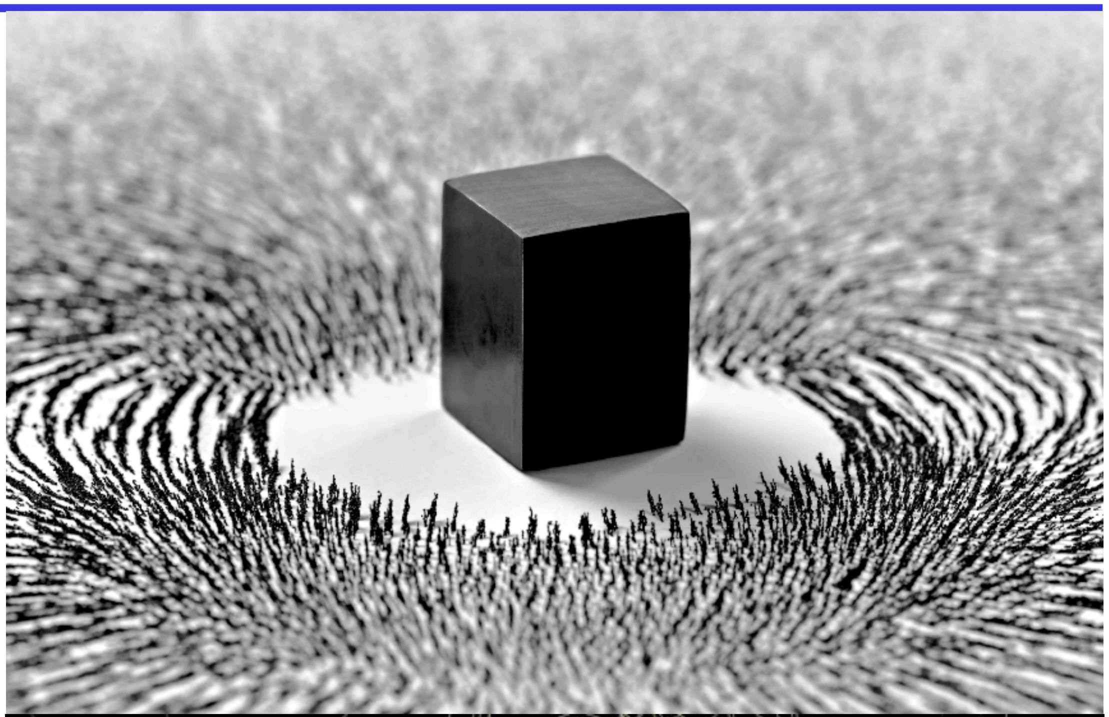
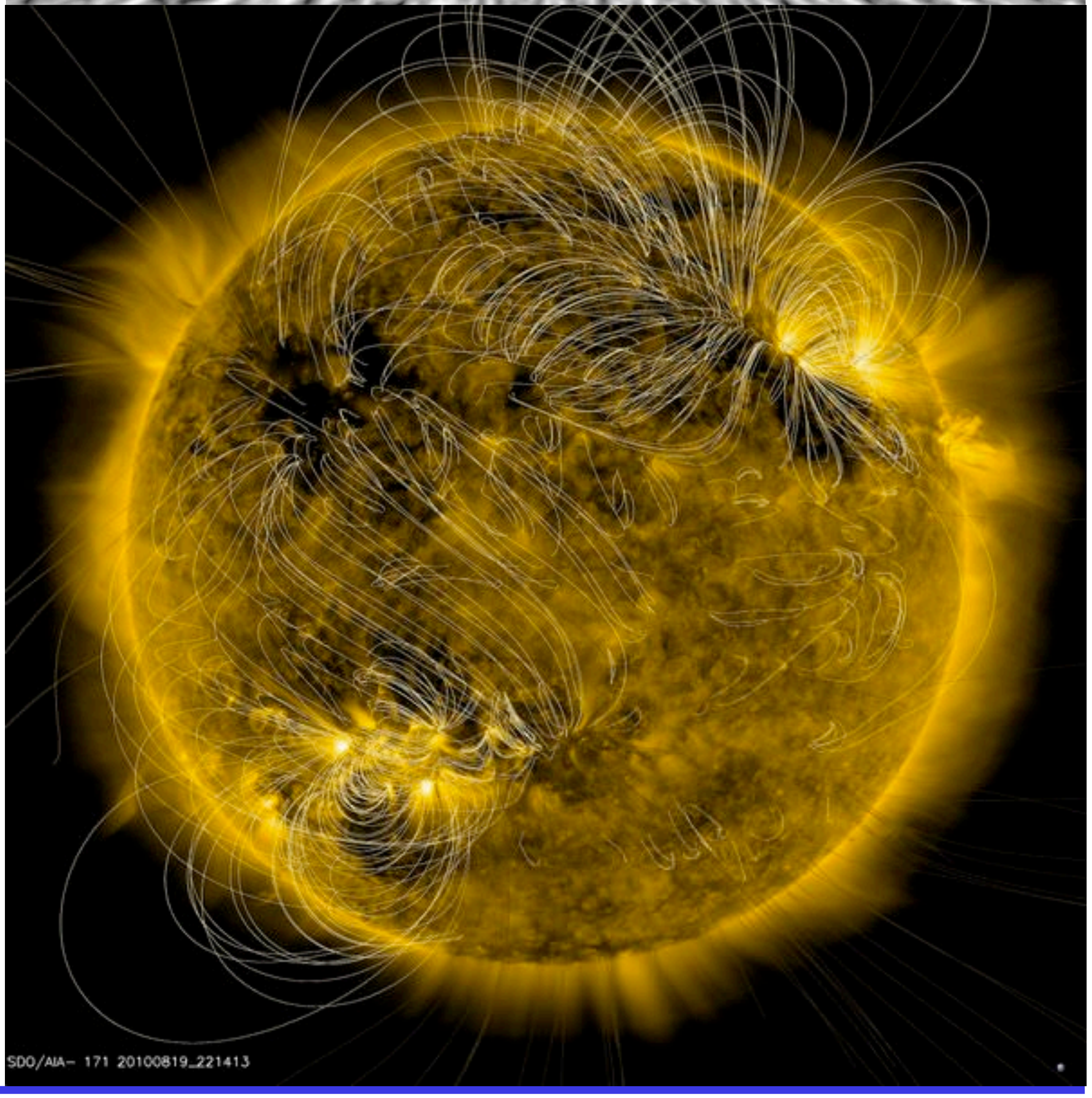
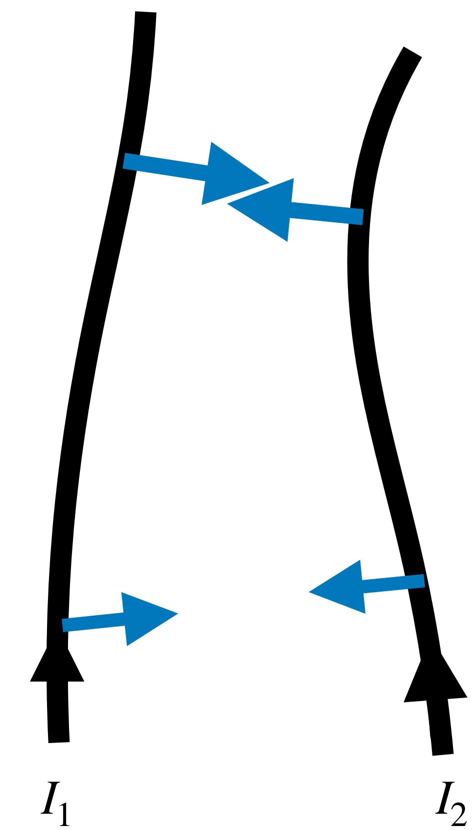
Magnetism

⚡ The magnetic “force"
⚡ The Ampère and Biot-Savart laws
⚡ The vector potential
⚡ Boundary conditions
⚡ Multipole expansion





The magnetic “force"



- When studying magnetism we never start by looking at magnets: their physics is actually quite complex!
- Instead, we look at wires that carry some current. It is observed that a pair of current-carrying wires that are placed near each other exert a mutual force that depends not only on the **intensity of the current** and the **orientation** of the wires, but also on how **far** are the two pieces of wire.
- Since the force is perpendicular to the direction of the current, it is inevitable that we use a **force law** and a notion of **field** that involves the **vector product.**
- The idea is that the **current**, the **field** and the **force** are all related through vector products:

$$I\,d\vec{l} \perp d\vec{B} \perp d\vec{F}$$

- We may start with the force, and try something like:

$$d\vec{F} = (I\,d\vec{l}) \times \vec{B} \quad ,$$

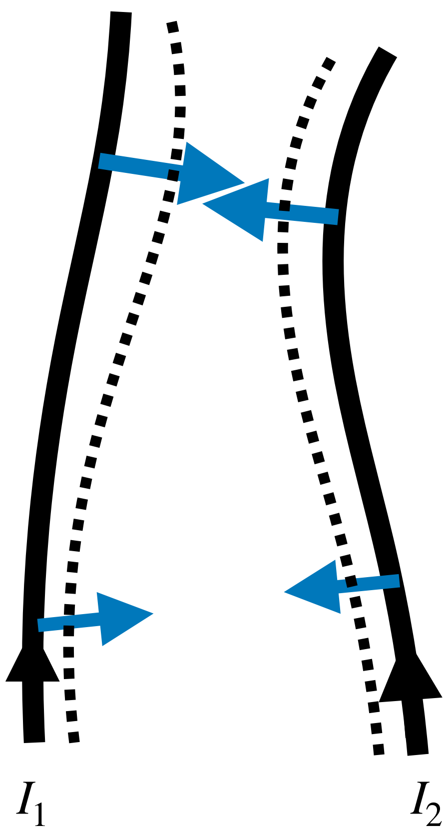
which we recognize as the magnetic part of the Lorentz force. For a general current density we should in fact write:

$$d\vec{F} = (\vec{J}\,dV) \times \vec{B}$$

- For a point charge this then reduces to:

$$\vec{F}_B = q\,\vec{v} \times \vec{B} \quad \Rightarrow \quad \vec{F}_L = q\,(\vec{E} + \vec{v} \times \vec{B})$$ which is, of course, just the Lorentz force.

The magnetic "force"



- From the start it is important to get something clear: the magnetic force never does actual **work**.
- This is self-evident from the Lorentz force:

$$\vec{F}_L = q(\vec{E} + \vec{v} \times \vec{B})$$

$$\Rightarrow \quad dW = \vec{F}_L \cdot d\vec{x} = q\vec{E} \cdot d\vec{x} + q\left(\frac{d\vec{x}}{dt} \times \vec{B}\right) \cdot d\vec{x} \; = \; q\vec{E} \cdot d\vec{x}$$

- Hence, although the magnetic field can exert a **torque**, it cannot do any actual **work**.
- Although we *know* this, it is easy to forget and to mindlessly blame the magnetic field for effects which are not its fault!
- Remember: every time that **work is being done** to/by a system, it is ultimately the **electric fields** doing the job. We will come back to this later.

Magnetostatics

- We will start with the simplest possible case of **stationary currents**. This means that the current density is such that **no charges accumulate** anywhere.
- In other words, from the continuity equation we have:

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Since the charge density is static, we have that:

$$\vec{\nabla} \cdot \vec{J} = 0$$

- This will be our working assumption for now — later we will relax this condition.
- Under the assumption of stationary currents, it was found first by Ampère and Biot-Savart that the magnetic field (the one in the Lorentz force!) is described by the laws:

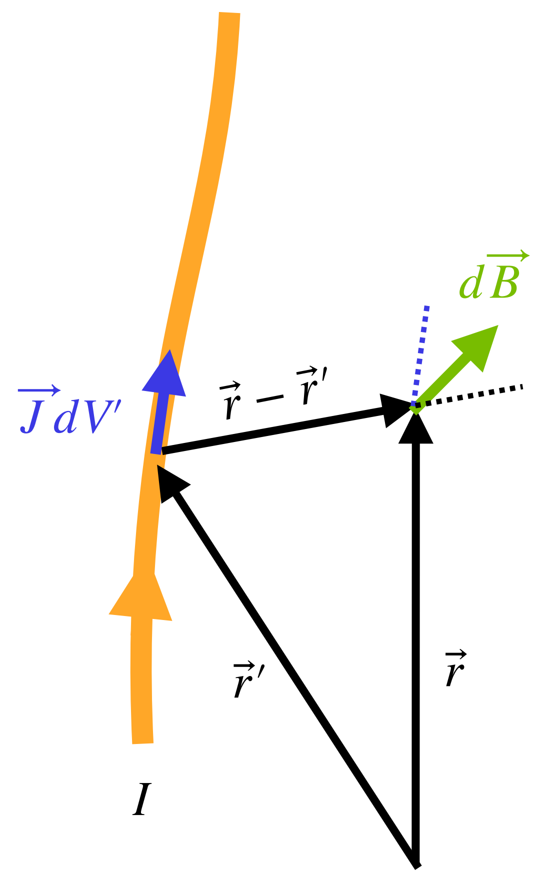
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$ [Ampère's law] , and

$$\vec{\nabla} \cdot \vec{B} = 0$$

Magnetostatics

- Historically, before Ampère (1826) there was a law derived by Biot-Savart (1820) that worked perfectly well — it just wasn't general enough.
- The expression from Biot-Savart is the following:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int I(\vec{r}\,')\,d\vec{l}\,' \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = \frac{\mu_0}{4\pi}\int dV'\,\vec{J}(\vec{r}\,') \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$



- In order to show that the Biot-Savart law follows from Ampère's law we will make use of the vector potential. Since

$\vec{\nabla}\cdot\vec{B} = 0$, we are able to write (remember the Helmholtz Theorem!):

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

- Substituting this into Ampère's law we get:

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J}$$

$$\Rightarrow \quad \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \nabla^2\vec{A} = \mu_0\vec{J}$$

Gauge invariance

- At this point we can make use of the fact that the physical fields ($\vec{E}$ and $\vec{B}$) are invariant under the ***gauge transformation***:

$$\vec{A} \to \vec{A} + \vec{\nabla} f \quad , \quad \text{and}$$

$$\phi \to \phi - \frac{\partial f}{\partial t}$$

- For now only the first part really matters — the gauge invariance for the vector potential. It means that we can choose any function f such that:

$$\vec{\nabla} \cdot \vec{A} \to \vec{\nabla} \cdot \left(\vec{A} + \vec{\nabla} f \right) = 0 \qquad \text{(this is known as } \textbf{Coulomb gauge}\text{)}$$

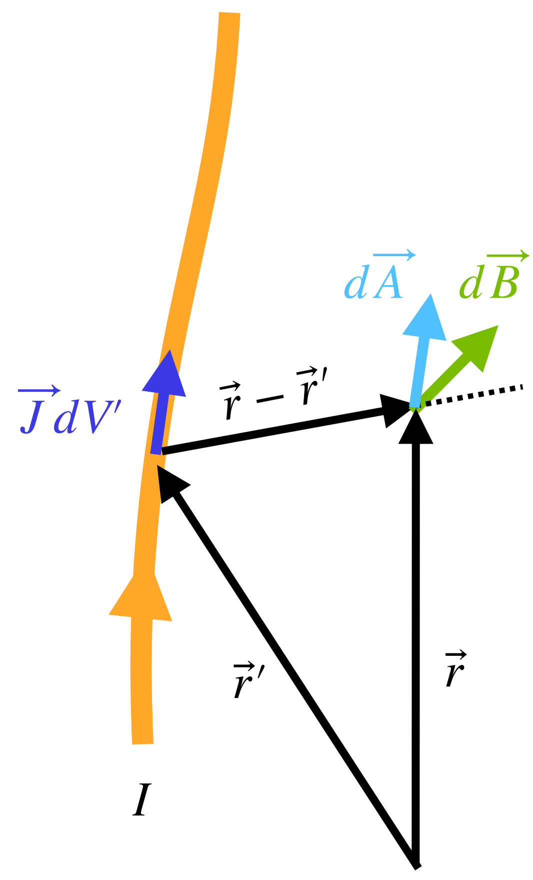
- This means that we can now look at a simple equation for the vector potential in terms of the current:

$$\nabla^2 \vec{A} = - \mu_0 \vec{J}$$

- In Cartesian coordinates the solution for each component, A_x, A_y and A_z is exactly the same as the one we found for the Poisson equation (and the Green function, etc. etc.):

$$\vec{A} = \frac{\mu_0}{4\pi} \int dV' \, \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \quad ,$$

where we are ignoring many subtleties related to boundary conditions. But you get the idea!

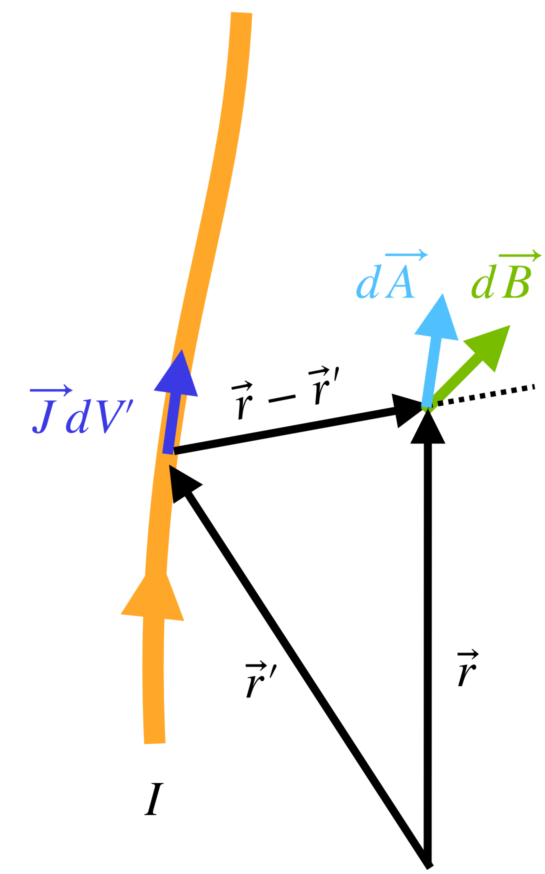


Gauge invariance

- Using the solution we obtained above it is now possible to compute the magnetic field:

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi} \int dV' \, \vec{\nabla} \times \left[\frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \right] \quad , \quad \text{which simplifies to:}$$

$$\vec{B} = \frac{\mu_0}{4\pi} \int dV' \, \vec{J}(\vec{r}\,') \times \left(\frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \right) \quad , \quad \text{i.e., the Biot-Savart law.}$$



- It is instructive to check explicitly that the field in this expression obeys our initial assumptions:

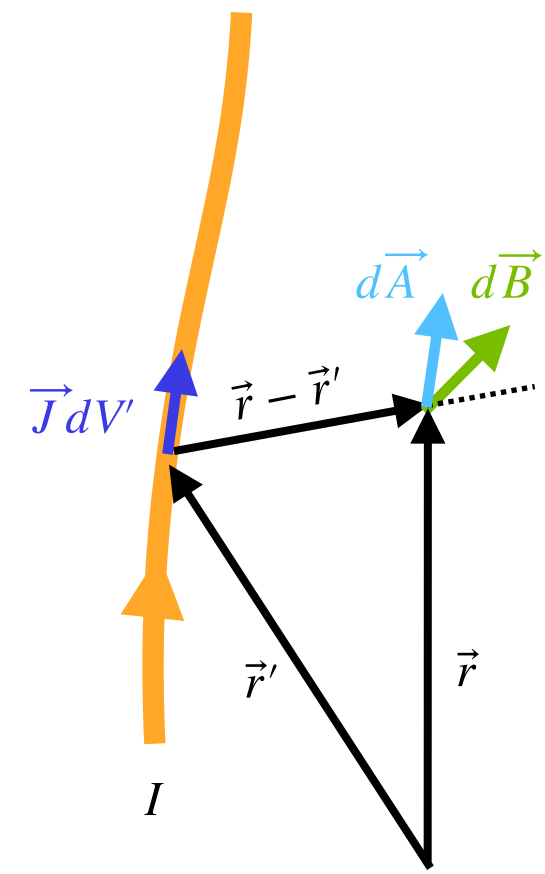
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \text{and} \quad \vec{\nabla} \cdot \vec{B} = 0$$

- Let's start with the second one, which is simpler. We have:

$$\vec{\nabla} \cdot \vec{B} = \frac{\mu_0}{4\pi} \int dV' \, \vec{\nabla} \cdot \left[\vec{J}(\vec{r}\,') \times \left(\frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \right) \right]$$

$$= \frac{\mu_0}{4\pi} \int dV' \left\{ \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \cdot \left[\vec{\nabla} \times \vec{J}(\vec{r}\,') \right] - \vec{J}(\vec{r}\,') \left[\vec{\nabla} \times \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \right] \right\} = 0$$

Gauge invariance



- Obtaining the second identity (Ampère's law) involves just a bit more algebra:

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int dV'\,\vec{\nabla}\times\left[\vec{J}(\vec{r}')\times\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)\right]$$

$$= \frac{\mu_0}{4\pi}\int dV'\left\{\vec{J}(\vec{r}')\left[\vec{\nabla}\cdot\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right]-\left[\vec{J}(\vec{r}')\cdot\vec{\nabla}\right]\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right\}$$

- The first term inside square brackets is an "old acquaintance":

$$\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right) = 4\pi\,\delta(\vec{r}-\vec{r}')$$

- The second term can be shown to become a surface term that integrates out to zero plus another term proportional to $\nabla\cdot\vec{J}=0$ (check!), with the expected result that:

$$\vec{\nabla}\times\vec{B} = \mu_0\,\vec{J}$$

Boundary conditions for magnetostatics

- We should now go back and try to be a bit more careful about the **boundary conditions** for the magnetic field and the vector potential.
- We can first select any **volume** that includes a section of the interface, and is just thick enough to include a section of the interface that runs parallel to its upper and lower sides. We then have:

$$0 = \int dV\, \vec{\nabla} \cdot \vec{B} = \oint d\vec{S} \cdot \vec{B}$$

- Neglecting the field on the sides of the box (small areas!) we get:

$$\boxed{\Delta B_\perp = 0}$$

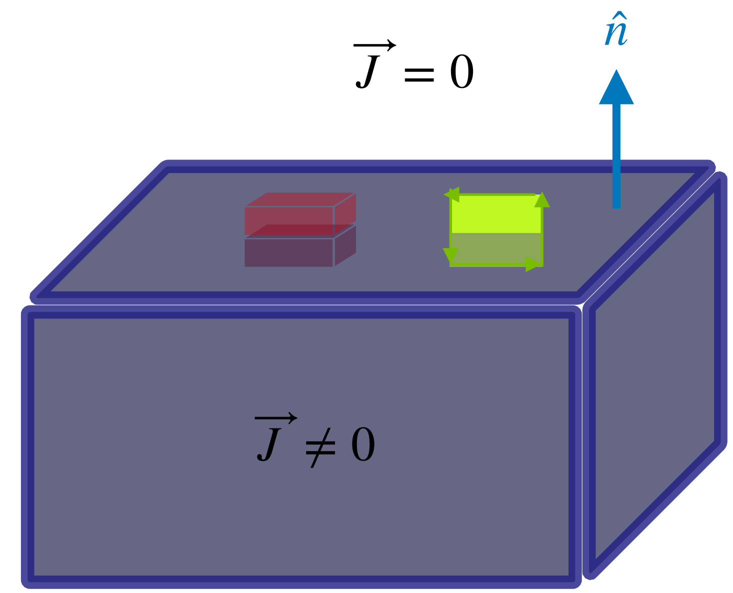
- If we now integrate the field along a circuit that crosses that same interface we get that:

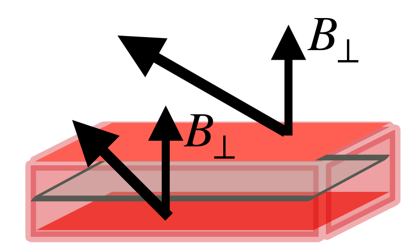
$$\oint d\vec{l} \cdot \vec{B} = \int d\vec{S} \cdot \left(\vec{\nabla} \times \vec{B}\right) = \int d\vec{S} \cdot \left(\mu_0 \vec{J}\right)$$

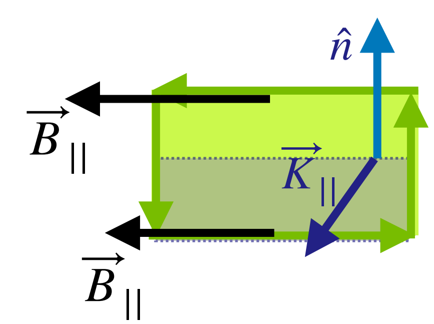
- Computing the left-hand side explicitly for circuits such as this, and defining the **surface currents**:

$\vec{K}_{||} = \int dS\, \vec{J}_{||}$ (N.B.: oriented *perpendicular* to the sides of the circuit), we obtain:

$\boxed{\Delta \vec{B}_{||} = \mu_0 \vec{K}_{||} \times \hat{n}}$, where $\hat{n}$ is the normal to the interface (not to the surface of the circuit!)







Boundary conditions for magnetostatics

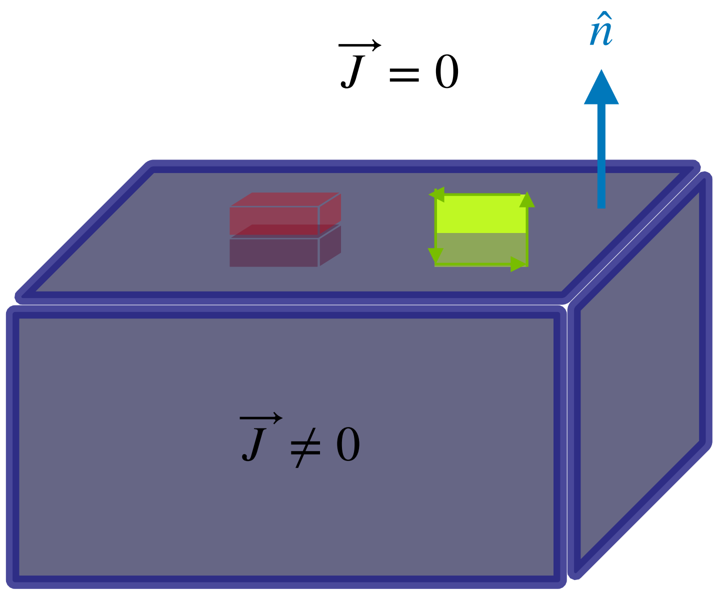
- The previous discussion implies that the magnetic field obeys the boundary conditions:

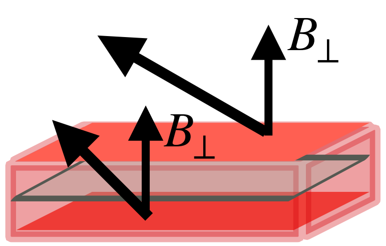
$$\Delta B_\perp = 0 \quad , \quad \text{and} \quad \Delta \vec{B}_{||} = \mu_0 \vec{K}_{||} \times \hat{n}$$

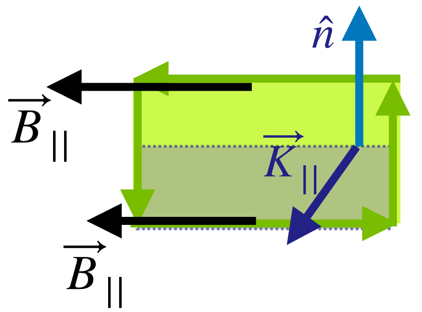
- This can be translated into boundary conditions for the vector potential:

$$\Delta \vec{A} = 0 \quad , \quad \text{and} \quad \Delta \left(\frac{\partial \vec{A}}{\partial n} \right) = - \mu_0 \vec{K} \quad ,$$

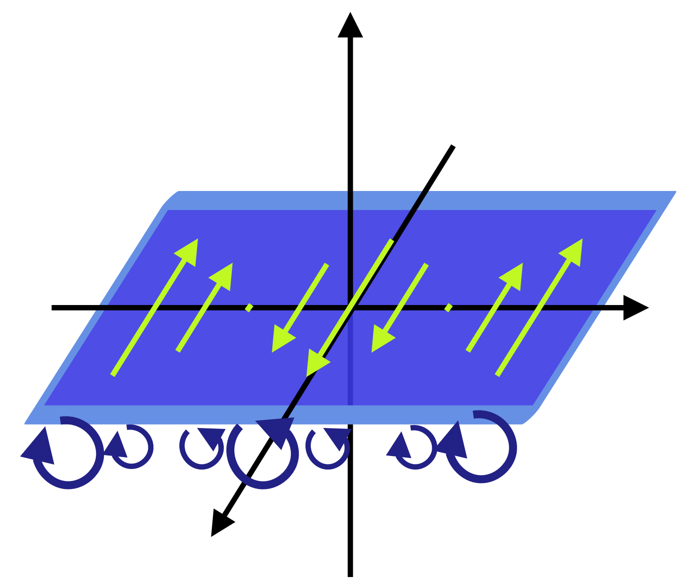
where n is the direction normal to the interface.







Boundary conditions: example



- Consider a surface current in the $z = 0$ plane that is given by:

$$\vec{K} = K_0 \, \hat{x} \, \cos ky \quad , \quad \text{which means the current is:}$$

$$\vec{J} = K_0 \, \hat{x} \, \cos ky \, \delta(z)$$

- We want to solve for the Laplace equation, $\nabla^2 \vec{A} = 0$. In full analogy to what was done for the case of the electric potential, we try a solution of type:

$$\vec{A} = \vec{A}_0 \, Y(y) \, Z(z) \quad ,$$

where we have no dependence on the direction x due to the symmetry of this problem.

- The Laplace equation for this ansatz leads to a familiar equation:

$$\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \quad \Rightarrow \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} = -\alpha^2$$

which leads to the solutions:

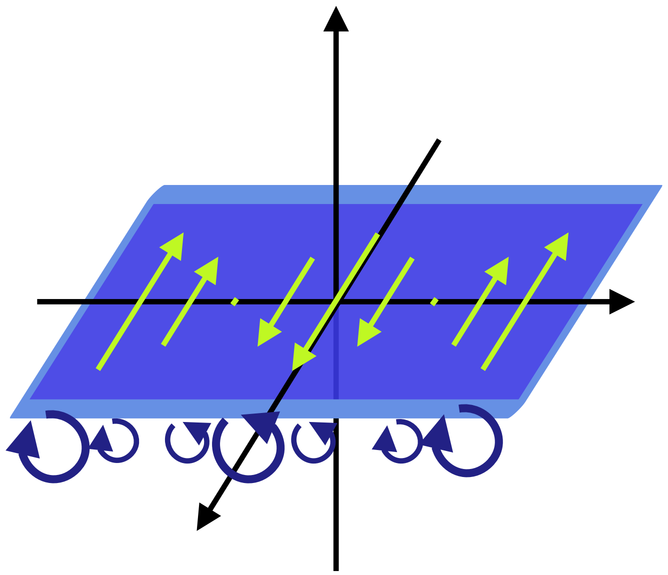
$$Y = \{\sin \alpha y \, , \, \cos \alpha y\} \quad \text{and} \quad Z = \{e^{\alpha z} \, , e^{-\alpha z}\}$$

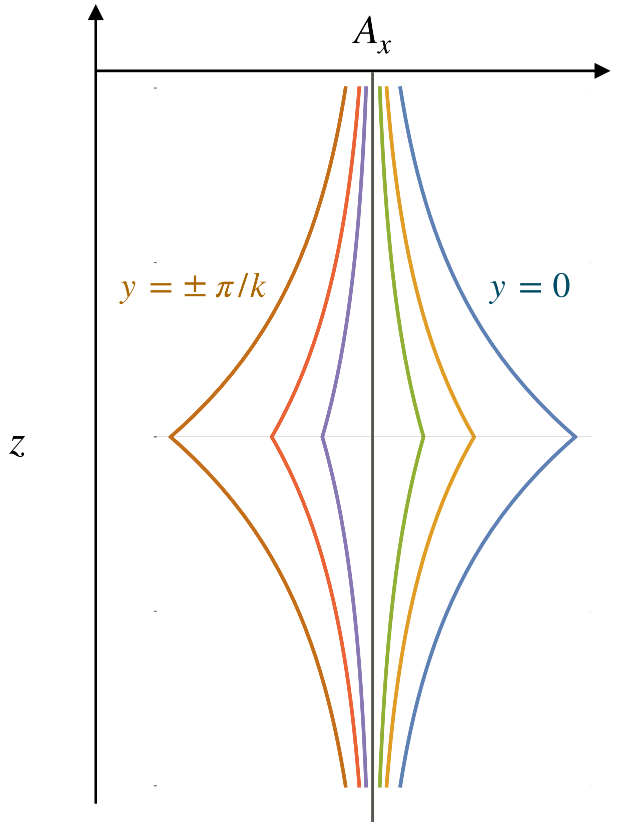
Boundary conditions: example

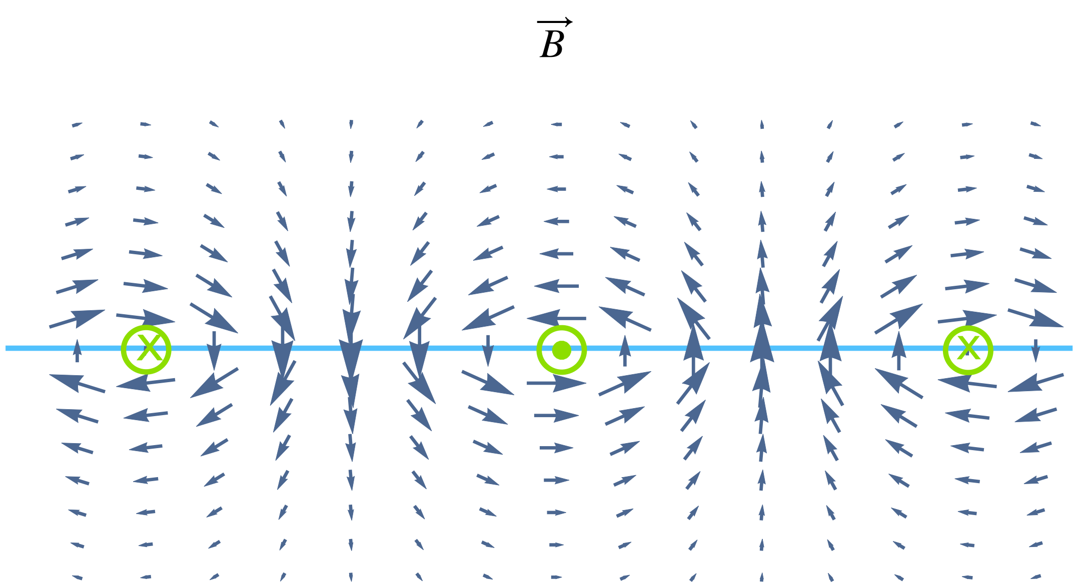
- Now we impose the boundary conditions: $\vec{A}$ is continuous on $z = 0$, and $\Delta(d\vec{A}/dz)_{z=0} = -\mu_0 \vec{K}$.
- It is immediately clear that $\alpha \to k$.
- By construction $\vec{A}$ is continuous, and the second condition leads to:

$$\vec{A} = \frac{\mu_0 K_0}{2k} \hat{x} \cos ky \, e^{\mp kz} \quad , \qquad \text{which leads to}$$

$$\vec{B} = \frac{\mu_0 K_0}{2} \left[\mp \hat{y} \cos ky \, e^{\mp kz} + \hat{z} \sin ky \, e^{\mp kz} \right]$$







Multipole expansion of the vector potential

- Let's now get back to our explicit solution for the vector potential in terms of the current density:

$$\vec{A} = \frac{\mu_0}{4\pi}\int dV' \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$ (where, remember, we used the Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$!)

- Recall that the term $1/|\vec{r}-\vec{r}\,'|$ can be expanded in terms of the multipole expansion:

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\hat{n}\cdot\hat{n}') \quad ,$$ or even better:

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell,m} \frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}} Y_\ell^{(m)}(\hat{n})\, Y_\ell^{(m)*}(\hat{n}')$$

- Substituting in the expression for the vector potential we get:

$$\vec{A} = \frac{\mu_0}{4\pi}\int dV'\, \vec{J}(\vec{r}\,') \sum_{\ell,m} \frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}} Y_\ell^{(m)}(\hat{n})\, Y_\ell^{(m)*}(\hat{n}')$$

- Assuming for the moment that we have an inner distribution of currents and we want to compute the vector potential in the exterior region, we get:

$$\vec{A} = \frac{\mu_0}{4\pi}\sum_{\ell,m} \frac{4\pi}{2\ell+1} r^{-\ell-1}\, Y_\ell^{(m)}(\hat{n})\, \vec{J}_\ell^{(m)} \quad ,$$ where

$$\vec{J}_\ell^{(m)} = \int dV'\, r'^{\ell}\, Y_\ell^{(m)*}(\hat{n}')\, \vec{J}(\vec{r}\,')$$ are the **multipoles** of the **current density**.

Multipole expansion of the vector potential

- There are some subtleties with the multipole expansion of a vector field that we will discuss later. But let's go with what we have for now, and consider the multipoles of the current density as they were defined above:

$$\vec{J}_{\ell}^{(m)} = \int dV' \, r'^{\ell} \, Y_{\ell}^{(m)*}(\hat{n}') \, \vec{J}(\vec{r}\,')$$

- Let's look at the very first term that appears here: the **monopole** ($\ell = 0$). Since $Y_0^{(0)} = 1/\sqrt{4\pi}$ we have:

$$\vec{J}_{0}^{(0)} = \int dV' \, \frac{1}{\sqrt{4\pi}} \, \vec{J}(\vec{r}\,')$$

- Now, this is obviously zero. You *know* it's zero, right?… But how do you go about **proving** this?
- Think about it: the current integrated inside a volume should only give you a non-zero result if charges are **coming in** or **out** of that volume. In other words, the result is linked to the time variation of the charge — and if all the charges and currents are well inside that volume, then there is no current exiting that volume.
- So, let's ask for the help of the equation that relates the spatial variation of the current with the time variation of the charge — the continuity equation:

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

- Therefore, if the variation of the charge inside a volume is:

$$\frac{\partial Q_V}{\partial t} = \frac{\partial}{\partial t} \int_V dV \, \rho \;\to\; 0$$ as long as we specify that all charges remain inside that volume.

Multipole expansion of the vector potential

- Now, let's go back to our integral and write each component of it as:

$$\int_V dV\, J^i \;=\; \int_V dV \,\sum_j J^j \frac{\partial x^i}{\partial x^j} \;=\; \int_V dV \,\sum_j \left[\frac{\partial}{\partial x^j}\left(x^i J^j\right) - x^i \frac{\partial J^j}{\partial x^j} \right]$$

- The last term is the divergence of $\vec{J}$, which is zero due to our previous assumption that no charges are leaving that volume. Only the first term remain, but for that we can use the Divergence theorem to write:

$$\int_V dV\, J^i \;=\; \int_V dV \,\sum_j \frac{\partial}{\partial x^j}\left(x^i J^j\right) \;=\; \sum_j \oint_{S(V)} dS_j \, x^i \, J^j$$

- Taking a volume that includes all charges and currents means that, on the surface that encloses that volume, the currents vanish, so all components above must vanish. Therefore, *both terms vanish* and we have that the **monopole vanishes**:

$$\int_V dV\, J^i \;=\; 0$$

- Basically the same argument can be used to show that:

$$\int_V dV \,\left(r^i J^j - r^j J^i\right) \;=\; 0 \qquad ,$$ which we will use in a moment.

Multipole expansion of the vector potential

- OK, so the monopole vanishes. But what about the dipole?

$$\vec{J}_1^{(m)} = \int dV' \, r' \, Y_1^{(m)*}(\hat{n}') \, \vec{J}(\vec{r}') \quad , \quad \text{and the vector potential for the dipole term is:}$$

$$\vec{A}_1 = \frac{\mu_0}{4\pi} \frac{1}{r^2} \sum_{m=-1}^{1} \frac{4\pi}{2 \times 1 + 1} Y_1^{(m)}(\hat{n}) \, \vec{J}_1^{(m)} = \frac{\mu_0}{4\pi} \frac{4\pi}{3} \frac{1}{r^3} \int dV' \, r \, r' \, \vec{J}(\vec{r}') \sum_{m=-1}^{1} Y_1^{(m)}(\hat{n}) \, Y_1^{(m)*}(\hat{n}')$$

$$= \frac{\mu_0}{4\pi} \frac{4\pi}{3} \frac{1}{r^3} \int dV' \, r \, r' \, \vec{J}(\vec{r}') \frac{3}{4\pi} P_1(\hat{n} \cdot \hat{n}')$$

$$= \frac{\mu_0}{4\pi} \frac{1}{r^3} \int dV' \, \left[(r\,\hat{n}) \cdot (r' \hat{n}') \right] \, \vec{J}(\vec{r}') = \frac{\mu_0}{4\pi} \frac{1}{r^3} \int dV' \, [\vec{r} \cdot \vec{r}'] \, \vec{J}(\vec{r}')$$

- This can be written in terms of the spatial components as:

$$A_1^i = \frac{\mu_0}{4\pi} \frac{1}{r^3} \sum_j r^j \int dV' \, r'^j \, J^i(\vec{r}') = \frac{\mu_0}{4\pi} \frac{1}{r^3} \sum_j r^j \int dV' \, \frac{1}{2} \left(r'^j J^i - r'^i J^j \right) \quad , \quad \text{where we use the identity in the previous slide.}$$

- This means that we can write the dipole of the vector potential as a vector product:

$$\vec{A}_1(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3} \quad , \quad \text{where} \quad \vec{m} = \frac{1}{2} \int dV' \, \vec{r}' \times \vec{J}(\vec{r}') \quad \text{is the magnetic dipole moment of the current density } \vec{J} \, .$$

Multipole expansion of the vector potential

- It is interesting to compare the electric and magnetic dipoles.
- For the **electric dipole** we have:

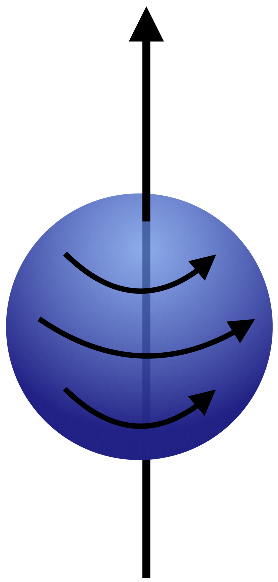
$$\phi_1(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} \quad , \quad \text{with} \quad \vec{p} = \int dV'\, \vec{r}'\rho(\vec{r}')$$

- While for the **magnetic dipole** we have, in almost perfect analogy:

$$\vec{A}_1(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3} \quad , \quad \text{with} \quad \vec{m} = \frac{1}{2}\int dV'\, \vec{r}'\times\vec{J}(\vec{r}')$$

- Notice, however, that the electric potential/field have a monopole, but the magnetic potential/field do not. This is, of course, due ultimately to the fact that there are electric monopoles (the charges), while magnetic monopoles do not exist in nature.

Multipole expansion: example



- As an example of an application of the notions we developed so far, let's look into a simple situation: a **charged sphere** (surface charge density σ_0 and radius R_0) that **rotates** with a constant angular velocity ω_0.
- The surface charge density and the current density at the surface of the sphere are easy to derive:

$$\rho = \sigma_0 \, \delta(r - R_0) \quad , \quad \text{and} \quad \vec{J} = \sigma_0 \omega_0 R_0 \, \sin\theta \, \delta(r - R_0) \, \hat{\varphi}$$

(You can easily check that this obeys the continuity equation!)

- The **multipoles** of the current density for this rotating charge sphere are:

$$\vec{J}_\ell^{(m)} = \int dV' \, r'^\ell \, Y_\ell^{(m)*}(\hat{n}') \, \vec{J}(\vec{r}\,') = \int dV' \, r'^\ell \, Y_\ell^{(m)*}(\hat{n}') \, \left[\sigma_0 \omega_0 R_0 \, \delta(r' - R_0) \, \sin\theta' \, \hat{\varphi}'\right]$$

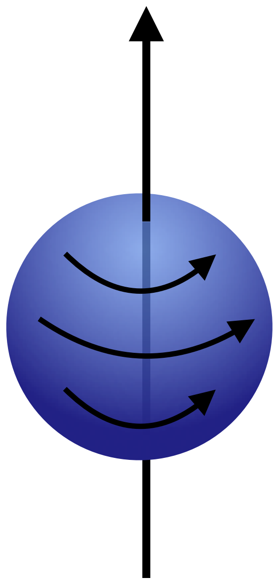
- Notice that the unit vector $\hat{\varphi}'$ is not constant as we move around the surface of the sphere — in fact, it rotates as:

$$\hat{\varphi}' = -\sin\varphi' \, \hat{x} + \cos\varphi' \, \hat{y} = \left(\frac{e^{-i\varphi'} - e^{i\varphi'}}{2i}\right)\hat{x} + \left(\frac{e^{-i\varphi'} + e^{i\varphi'}}{2}\right)\hat{y}$$

- Using $dV' = r'^2 dr' \, d(\cos\theta') \, d\varphi'$ and integrating the delta function, the multipoles become:

$$\vec{J}_\ell^{(m)} = \sigma_0 \omega_0 R_0^{\ell+3} \int d(\cos\theta') \int d\varphi' \, Y_\ell^{(m)*}(\theta', \varphi') \, \sin\theta' \left[\left(\frac{e^{-i\varphi'} - e^{i\varphi'}}{2i}\right)\hat{x} + \left(\frac{e^{-i\varphi'} + e^{i\varphi'}}{2}\right)\hat{y}\right]$$

Multipole expansion: example



- Now look at the integral over $d\varphi'$. The terms look like:

$$\int_0^{2\pi} d\varphi'\, Y_\ell^{(m)*}(\theta',\varphi')\, e^{\pm i\varphi'} \sim \int_0^{2\pi} d\varphi'\, e^{-im\varphi'}\, e^{\pm i\varphi'} \quad ,$$

hence the only way this integral does not vanish is if $m = \mp 1$.

- Now, consider the integral over $d(\cos\theta')$:

$$\int_{-1}^{1} d(\cos\theta')\, P_\ell^{(m)}(\cos\theta')\, \sin\theta'$$

- Remember the orthogonality condition for the associated Legendre functions ($\mu = \cos\theta$):

$$\int_{-1}^{1} d\mu\, P_\ell^{(m)}(\mu)\, P_{\ell'}^{(m)}(\mu) = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'} \quad ,$$ where recall also that:

$$P_\ell^{(m)}(\mu) = (-1)^m (1-\mu^2)^{m/2}\frac{d^m}{d\mu^m}P_\ell(\mu) \quad (m \geq 0) \quad \text{and} \quad P_\ell^{(-m)}(\mu) = (-1)^m \sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^{(m)}$$

- In particular, notice that:

$$P_0^{(0)}(\mu) = 1 \quad ,$$

$$P_1^{(0)}(\mu) = P_1(\mu) = \mu = \cos\theta \quad , \quad \text{and}$$

$$P_1^{(1)}(\mu) = (-1)(1-\mu^2)^{1/2}\frac{d}{d\mu}P_1(\mu) = -\sin\theta \quad , \quad \text{and} \quad P_1^{(-1)}(\mu) = +\sin\theta$$

Multipole expansion: example

- This discussion implies that we only have the terms $\ell = 1\,, m = \pm 1$:

$$\vec{J}_1^{(m)} = \sigma_0 \omega_0 R_0^4 \int d(\cos\theta') \int d\varphi'\, Y_1^{(m)*}(\theta',\varphi')\, \sin\theta' \left[\left(\frac{e^{-i\varphi'} - e^{i\varphi'}}{2i} \right) \hat{x} + \left(\frac{e^{-i\varphi'} + e^{i\varphi'}}{2} \right) \hat{y} \right]$$

$$= \sigma_0 \omega_0 R_0^4 \int d(\cos\theta') \int d\varphi'\, (-1)^{\frac{m+1}{2}} \frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin\theta'\, e^{-im\varphi} \times \sin\theta' \left[\left(\frac{e^{-i\varphi'} - e^{i\varphi'}}{2i} \right) \hat{x} + \left(\frac{e^{-i\varphi'} + e^{i\varphi'}}{2} \right) \hat{y} \right]$$

- The integral over $d(\cos\theta')$ gives:

$$\int_{-1}^{1} d(\cos\theta')\, \sin^2\theta' = \int_{-1}^{1} d\mu'\, (1 - \mu'^2) = 2 - \frac{2}{3} = \frac{4}{3}$$

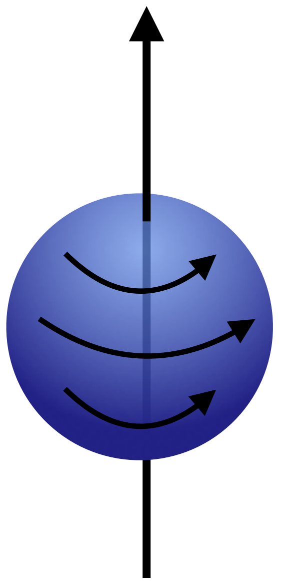
- Hence, we find that:

$$J_1^{(1)} = -\, \sigma_0 \omega_0 R_0^4 \sqrt{\frac{3}{2\pi}} \frac{4\pi}{3} \left(-\frac{1}{2i} \hat{x} + \frac{1}{2} \hat{y} \right) \quad , \quad \text{and}$$

$$J_1^{(-1)} = +\, \sigma_0 \omega_0 R_0^4 \sqrt{\frac{3}{2\pi}} \frac{4\pi}{3} \left(+\frac{1}{2i} \hat{x} + \frac{1}{2} \hat{y} \right)$$

- Substituting this into the expression for the vector potential we have:

$$\vec{A} = \frac{\mu_0}{4\pi} \sum_{\ell,m} \frac{4\pi}{2\ell + 1} r^{-\ell-1} Y_\ell^{(m)}(\hat{n})\, \vec{J}_\ell^{(m)} = \frac{\mu_0}{4\pi} \frac{4\pi}{3} \frac{1}{r^2} \left[Y_1^{(1)}\, \vec{J}_1^{(1)} + Y_1^{(-1)}\, \vec{J}_1^{(-1)} \right]$$



Multipole expansion: example

- Plugging the expressions for $J_1^{(\pm 1)}$ into the vector potential we obtain:

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{4\pi}{3}\frac{1}{r^2}\,\sigma_0\omega_0 R_0^4\,\sqrt{\frac{3}{2\pi}}\frac{4\pi}{3}\left[\frac{-1}{2}\sqrt{\frac{3}{2\pi}}\,\sin\theta\, e^{i\varphi}\left(\frac{\hat{x}}{2i}-\frac{\hat{y}}{2}\right) + \frac{1}{2}\sqrt{\frac{3}{2\pi}}\,\sin\theta\, e^{-i\varphi}\left(\frac{\hat{x}}{2i}+\frac{\hat{y}}{2}\right)\right]$$

- Simplifying all that we get that:

$$\vec{A} = \frac{\mu_0}{3}\frac{1}{r^2}\,\sigma_0\omega_0 R_0^4\,\sin\theta\left[e^{i\varphi}\left(-\frac{\hat{x}}{2i}+\frac{\hat{y}}{2}\right) + e^{-i\varphi}\left(+\frac{\hat{x}}{2i}+\frac{\hat{y}}{2}\right)\right]$$

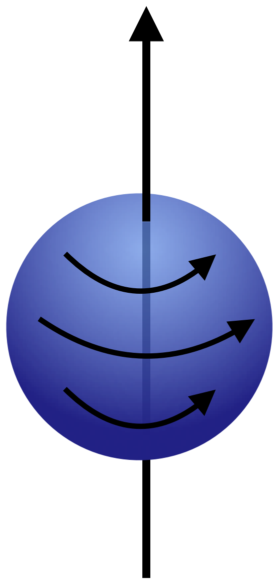
$$= \frac{\mu_0}{3}\frac{1}{r^2}\,\sigma_0\omega_0 R_0^4\,\sin\theta\,\left[-\sin\varphi\,\hat{x}+\cos\varphi\,\hat{y}\right]$$

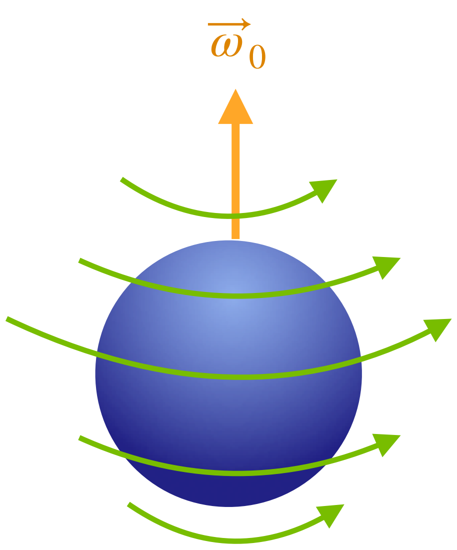
$$= \frac{\mu_0}{3}\,\sigma_0\omega_0 R_0^4\,\frac{1}{r^2}\,\sin\theta\,\hat{\varphi}$$

- The solution above is for the region outside the sphere; in the interior it is completely analogous, with:

$$\vec{A}_< = \frac{\mu_0}{3}\,\sigma_0\omega_0 R_0\, r\,\sin\theta\,\hat{\varphi}$$, which you can use to check that:

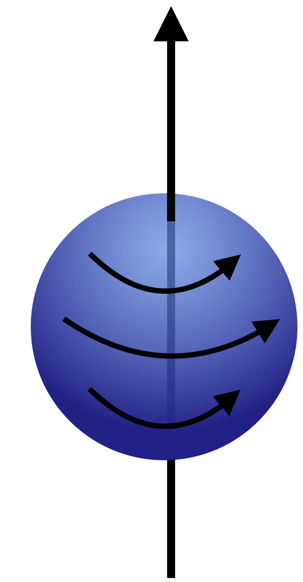
$$\Delta(\partial\vec{A}/\partial r)_{r=R_0} = -\,\mu_0\,\vec{K}\,.$$





Multipole expansion: example

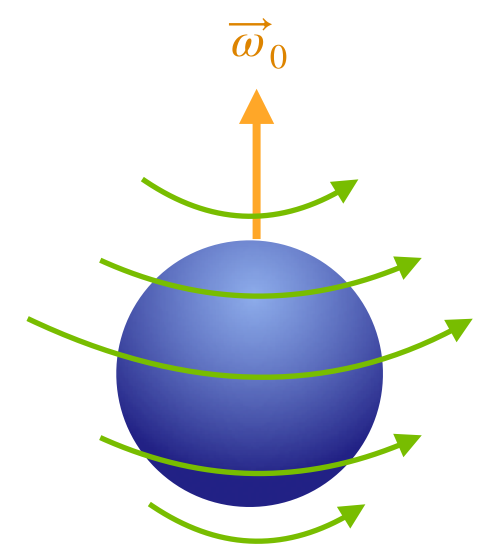
- Now, I obviously took you out for a long walk "around" this problem: we could have used the axial symmetry from the outset, then realized that it was only a problem with a dipole $(\ell = 1)$, and skipped the long calculations with the spherical harmonics!



- In fact, it is easy to see that the magnetic dipole of this sphere is:

$$\vec{m} = \frac{4\pi}{3}\sigma_0\omega_0 R_0^4\,\hat{z} = \frac{4\pi}{3}\sigma_0 R_0^4\,\vec{\omega}_0$$, such that:

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}$$ (outside the sphere).



- But I wanted to show that, in this simple problem, you *can* work with the spherical harmonics — no problem! And for future problems, you will *have to*!

Next class:

- Magnetostatics: examples and applications
- Magnetic dipoles — and quadrupoles!
- Magnetic force and torque
- Jackson, Ch. 5