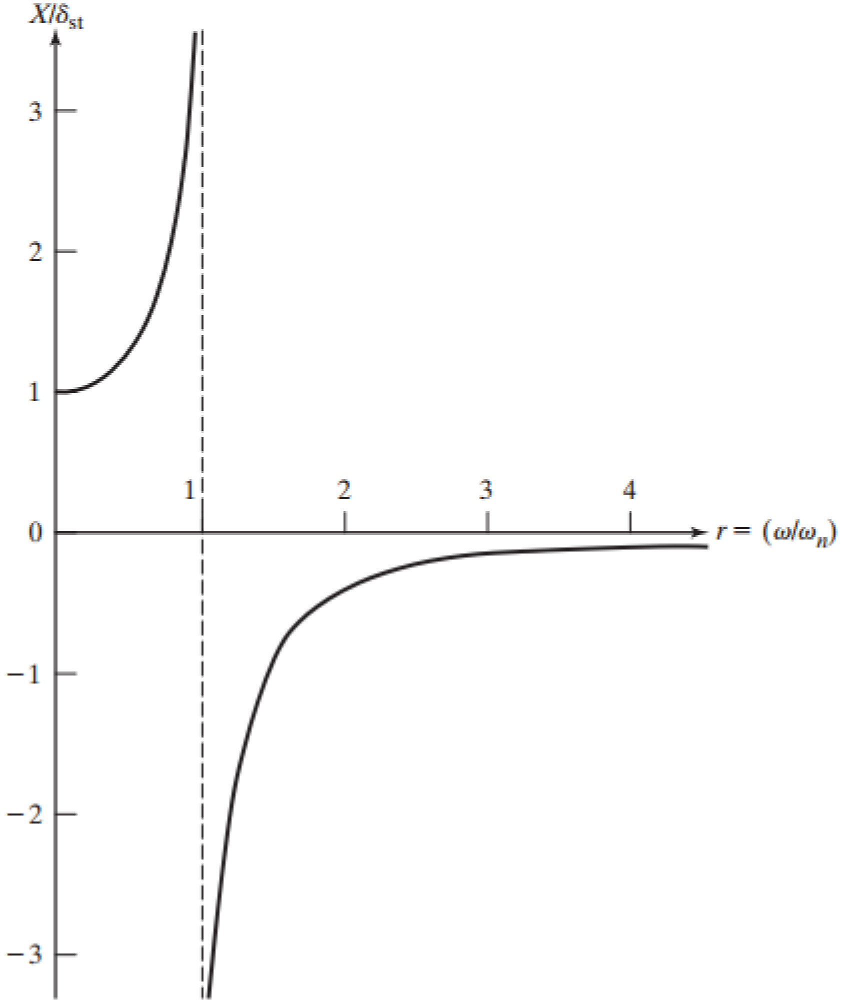


**FIGURE 3.3** Magnification factor of an undamped system, Eq. (3.10).

https://www.youtube.com/watch?v=sH7XSX10QkM glass break

https://www.youtube.com/watch?v=BE827gwnnk4 glass break

**Case 1.** When $0 < \omega/\omega_n < 1$, the denominator in Eq. (3.10) is positive and the response is given by Eq. (3.5) without change. The harmonic response of the system $x_p(t)$ is said to be in phase with the external force as shown in Fig. 3.4.

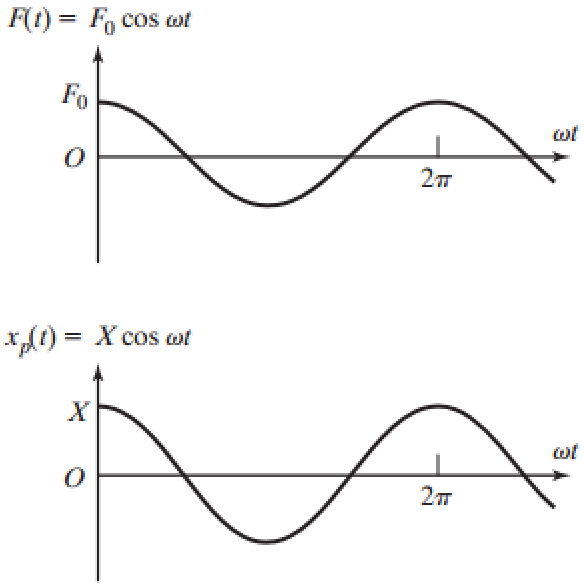


**FIGURE 3.4** Harmonic response when $0 < \omega/\omega_n < 1$.

**Case 2.** When $\omega/\omega_n > 1$, the denominator in Eq. (3.10) is negative, and the steady-state solution can be expressed as

$$x_p(t) = -X \cos \omega t \tag{3.11}$$

where the amplitude of motion $X$ is redefined to be a positive quantity as

$$X = \frac{\delta_{st}}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} \tag{3.12}$$

The variations of $F(t)$ and $x_p(t)$ with time are shown in Fig. 3.5. Since $x_p(t)$ and $F(t)$ have opposite signs, the response is said to be 180° out of phase with the external force. Further, as $\omega/\omega_n \to \infty$, $X \to 0$. Thus the response of the system to a harmonic force of very high frequency is close to zero.

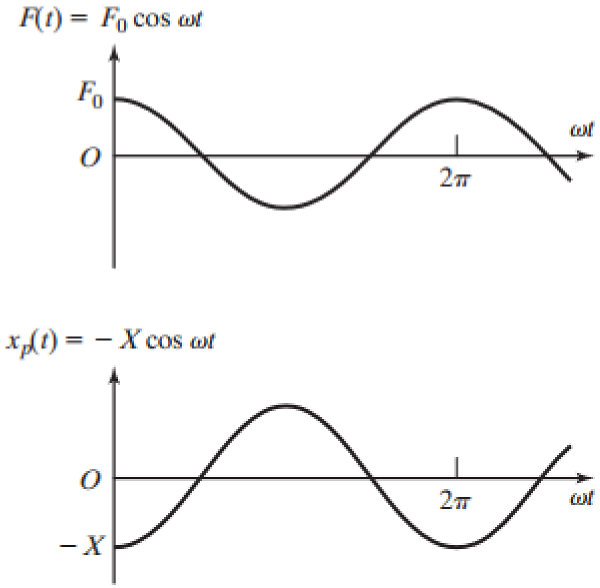


**FIGURE 3.5** Harmonic response when $\omega/\omega_n > 1$.

**Case 3.** When $\omega/\omega_n = 1$, the amplitude $X$ given by Eq. (3.10) or (3.12) becomes infinite. This condition, for which the forcing frequency $\omega$ is equal to the natural frequency of the system $\omega_n$, is called *resonance*. To find the response for this condition, we rewrite Eq. (3.9) as

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \delta_{st} \left[ \frac{\cos \omega t - \cos \omega_n t}{1 - \left( \frac{\omega}{\omega_n} \right)^2} \right] \tag{3.13}$$

Since the last term of this equation takes an indefinite form for $\omega = \omega_n$, we apply L'Hospital's rule [3.1] to evaluate the limit of this term:

$$\lim_{\omega \to \omega_n} \left[ \frac{\cos \omega t - \cos \omega_n t}{1 - \left( \frac{\omega}{\omega_n} \right)^2} \right] = \lim_{\omega \to \omega_n} \left[ \frac{\frac{d}{d\omega}(\cos \omega t - \cos \omega_n t)}{\frac{d}{d\omega}\left(1 - \frac{\omega^2}{\omega_n^2}\right)} \right]$$

$$= \lim_{\omega \to \omega_n} \left[ \frac{t \sin \omega t}{2 \frac{\omega}{\omega_n^2}} \right] = \frac{\omega_n t}{2} \sin \omega_n t \tag{3.14}$$

Thus the response of the system at resonance becomes

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st} \omega_n t}{2} \sin \omega_n t \tag{3.15}$$

It can be seen from Eq. (3.15) that at resonance, $x(t)$ increases indefinitely. The last term of Eq. (3.15) is shown in Fig. 3.6, from which the amplitude of the response can be seen to increase linearly with time.

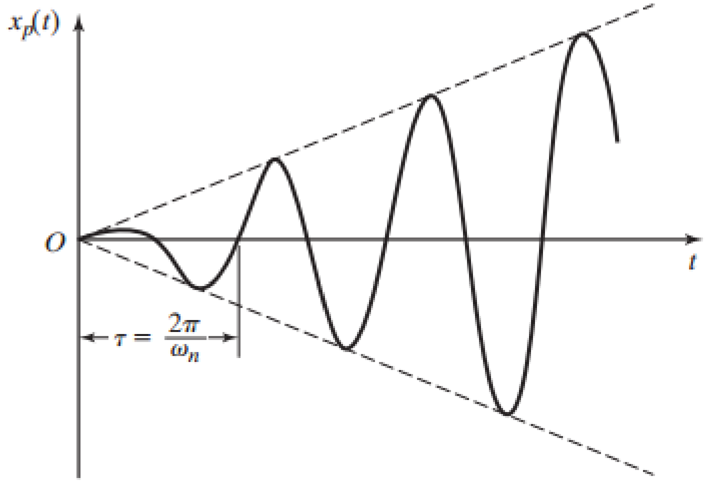


**FIGURE 3.6** Response when $\omega/\omega_n = 1$.

The total response of the system, Eq. (3.7) or Eq. (3.9), can also be expressed as

$$x(t) = A\cos(\omega_n t - \phi) + \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\cos\omega t; \quad \text{for } \frac{\omega}{\omega_n} < 1 \tag{3.16}$$

$$x(t) = A\cos(\omega_n t - \phi) - \frac{\delta_{st}}{-1 + \left(\frac{\omega}{\omega_n}\right)^2}\cos\omega t; \quad \text{for } \frac{\omega}{\omega_n} > 1 \tag{3.17}$$

where $A$ and $\phi$ can be determined as in the case of Eq. (2.21). Thus the complete motion can be expressed as the sum of two cosine curves of different frequencies. In Eq. (3.16), the forcing frequency $\omega$ is smaller than the natural frequency, and the total response is shown in Fig. 3.7(a). In Eq. (3.17), the forcing frequency is greater than the natural frequency, and the total response appears as shown in Fig. 3.7(b).

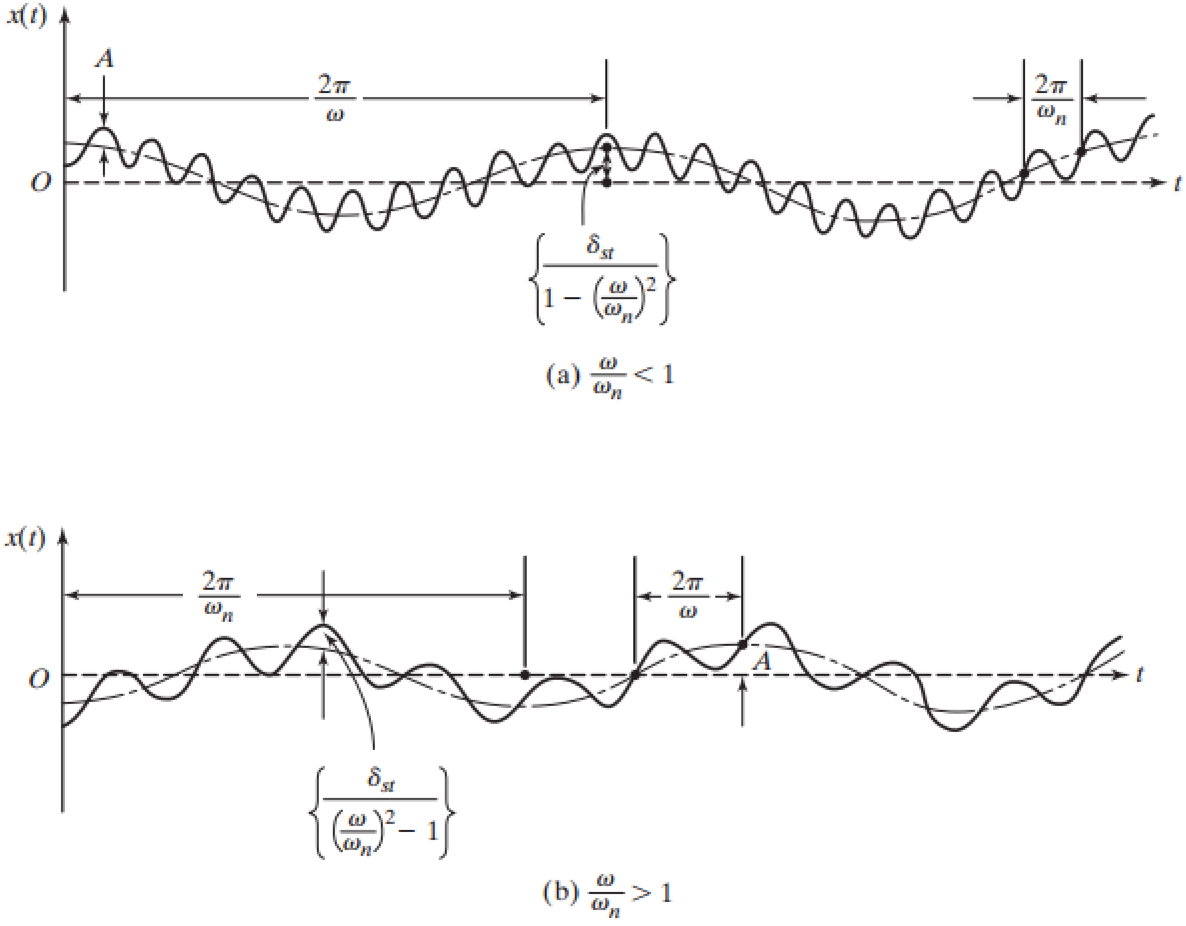


**FIGURE 3.7** Total response.

# Resposta de um sistema com amortecimento e de 1GL à uma excitação harmônica

If the forcing function is given by $F(t) = F_0 \cos \omega t$, the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t \tag{3.24}$$

The particular solution of Eq. (3.24) is also expected to be harmonic; we assume it in the form[1]

$$x_p(t) = X \cos(\omega t - \phi) \tag{3.25}$$

where $X$ and $\phi$ are constants to be determined. $X$ and $\phi$ denote the amplitude and phase angle of the response, respectively. By substituting Eq. (3.25) into Eq. (3.24), we arrive at

$$X[(k - m\omega^2)\cos(\omega t - \phi) - c\omega \sin(\omega t - \phi)] = F_0 \cos \omega t \tag{3.26}$$

Using the trigonometric relations

$$\cos(\omega t - \phi) = \cos \omega t \cos \phi + \sin \omega t \sin \phi$$
$$\sin(\omega t - \phi) = \sin \omega t \cos \phi - \cos \omega t \sin \phi$$

in Eq. (3.26) and equating the coefficients of $\cos \omega t$ and $\sin \omega t$ on both sides of the resulting equation, we obtain

$$\begin{aligned} X[(k - m\omega^2)\cos \phi + c\omega \sin \phi] &= F_0 \\ X[(k - m\omega^2)\sin \phi - c\omega \cos \phi] &= 0 \end{aligned} \tag{3.27}$$

Solution of Eq. (3.27) gives

$$X = \frac{F_0}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}} \tag{3.28}$$

and

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \tag{3.29}$$

By inserting the expressions of $X$ and $\phi$ from Eqs. (3.28) and (3.29) into Eq. (3.25), we obtain the particular solution of Eq. (3.24). Figure 3.10(a) shows typical plots of the forcing function and (steady-state) response. The various terms of Eq. (3.26) are shown vectorially in Fig. 3.10(b). Dividing both the numerator and denominator of Eq. (3.28) by $k$ and making the following substitutions

$$\omega_n = \sqrt{\frac{k}{m}} = \text{undamped natural frequency,}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{mk}}; \qquad \frac{c}{m} = 2\zeta\omega_n,$$

$$\delta_{st} = \frac{F_0}{k} = \text{deflection under the static force } F_0, \text{ and}$$

$$r = \frac{\omega}{\omega_n} = \text{frequency ratio}$$

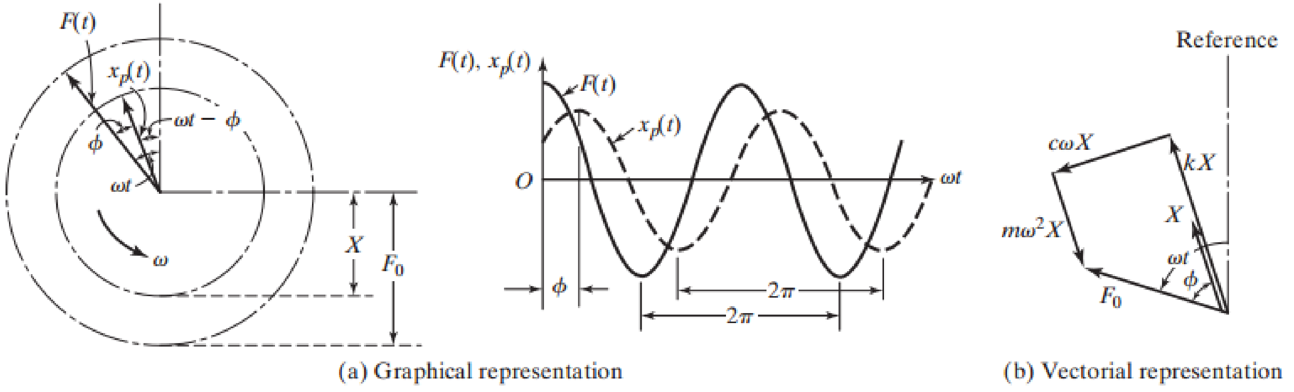


**FIGURE 3.10** Representation of forcing function and response.

we obtain

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{3.30}$$

and

$$\phi = \tan^{-1}\left\{\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\} = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \tag{3.31}$$

As stated in Section 3.3, the quantity $M = X/\delta_{st}$ is called the *magnification factor, amplification factor*, or *amplitude ratio*. The variations of $X/\delta_{st}$ and $\phi$ with the frequency ratio $r$ and the damping ratio $\zeta$ are shown in Fig. 3.11.

The following characteristics of the magnification factor ($M$) can be noted from Eq. (3.30) and Fig. 3.11(a):

1. For an undamped system ($\zeta = 0$), Eq. (3.30) reduces to Eq. (3.10), and $M \to \infty$ as $r \to 1$.
2. Any amount of damping ($\zeta > 0$) reduces the magnification factor ($M$) for all values of the forcing frequency.
3. For any specified value of $r$, a higher value of damping reduces the value of $M$.
4. In the degenerate case of a constant force (when $r = 0$), the value of $M = 1$.
5. The reduction in $M$ in the presence of damping is very significant at or near resonance.
6. The amplitude of forced vibration becomes smaller with increasing values of the forcing frequency (that is, $M \to 0$ as $r \to \infty$).

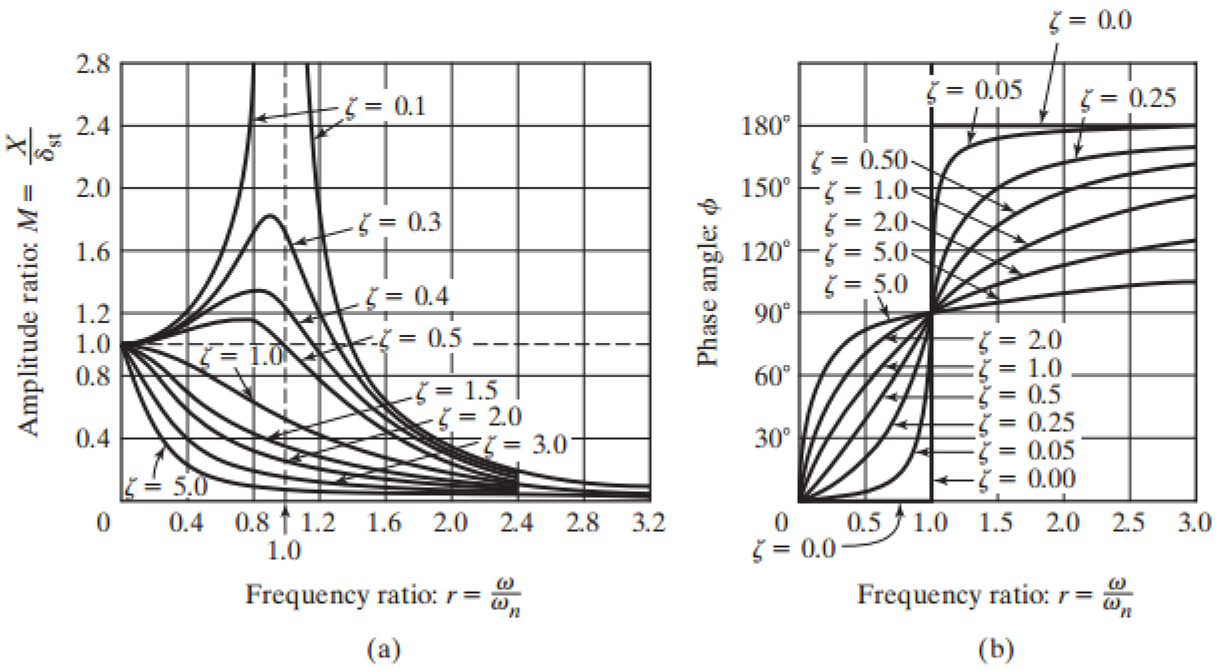


**FIGURE 3.11** Variation of $X$ and $\phi$ with frequency ratio $r$.

7. For $0 < \zeta < \frac{1}{\sqrt{2}}$, the maximum value of $M$ occurs when (see Problem 3.32)

$$r = \sqrt{1 - 2\zeta^2} \quad \text{or} \quad \omega = \omega_n \sqrt{1 - 2\zeta^2} \tag{3.32}$$

which can be seen to be lower than the undamped natural frequency $\omega_n$ and the damped natural frequency $\omega_d = \omega_n \sqrt{1 - \zeta^2}$.

8. The maximum value of $X$ (when $r = \sqrt{1 - 2\zeta^2}$) is given by

$$\left(\frac{X}{\delta_{st}}\right)_{max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \tag{3.33}$$

and the value of $X$ at $\omega = \omega_n$ by

$$\left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta} \tag{3.34}$$

Equation (3.33) can be used for the experimental determination of the measure of damping present in the system. In a vibration test, if the maximum amplitude of the response $(X)_{max}$ is measured, the damping ratio of the system can be found using Eq. (3.33). Conversely, if the amount of damping is known, one can make an estimate of the maximum amplitude of vibration.

9. For $\zeta = \frac{1}{\sqrt{2}}$, $\frac{dM}{dr} = 0$ when $r = 0$. For $\zeta > \frac{1}{\sqrt{2}}$, the graph of $M$ monotonically decreases with increasing values of $r$.

The following characteristics of the phase angle can be observed from Eq. (3.31) and Fig. 3.11(b):

1. For an undamped system ($\zeta = 0$), Eq. (3.31) shows that the phase angle is 0 for $0 < r < 1$ and 180° for $r > 1$. This implies that the excitation and response are in phase for $0 < r < 1$ and out of phase for $r > 1$ when $\zeta = 0$.
2. For $\zeta > 0$ and $0 < r < 1$, the phase angle is given by $0 < \phi < 90°$, implying that the response lags the excitation.
3. For $\zeta > 0$ and $r > 1$, the phase angle is given by $90° < \phi < 180°$, implying that the response leads the excitation.
4. For $\zeta > 0$ and $r = 1$, the phase angle is given by $\phi = 90°$, implying that the phase difference between the excitation and the response is 90°.
5. For $\zeta > 0$ and large values of $r$, the phase angle approaches 180°, implying that the response and the excitation are out of phase.

$X$ and $\phi$ are given by Eqs. (3.30) and (3.31), respectively, and $X_0$ and $\phi_0$ [different from those of Eq. (2.70)] can be determined from the initial conditions. For the initial conditions, $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, Eq. (3.35) yields

$$\begin{aligned} x_0 &= X_0 \cos \phi_0 + X \cos \phi \\ \dot{x}_0 &= -\zeta\omega_n X_0 \cos \phi_0 + \omega_d X_0 \sin \phi_0 + \omega X \sin \phi \end{aligned} \tag{3.36}$$

The solution of Eq. (3.36) gives $X_0$ and $\phi_0$ as

$$\left.\begin{aligned} X_0 &= \left[ (x_0 - X \cos \phi)^2 + \frac{1}{\omega_d^2}(\zeta\omega_n x_0 + \dot{x}_0 - \zeta\omega_n X \cos \phi - \omega X \sin \phi)^2 \right]^{\frac{1}{2}} \\ \tan \phi_0 &= \frac{\zeta\omega_n x_0 + \dot{x}_0 - \zeta\omega_n X \cos \phi - \omega X \sin \phi}{\omega_d (x_0 - X \cos \phi)} \end{aligned}\right\} \tag{3.37}$$

## 3.4.2 Quality Factor and Bandwidth

For small values of damping ($\zeta < 0.05$), we can take

$$\left(\frac{X}{\delta_{st}}\right)_{max} \simeq \left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta} = Q \tag{3.38}$$

The value of the amplitude ratio at resonance is also called *Q factor* or *quality factor* of the system, in analogy with some electrical-engineering applications, such as the tuning circuit of a radio, where the interest lies in an amplitude at resonance that is as large as possible [3.2]. The points $R_1$ and $R_2$, where the amplification factor falls to $Q/\sqrt{2}$, are called *half power points* because the power absorbed ($\Delta W$) by the damper (or by the resistor in an electrical circuit), responding harmonically at a given frequency, is proportional to the square of the amplitude (see Eq. (2.94)):

$$\Delta W = \pi c \omega X^2$$

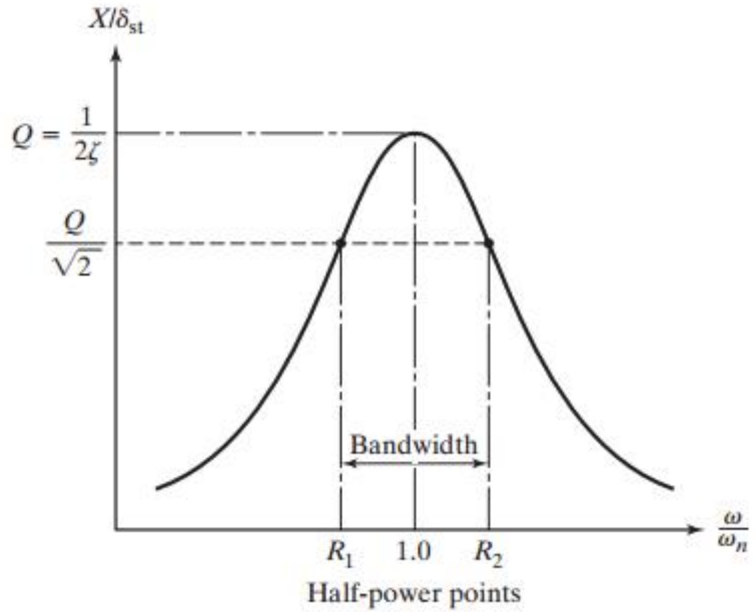


**FIGURE 3.12** Harmonic-response curve showing half-power points and bandwidth.

The difference between the frequencies associated with the half-power points $R_1$ and $R_2$ is called the *bandwidth* of the system (see Fig. 3.12). To find the values of $R_1$ and $R_2$, we set $X/\delta_{st} = Q/\sqrt{2}$ in Eq. (3.30) so that

$$\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{Q}{\sqrt{2}} = \frac{1}{2\sqrt{2}\zeta}$$

or

$$r^4 - r^2(2-4\zeta^2) + (1-8\zeta^2) = 0 \tag{3.40}$$

The solution of Eq. (3.40) gives

$$r_1^2 = 1 - 2\zeta^2 - 2\zeta\sqrt{1+\zeta^2}, \qquad r_2^2 = 1 - 2\zeta^2 + 2\zeta\sqrt{1+\zeta^2} \tag{3.41}$$

For small values of $\zeta$, Eq. (3.41) can be approximated as

$$r_1^2 = R_1^2 = \left(\frac{\omega_1}{\omega_n}\right)^2 \simeq 1 - 2\zeta, \qquad r_2^2 = R_2^2 = \left(\frac{\omega_2}{\omega_n}\right)^2 \simeq 1 + 2\zeta \tag{3.42}$$

where $\omega_1 = \omega|_{R_1}$ and $\omega_2 = \omega|_{R_2}$. From Eq. (3.42),

$$\omega_2^2 - \omega_1^2 = (\omega_2+\omega_1)(\omega_2-\omega_1) = (R_2^2 - R_1^2)\omega_n^2 \simeq 4\zeta\omega_n^2 \tag{3.43}$$

Using the relation

$$\omega_2 + \omega_1 = 2\omega_n \tag{3.44}$$

in Eq. (3.43), we find that the bandwidth $\Delta\omega$ is given by

$$\Delta\omega = \omega_2 - \omega_1 \simeq 2\zeta\omega_n \tag{3.45}$$

Combining Eqs. (3.38) and (3.45), we obtain

$$Q \simeq \frac{1}{2\zeta} \simeq \frac{\omega_n}{\omega_2 - \omega_1} \tag{3.46}$$

It can be seen that the quality factor $Q$ can be used for estimating the equivalent viscous damping in a mechanical system.[2]

# Exercícios selecionados [RAO]

match the items in the two columns below:

| | |
|---|---|
| **1.** Magnification factor of an undamped system | **a.** $\dfrac{2\pi}{\omega_n - \omega}$ |
| **2.** Period of beating | **b.** $\left[\dfrac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2}$ |
| **3.** Magnification factor of a damped system | **c.** $\dfrac{\omega_n}{\omega_2 - \omega_1}$ |
| **4.** Damped frequency | **d.** $\dfrac{1}{1 - r^2}$ |
| **5.** Quality factor | **e.** $\omega_n\sqrt{1 - \zeta^2}$ |
| **6.** Displacement transmissibility | **f.** $\left[\dfrac{1}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2}$ |

**3.1** A weight of 50 N is suspended from a spring of stiffness 4000 N/m and is subjected to a harmonic force of amplitude 60 N and frequency 6 Hz. Find (a) the extension of the spring due to the suspended weight, (b) the static displacement of the spring due to the maximum applied force, and (c) the amplitude of forced motion of the weight.

**3.2** A spring-mass system is subjected to a harmonic force whose frequency is close to the natural frequency of the system. If the forcing frequency is 39.8 Hz and the natural frequency is 40.0 Hz, determine the period of beating.

**3.3** Consider a spring-mass system, with $k = 4000$ N/m and $m = 10$ kg, subject to a harmonic force $F(t) = 400 \cos 10t$ N. Find and plot the total response of the system under the following initial conditions:

**a.** $x_0 = 0.1$ m, $\dot{x}_0 = 0$
**b.** $x_0 = 0$, $\dot{x}_0 = 10$ m/s
**c.** $x_0 = 0.1$ m, $\dot{x}_0 = 10$ m/s

**3.8** A mass $m$ is suspended from a spring of stiffness 4000 N/m and is subjected to a harmonic force having an amplitude of 100 N and a frequency of 5 Hz. The amplitude of the forced motion of the mass is observed to be 20 mm. Find the value of $m$.

**3.26** Consider a spring-mass-damper system with $k = 4000$ N/m, $m = 10$ kg, and $c = 40$ N-s/m. Find the steady-state and total responses of the system under the harmonic force $F(t) = 200 \cos 10t$ N and the initial conditions $x_0 = 0.1$ m and $\dot{x}_0 = 0$.

**3.37** A video camera, of mass 2.0 kg, is mounted on the top of a bank building for surveillance. The video camera is fixed at one end of a tubular aluminum rod whose other end is fixed to the building as shown in Fig. 3.50. The wind-induced force acting on the video camera, $f(t)$, is found to be harmonic with $f(t) = 25 \cos 75.3984t$ N. Determine the cross-sectional dimensions of the aluminum tube if the maximum amplitude of vibration of the video camera is to be limited to 0.005 m.

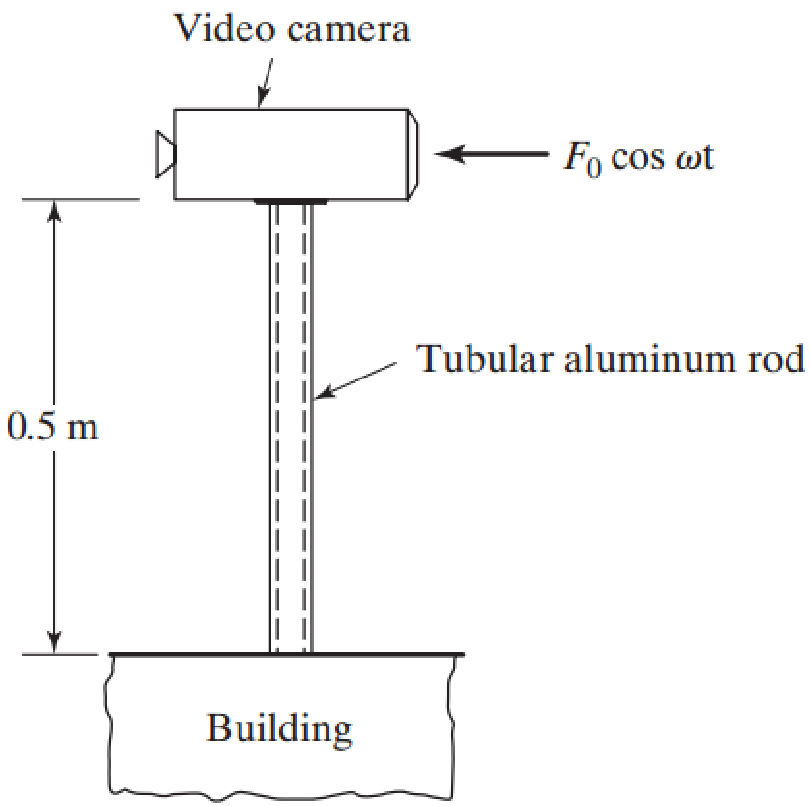


FIGURE 3.50

**3.44** The landing gear of an airplane can be idealized as the spring-mass-damper system shown in Fig. 3.52. If the runway surface is described $y(t) = y_0 \cos \omega t$, determine the values of $k$ and $c$ that limit the amplitude of vibration of the airplane ($x$) to 0.1 m. Assume $m = 2000$ kg, $y_0 = 0.2$ m, and $\omega = 157.08$ rad/s.

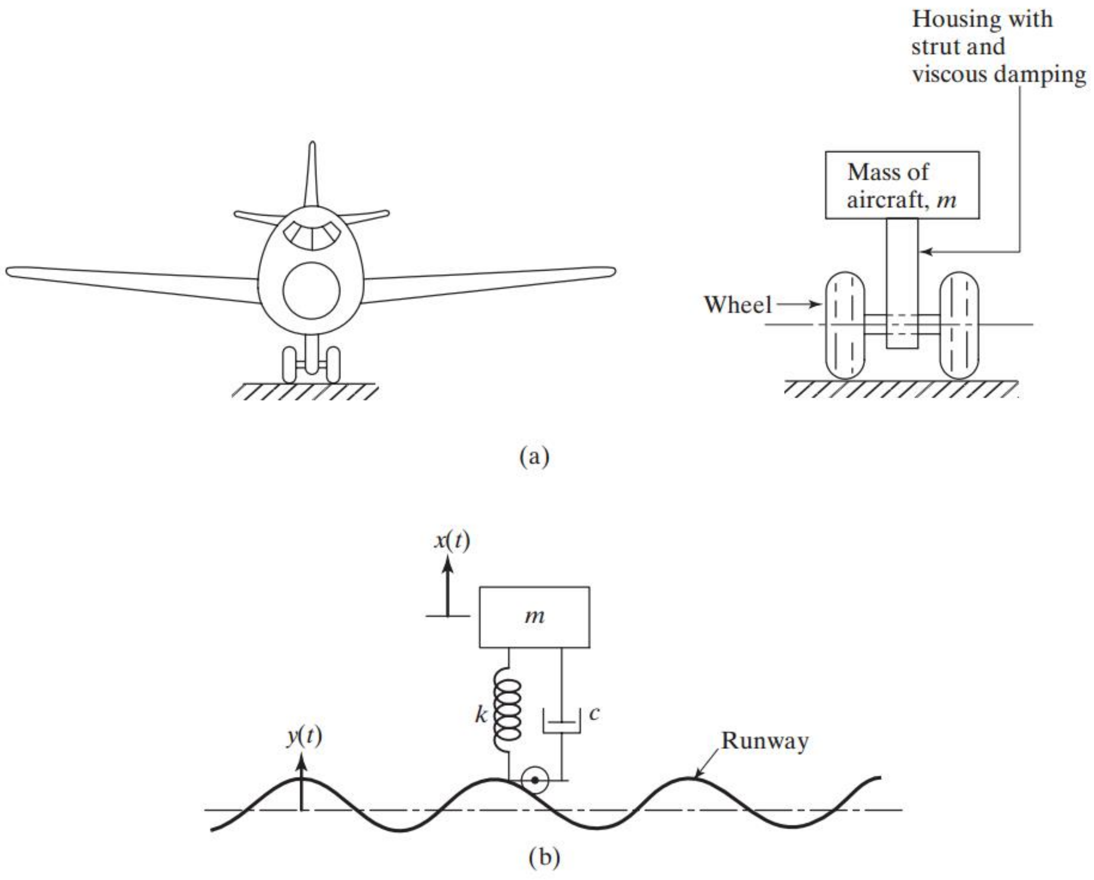


**FIGURE 3.52** Modeling of landing gear.