UNIVERSITY OF SOUTHAMPTON

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Analysis of Mathematical Discourse: Multiple Perspectives

by

Seiji Hariki

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ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

Doctor of Philosophy

ANALYSIS OF MATHEMATICAL DISCOURSE: MULTIPLE PERSPECTIVES

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Textbooks have a powerful influence on mathematical education in developing countries like Brazil, where many poorly prepared teachers use them as the only source of information and activities for their teaching. In this context the investigation of the discourse conveyed by mathematics textbooks is very important.

As my interest at the present moment is on mathematics education for the undergraduate level, I have chosen the discourse of undergraduate mathematics textbooks as the theme of investigation, giving particular attention to the discourse of Complex Analysis textbooks.

In the first two parts of this thesis, I develop the theoretical frameworks necessary to make discourse analysis into an effective educational tool. The philosophical perspectives and analytical tools developed there are expected to be helpful for the discursive investigation of textbooks on other areas of mathematics and other levels of teaching as well.

In the third part, I show that, contrary to common expectation, the discourse of Complex Analysis is neither standardized, nor objectively neutral. By applying the theoretical tools developed in the first two parts of this thesis, I demonstrate that this discourse is, globally and locally, permeated by frequent conflicts between logical, heuristic, and rhetorical schemes.

The main aim of the discourse analysis I have developed is to give awareness to the teachers that in mathematical discourse heuristic, intuition and rhetoric are necessary complements of logic, and, consequently, mathematical discourse cannot be reduced to logical reasoning at all.

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PART II NEGOTIATION OF MATHEMATICAL KNOWLEDGE

To Teluko and Mariana

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INTRODUCTION

The main aim of this thesis is to develop and apply theoretical frameworks for the analysis of the discourse conveyed by undergraduate mathematics textbooks.

Mathematical discourse seems not to have been given a great deal of attention in mathematics education; few books ([Pimm 87], [Davis & Hersh 88]) and few papers ([Fauvel 88], [Fauvel 89], [McBride 89]) deal with this theme.

David Pimm in his book **Speaking Mathematically: Communications in Mathematics Classrooms** (1987) has focused on spoken communication in mathematics classrooms, that is, on classroom discourse. He has also treated some features of the mathematical writing system, however his overall approach is delimited by the metaphor he wants to explore: mathematics as a language. In my view, if metaphor is to be used, mathematics is a discourse rather than a language.

Philip J. Davis and Reuben Hersh wrote in their book **Descartes' Dream** (1988) an interesting section on the relation between mathematics and rhetoric. They argue that rhetoric is present in mathematical papers and treatises:

> ... within the practise of mathematics itself, among the professional mathematicians, continual and essential use is made of rhetorical modes of argument and persuasion in addition to purely formal or logical procedures.
>
> [Davis & Hersh 88, p. 58]

This suggests that authors of undergraduate mathematics textbooks, with greater reason, will resort to rhetorical devices as well.

John Fauvel's perspective is historical: he has made rhetorical criticism of ancient mathematical texts and textbooks, analysing texts written by Euclid, Descartes and Robert Recorde.

Maggie McBride is concerned with a feministic critique of mathematical discourse; she has denounced the domination of mathematical discourse by the 'masculine' point of view:

> Within the four discursive practices I have described [textbooks, teaching methods, examination process, and use of space in classroom], masculine gender biases clearly predominate. Students and teachers usually are not conscious of these tendencies and many times cannot articulate gender as an issue and its connection to power relations.
>
> [McBride 89, p. 45]

The understanding of the discourse of mathematics textbooks is of paramount importance in the context of developing countries like Brazil, where I am from.

In Brazil, textbooks, for several reasons (government educational policy, influence of the cultural industry, poor education and training of teachers, teachers' conformism, etc.), have a central role at all levels of mathematics education. Textbooks are at the heart of mathematics instruction, moulding practically all aspects of teaching and learning of mathematics, in schools and at universities.

Just to have a picture of the 'industrial' dimension of textbooks, it should be known that nearly 240 million books are published each year in Brazil; half of them are school textbooks for a population of one million teachers and 30 million pupils [Freitag et al 89, p. 62]. For the sake of numerical comparison, in the United Kingdom nearly 30 million school textbooks are sold each year, for a population of 400,000 teachers and 8 million pupils [Newton 90, p. 9].

Many Brazilian educationalists are concerned with the role of textbooks on the modelling of educational practices and their influence on teachers and pupils. For example, Bárbara Freitag and her colleagues say that

> School textbook does not function in the classroom just as a tool of teaching and transmission of knowledge, but as the standard model, the supreme authority, the ultimate criterion of truth.
>
> [Freitag et al 89, p. 111]

Researchers have shown that the majority of teachers of primary and secondary schools in Brazil are uncritical and conformist; they appreciate the adopted textbooks (all educationalists recognise that textbooks have a great amount of errors and are educationally and ideologically biased), and regard them as the vehicle of ultimate truth:

> The teacher is not only satisfied with the textbook she has but idealizes it, using it not as one among many pedagogic tools, but as the only tool. Textbook is considered as the definitive source of knowledge in the area, being treated in classroom as the vehicle of the absolute truth. Lack of information, self-indulgence, and conformism of the majority of teachers give strength to the 'omnipotence' and 'omniscience' of the textbook. Teachers proceed to respect the written word in the textbook considering it as the great authority, submitting themselves docilely to its psycho-pedagogic and ideological contents.
>
> [Freitag et al 89, p. 131]

Textbooks can become (in Brazil I believe they are) an instrument of cultural domination and imposition, even in the first world countries. I interpret the following passage of the Cockcroft report, **Mathematics Counts** (1982), as a warning against uncritical acceptance of textbooks:

> ... it is always necessary to use any textbook with discrimination, and selections should be made to suit the varying needs of different children ... It should not be expected that any textbook, however good, can provide a complete course or meet the needs of all children; additional activities of various kinds need to be provided.
>
> [Cockcroft 82, p. 91]

Therefore, it seems to me educationally relevant to undertake the analysis of the discourse of mathematics textbooks: how authors negotiate meanings, perspectives, and values with the readers.

As my interest is on undergraduate mathematics education, I have delimited the investigation of mathematical discourse to undergraduate mathematics textbooks.

This thesis is divided into three distinct parts. In the first part, I will develop the general theoretical features of mathematical discourse aiming to establish theoretical foundations for the practice of discourse analysis.

In the second part, I will make a detailed analysis of the negotiation author-reader: negotiation of mathematical objects, truth, comprehension, activities and ambiguity.

In the third part, I will apply the perspectives and methods developed in the first two parts to the analysis of the discourse on complex numbers, and the discourse on Riemann surfaces. This section ends with the discourse analysis of a whole textbook on Complex Analysis.

PART I

THEORETICAL FRAMEWORKS

FOR THE

ANALYSIS OF MATHEMATICAL DISCOURSE

CHAPTER 1

THEORETICAL PRELIMINARIES

Introduction

Mathematical discourse is the discourse that moulds communication of mathematical knowledge.

I distinguish three varieties of mathematical discourse:
(i) the discourse of *mathematicians* (scientific discourse),
(ii) the discourse of *teachers and learners of mathematics* (pedagogic discourse), and
(iii) the discourse of *authors of mathematics textbooks.*

In this thesis I am particularly interested in establishing perspectives and developing techniques for the analysis of the latter discourse. This discourse reveals itself to be a complex amalgamation of the scientific and pedagogic discourses.

In the first section I will discuss the general concept of discourse, in order to make clear what I understand by discourse and discourse analysis.

In section 2 I will deal with the functions of discourse. After presenting several classificatory schemes of the functions of discourse, I will propound a functional perspective for the analysis of mathematical discourse.

In section 3, as a preparation for the investigation of the discourse conveyed by mathematics textbooks, I will point out similarities and differences between scientific and pedagogic discourses.

1.1 THE CONCEPT OF DISCOURSE

Discourse is a concept used in several areas such as linguistics, semiotics, rhetoric, literary studies, philosophy, sociology, social psychology, and education. It is a controversial concept, perhaps due to the fact that specialists have biased perspectives whatever the issue.

THE CONCEPT OF DISCOURSE IS CONTROVERSIAL

According to the linguist János S. Petöfi, there is no consensus at all about the use of the terms *discourse* and *text* in linguistics; the panorama, even within this restricted area, is blurred:

> The research field, the objects of which are generally called *text* or *discourse*, involves a two-fold terminological problem: neither the research objects nor the research field itself has generally accepted, unequivocal denomination. "Being" text or "being" discourse is not an inherent property of certain objects, but is rather a property assigned to objects by those producing or analyzing them.
> [Petöfi 86, p. 1080]

In order to illustrate this chaotic situation, Petöfi presents a list of fields that deal with discourse or text as a specific subject matter: language processing, text processing, discourse processing, text analysis, discourse analysis, analysis of conversation, text linguistics, text grammar, discourse grammar, text theory, theory of discourse, science of texts, text semiotics.

In short, linguists do not agree about what discourse is and what discourse analysis is.

DISCOURSE HAS NON-LINGUISTIC FEATURES

In my view, discourse analysis should be broader than the analysis of linguistic features of written and spoken texts: discourse transcends linguistics. Discourse is intimately related to text and language, but cannot be reduced to any of them.

I agree with the philosopher John B. Thompson who says that 'to introduce the concept of *discourse* is to open an avenue for the investigation of the relation between language and ideology' [Thompson 84, p. 8]. In other words, discourse has ideological features.

I also agree with the literary critic Tzvetan Todorov, who affirms that discourse should necessarily be linked to its social and cultural context:

> Starting from vocabulary and grammar rules, language produces sentences; but sentences are only the point of departure of discursive functioning. Sentences are articulated among themselves and uttered in a given sociocultural context; they are transformed into utterances, and language is transformed into discourse.
>
> [Todorov 90, p. 9]

ORDINARY MEANINGS OF DISCOURSE

The **Oxford Advanced Learner's Dictionary** (1989) gives two different meanings for the term *discourse*. First, in ordinary language, discourse means 'a LENGTHY AND SERIOUS treatment of a subject in speech or writing' [my emphasis]; second, in linguistics, discourse means 'a continuous piece of spoken or written language'.

Despite the tedium of some mathematical texts, I will try not to use the term *discourse* in the first derogatory meaning (lengthy and serious ...), but in a more 'objective' sense. However, the second meaning (a continuous piece ...) seems too neutral to be applied to discourse; it is perhaps more appropriate to text.

The **Chambers Dictionary of Synonyms and Antonyms** (1989) furnishes a list of sixteen synonyms of discourse: address, chat, communication, conversation, converse, dialogue, discussion, dissertation, essay, homily, lecture, oration, sermon, speech, talk, treatise.

According to this dictionary, discourse can be formal like a sermon or a treatise, or otherwise informal like a chat; it is not necessarily 'lengthy and serious'.

Thus, the term *discourse* can be used to designate (a) SPEECH EVENTS such as address, chat, conversation, converse, dialogue, discussion, homily, lecture, oration, sermon, speech, and talk as well as (b) PIECES OF WRITING such as dissertation, essay, and treatise. All these terms are in fact instances of discourse and not 'synonyms', except the term *communication*.

DISCOURSE IS COMMUNICATION

Communication seems to be the keyword for the deciphering of the concept of discourse: it is a concept sufficiently general and abstract to incorporate all other cited items as its instances as well. Therefore it seems quite natural at this juncture to regard discourse as a synonym of communication, or at least intimately related to communication. In the following I will attempt to link the concept of discourse to the concept of communication.

WHAT IS COMMUNICATION?

According to the media expert John Fiske, *communication means social interaction through messages* [Fiske 90, p.2]. He points out the existence of two divergent schools in the study of communication.

The first school sees communication as the *transmission of messages*. It defines social interaction as the process by which one person (the sender) relates to others, or affects the behaviour, state of mind or emotional response of another. It sees a message as that which is transmitted by the communication process. Many of its followers believe that intention is a crucial factor in deciding what constitutes a message. This school emphasizes transmission and the addresser's role [*Ibidem*, p. 2].

The second school sees communication as the *production and exchange of meanings*. It is concerned with how messages, or texts, interact with people in order to produce meanings; that is, it is concerned with the role of texts in our culture. The message is a construction of signs which, through interacting with the receivers, produce meanings. The sender, defined as transmitter of the message, declines in importance. This school emphasizes reception and text's role [*Ibidem*, p.3].

The opposition between these two conceptions of communication recalls the analogous opposition between two conceptions of education: one centred on the teacher (transmission of messages) and the other centred on the learners (production and exchange of meanings).

CONNECTING DISCOURSE WITH COMMUNICATION

The dictionarist of stylistics Katie Wales gives us the clue to the desired connection between discourse and communication:

> One prominent and comprehensive sense [of discourse], for which there is indeed no other direct equivalent, covers all those aspects of *communication* which involve not only a message or text but also the addresser and addressee, and their immediate context of situation. Leech & Short (1981) emphasize its interpersonal or transactional nature, and also its social purpose.
>
> [Wales 89, p. 129]

Furthermore, Wales suggests that text can be conceived as a component of discourse:

> ... very commonly now discourse is used in a very comprehensive way for all those aspects of the situation or context of communication, not only the message (written or spoken), but also the relations between addressers and addressees. In this sense, discourse would subsume the text, as comparable to the 'message'.
>
> [*Ibidem*, p. 459]

DISCOURSE AND TEXT

According to the linguists G.N. Leech and M.H. Short, discourse and text are two different facets of linguistic communication:

> A distinction may be drawn between communication as discourse and as text. *Discourse* is linguistic communication seen as a transaction between speaker and hearer, as an interpersonal activity whose form is determined by its social purpose. *Text* is linguistic communication (either spoken or written) seen simply as a message coded in its auditory or visual medium.
>
> [Leech & Short 87, p. 209]

In my view, Leech & Short have dissolved the dichotomy posed by Fiske between communication as transmission of messages and communication as production and exchange of meanings. Instead of dichotomy, Leech & Short affirm the dual nature of communication: communication implies production and exchange of meanings, because it is (or has) a discourse; communication implies transmission of messages, because it is (or has) a text.

DISCOURSE IS A COMMUNICATION PROCESS

Leech & Short represent discourse as a process that works according to the flow chart shown in figure 1.1.

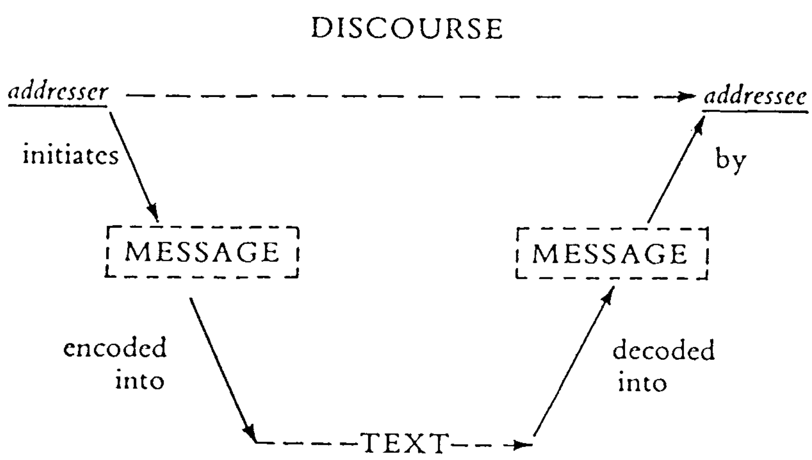


Figure 1.1 Leech & Short's model

Observe that this diagram does not reflect exactly what Leech & Short have said in the quotation above. Observe the role of discourse and text in the flow chart.

On the one hand, they use 'discourse' as the heading of the figure. This means that the notion of discourse was enlarged: discourse is no more a facet of communication; it is synonymous to linguistic communication; discourse is then any process of communication mediated by a text.

On the other hand, the notion of text was reduced: text is no more a facet of communication; it is only the linguistic (or semiotic) vehicle of discourse; it is a mere component of the discourse.

MY CONCEPT OF DISCOURSE

In this thesis I will simply adopt Leech & Short's concept of discourse, and Fiske's concept of communication. In short,

- **Discourse means social interaction through messages.**

Even though discourse is clearly a sociocultural process, I will consider it, for the convenience of expression, as an interpersonal activity, a negotiation between two 'persons': the addresser and the addressee. Therefore,

- **Discourse is a negotiation of messages between writer and reader (or between speaker and hearer).**

1.2 FUNCTIONS OF DISCOURSE

In linguistics, the word *function* usually means purpose, role, use. *Functions of language* are then the purposes for which people use language [Halliday & Hasan 90, p. 15], or the communicative roles of language [Wales 89, p. 195].

I will often use the expression *function of discourse* instead of *function of language* because discourse is nothing more than language seen as a system of communication. My long-term objective is the investigation of the *functions of mathematical discourse.*

In view of the great amount and variety of the purposes of language, researchers of different areas have attempted to classify them, according to their interests and perspectives. In the following I will describe briefly five general schemes of classifying linguistic functions due to Bühler, Popper, Jakobson, Leech and Halliday. They will be useful as a source of ideas for the construction of a functional perspective of mathematical discourse.

BUHLER'S SCHEME

The psychologist K. Bühler (1934) has propounded a triadic classification of linguistic functions: *Darstellung*, *Ausdruck*, and *Appell*.

- *Darstellung*: language is used to describe a certain state of affairs, to represent the real world, or to transmit information about something. The discourse is then orientated towards the context, the referent, or the subject matter. This function is called in English *representational*, *referential* or *informational function.*

- *Ausdruck*: language serves to express the addresser's feelings, emotions and thoughts, his or her attitude towards what he or she is speaking about. This function is called in English *expressive* or *emotive function.*

- *Appell*: the addresser uses language to appeal to the addressee, to influence the addressee in some way. This function is called in English *conative* or *directive function.*

POPPER'S SCHEME

The philosopher Karl Popper has devised a scheme with four functions of language: expressive, signalling, descriptive and argumentative functions [Popper 86, p. 234].

• *expressive* or *symptomatic* function: language consists of symptoms or expressions; it is symptomatic or expressive of the state of the organism which makes the linguistic signs. It corresponds to Bühler's *Ausdruck.*

• *signalling* or *releasing* function: the symptomatic expression of the sender releases or stimulates a reaction in the receiver, which responds to the sender's behaviour, thereby turning it into a signal. It corresponds to Bühler's *Appell.*

• *descriptive* function: description of facts, including the description of conjectured states of affairs, which we formulate in the form of theories or hypotheses. It corresponds to Bühler's *Darstellung.*

• *argumentative* function: presentation and comparison of arguments in connection with certain definite questions or problems.

The first two functions are common to the languages of animals and men; the last two functions are exclusive of the human language.

According to Popper, these functions constitute a hierarchy, in the sense that each of the higher ones cannot be present without all those which are lower, while the lower ones may be present without the higher ones. Popper illustrates the many purposes an argument can have:

> An argument serves as an EXPRESSION in so far it is an outward symptom of some internal state (whether physical or psychological is here irrelevant) of the organism. It is also a SIGNAL, since it may provoke a reply, or agreement. In so far as it is about something, and supports a view of some situation or state of affairs, it is DESCRIPTIVE. And lastly, there is its argumentative function, its giving REASONS for holding this view, e.g., by pointing out difficulties or even inconsistencies in an alternative view.
> [Popper 85, p. 295]

This viewpoint is interesting because it establishes that there are no 'neutral' descriptions, or 'objective' arguments, since descriptions and arguments are affected by its expressing and signalling components. My overall approach to discourse analysis is based on the assumption that even mathematical argumentation has 'subjective' or 'interactive' features.

JAKOBSON'S SCHEME

The linguist Roman Jakobson has developed a model of verbal communication, which involves six constituents. According to him,

> Language must be investigated in all the variety of its functions ... An outline of these functions demands a concise survey of the constitutive factors in any speech event, in any act of verbal communication. The ADDRESSER sends a MESSAGE to the ADDRESSEE. To be operative the message requires a CONTEXT referred to, seizable by the addressee, and either verbal or capable of being verbalized; a CODE fully, or at least partially, common to the addresser and the addressee; and, finally a CONTACT, a physical channel and psychological connection between the addresser and the addressee, enabling both of them to enter and stay in communication.
>
> [Jakobson 86, p. 150]

Jakobson schematizes his model using the diagram shown in figure 1.2.

CONTEXT

MESSAGE

ADDRESSER ———————————— ADDRESSEE

CONTACT

CODE

Figure 1.2 Jakobson's model of communication

Jakobson associates a linguistic function to each component of his communicative scheme. Jakobson's scheme of functions can be represented by the diagram shown in figure 1.3.

REFERENTIAL

POETIC

EMOTIVE CONATIVE

PHATIC

METALINGUAL

Figure 1.3 Jakobson's scheme

Referential, *emotive*, and *conative functions* are borrowed from Bühler's scheme.

- *poetic function*: the focus is on the MESSAGE for its own sake. Jakobson suggests that 'this function, by promoting the palpability of signs, deepens the fundamental dichotomy of signs and objects'.

- *phatic function*: the discourse is orientated towards the CONTACT. The addresser intends to keep the channels of communication open: he or she attracts addressee's attention, to confirm his or her continued attention, in order to prolong communication.

- *metalingual function*: the discourse is focused on the CODE. This means that language is used as a metalanguage, that is, to tell us something about the language itself.

In order to decipher Jakobson's abstract scheme I will compare it with the scheme devised by the linguist G. Leech in his book **Semantics** (1974).

LEECH'S SCHEME

The table 1.1 shows the relationship between Leech's and Jakobson's schemes [Leech 74, p. 42].

LEECH		JAKOBSON	
FUNCTION	COMPONENT	FUNCTION	COMPONENT
informational	subject-matter	referential	context
expressive	speaker/writer	emotive	addresser
directive	listener/reader	conative	addressee
aesthetic	message	poetic	message
phatic	channel	phatic	contact
-------	-------	metalingual	code

Table 1.1 Comparing Leech's and Jakobson's schemes

I interpret Jakobson's and Leech's schemes in the following way.

The discourse of a mathematics textbook can be analysed by focusing on different features. For example, when we focus on the subject-matter, that is, on the 'mathematical content' of the message, we are analysing its informational or referential function. When we focus on the author's personal viewpoints, we are analysing the expressive or emotive function of the messsage, etc. According to Jakobson, six features or components worth attention, while Leech chooses only five features.

Leech schematizes his model of communication with the diagram shown in figure 1.4.

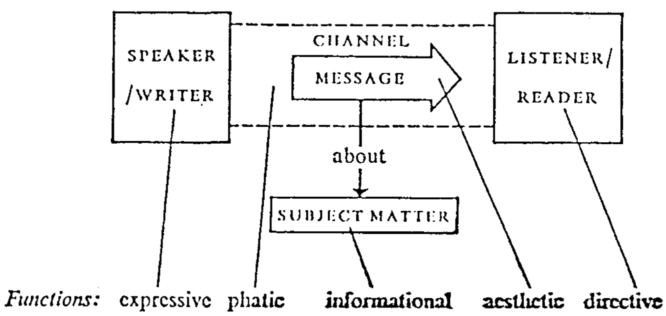


Figure 1.4 Leech's model of communication

HALLIDAY'S SCHEME

The linguist M.A.K. Halliday propounds only three functions:

(i) *ideational function* (split into *experiential* and *logical functions*), which corresponds to Bühler's referential function;

(ii) *interpersonal function* , which is the sum of Buhler's expressive and conative functions, and

(iii) *textual function* [Halliday 79, p. 48].

MY SCHEME

In the first section I have characterized discourse as an interaction between author and reader (or between speaker and listener) through messages. In this characterization, it is implicit that discourse (or communication) has two fundamental purposes: transmission of information and negotiation of meanings.

This means that I recognise the INFORMATIONAL function [propounded by Leech] and the INTERPERSONAL function [propounded by Halliday] as two fundamental functions of discourse.

In the next section, for the specific case of mathematical discourse, I will accept the HEURISTIC function as the third fundamental discursive function.

FUNCTIONAL APPROACH

Halliday summarises his functional approach to language in the following masterly way:

> Every sentence in a text is multifunctional; but not in such a way that you can point to one particular constituent or segment and say this segment has just this function. The meanings are woven together in a very dense fabric in such a way that, to understand them, we do not look separately at its different parts; rather, **we look at the whole thing simultaneously from a number of different angles, each perspective contributing towards the total interpretation.** That is the essential nature of a functional approach.
>
> [Halliday & Hasan 90, p. 23]

What I want to do is to analyse mathematical discourse using a functionalist approach. Until now several aspects of discourse have been seen at an abstract level. Later on they will acquire a richer meaning through the many examples and the uses of particular mathematical texts.

1.3 SCIENTIFIC DISCOURSE VS. PEDAGOGIC DISCOURSE

As I said in the introduction of this chapter, besides the discourse of mathematics textbooks' authors, there are two other mathematical discourses: (i) the *scientific discourse*, that is, the discourse by which mathematicians communicate with other members of their community, and (ii) the *pedagogic discourse*, that is, the discourse by which teachers and learners communicate with each other.

SCIENTIFIC MATHEMATICS VS. PEDAGOGIC MATHEMATICS

For ease of reference, I will distinguish two 'kinds' of mathematics: scientific mathematics and pedagogic mathematics.

Scientific mathematics is the mathematics of RESEARCH. It is the knowledge that mathematicians produce and communicate to their colleagues through the many specialized journals such as *Acta Mathematica*, *Journal of Algebra*, *Journal of Differential Equations*, etc., through treatises and monographs, and through research seminars and conference lectures. Only a small part of this new mathematics will be transformed in the future into pedagogic mathematics.

Pedagogic mathematics is the mathematics negotiated by INSTRUCTION, that is, by TEACHING AND LEARNING. This kind of mathematics is what teachers and pupils negotiate at primary and secondary schools, and lecturers and students negotiate at colleges and universities. It is also the mathematics propagated by undergraduate mathematics textbooks, in which I am interested.

It is obvious that the contents of scientific and pedagogic mathematics are very different, however it is not so clear whether their discourses are different.

1.3.1 Scientific Discourse

Scientific discourse is usually identified with the discourse of written texts: papers, treatises and monographs. However, in order to understand the characteristics of this textual discourse, it is necessary to take into account not only other forms of communication but the whole process of construction and communication of scientific mathematics.

CONSTRUCTION OF MATHEMATICS

Mathematicians solve problems and invent new theorems or algorithms, develop new concepts and theories; afterwards they transform these things into cultural products: papers, treatises, monographs.

However they do not create out of nothing. The process of construction of mathematics involves two aspects: personal aspects (personal inventiveness, personal experience; here I include previous education), and social aspects (contact with other mathematicians; here I include the reading of mathematical texts).

COMMUNICATION OF MATHEMATICS

After creating new theorems, concepts, or a new theory, the mathematician communicates them to other mathematicians in two forms: (i) orally, in seminars and congresses, and (ii) in written form, through papers in specialized journals or advanced books.

The process of construction and communication of scientific mathematics can then be represented by the flow chart shown in figure 1.5.

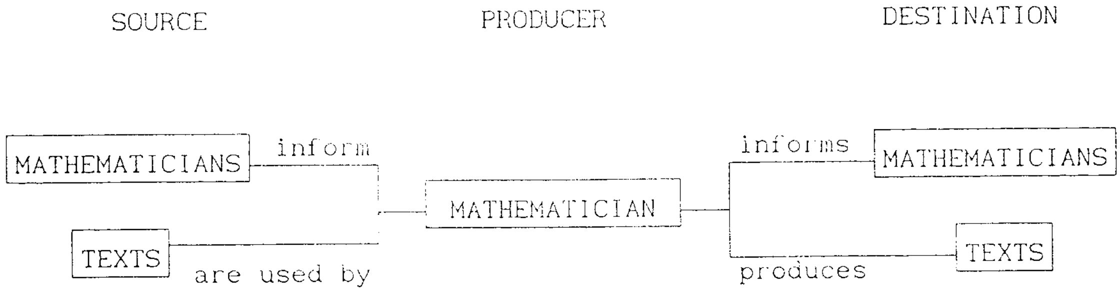


Figure 1.5 Scientific process

This scientific process has then three components: (a) construction of mathematics, (b) textual communication, and (c) oral communication.

Each subprocess is dominated by a particular logic.

HEURISTIC IS THE DOMINANT 'LOGIC' OF CONSTRUCTION OF MATHEMATICS

Mathematics is mainly constructed by the use of heuristic schemes (techniques, methods, rules). According to George Polya, 'the aim of heuristic is to study the methods and rules of discovery and invention' [Polya 71, p. 112]. Induction and analogy are the main heuristic schemes used by mathematicians.

FORMAL LOGIC IS THE DOMINANT 'LOGIC' OF TEXTUAL COMMUNICATION

The discourse conveyed by papers in journals and by advanced books on mathematics is basically the discourse of formal mathematics; it is governed by schemes of formal logic. Through this discourse, mathematicians introduce new concepts, and prove theorems about these concepts or otherwise they prove new theorems on known concepts.

Rigour in argumentation, generality of the results and conciseness of expression seem to be the most esteemed values in formal mathematics. In general, mathematicians write their papers aiming to maximize generality and rigour, with the minimum of description, explanation and argumentation. Details are often omitted and examples are scarcely given. A great educational problem is that some authors transfer these values to undergraduate mathematics textbooks.

RHETORIC IS THE DOMINANT 'LOGIC' OF ORAL COMMUNICATION

According to Polya: 'everybody prefers intuitive insight to formal logical arguments, including professional mathematicians' [Polya 81, p. 127]. However, mathematicians in general do not communicate intuitive insight in their papers, or treatises. This is only done in spoken communication.

As the interlocutors are at the same level of knowledge, they are allowed to exchange conjectures, or transmit intuitive knowledge and provide insight, by means of models, diagrams, figures, examples, particular cases, instead of formal proofs; specialists consider an excess of 'elementary' details boring or even offensive.

Mathematicians negotiate intuition and use rhetoric to persuade colleagues of the truth of a proposition without formally proving it. Therefore, in the context of oral communication between mathematicians, mathematics is never imposed, but negotiated.

Note: the model of production and communication of scientific mathematics I presented is a very simplified one. For example, it does not take into account the interplay between mathematicians and other scientists, such as physicists, economists, biologists, etc., or between mathematicians and their "disciples". Nevertheless, it is sufficient to show that three types of discursive schemes (logico-formal, heuristic and rhetorical) are used in scientific discourse.

1.3.2 Pedagogic Discourse

Here I call *pedagogic process* the process of teaching and learning mathematics. It can be seen as a process of communication and construction of mathematical knowledge, where meanings are produced and negotiated by the mutual interaction between the teacher, the learner, and the text, as it is illustrated in figure 1.6.

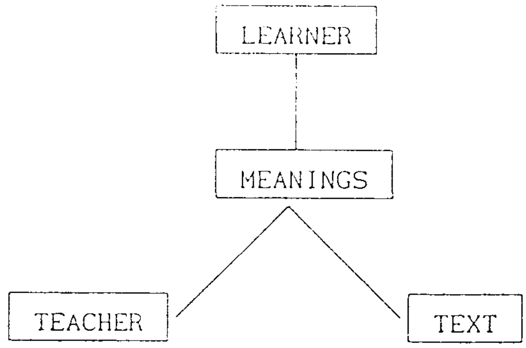


Figure 1.6 Pedagogic interactions

By analogy with the scientific process, we can model the pedagogic process by a flow chart (see figure 1.7).

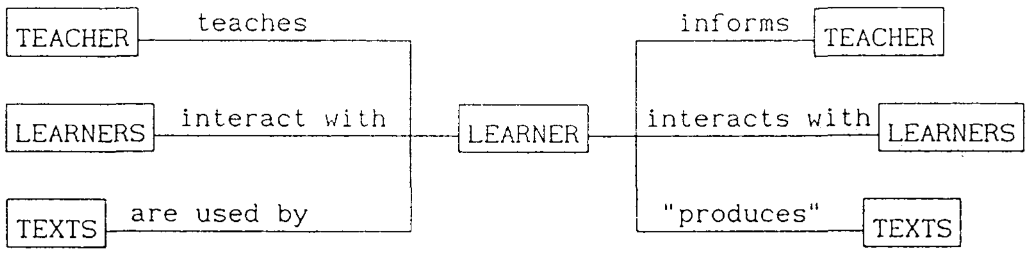


Figure 1.7 Pedagogic process

From the viewpoint of the learner, pedagogic process should then comprise the following three activities: (i) the learner negotiates with the teacher, and with their peers, (ii) the learner interacts with texts, and (iii) the learner does mathematical activities, propounded by the teacher or by the text.

NEGOTIATION BETWEEN TEACHER AND LEARNER

The negotiation teacher-learners is a face-to-face interaction: they negotiate meanings, behaviour and values in the classroom. Negotiation does not mean that there is equilibrium of forces between the participants. The imbalance of forces between teacher and learners is indeed one of the principal characteristics of the pedagogic process.

The mutual interaction between learners is nowadays very important for this process; it can be done in cooperation or in competition.

TEXTUAL COMMUNICATION

The text, besides chalk and talk, is the most used tool of pedagogic process. According to M. Kline,

> Curriculum and teachers are the most important factors in education. But there are also texts from which students might learn and which, at the very least, can reinforce the teachers' contribution.
>
> [Kline 77, p. 208]

As I said in the introduction to the thesis, in developing countries like Brazil, where the education and training of the majority of teachers are poor, the text becomes the ruler of the pedagogic activities; teacher is then just the 'tool' textbook authors use in order to affect the students.

In Brazil first year university students are often supposed to possess textbooks and use them, since they are adopted by the lecturer; this means that the lecturer uses them as the only source of knowledge and activities.

CONSTRUCTION OF KNOWLEDGE

In order to do mathematical activities the learner uses heuristic and logical techniques analogous to those used by the mathematician in the construction of original mathematics. According to J. Hadamard:

> Between the work of the student who tries to solve a problem in geometry or algebra and a work of invention, one can say that there is only a difference of degree, a difference of level, both works being of a similar nature.
>
> [Hadamard 54, p. 104]

Whatever the philosophical orientation of mathematics teaching, liberal or authoritarian, progressive or traditional, the learner is always in charge of some learning tasks, that is, the learner is to some extent constructing his or her own knowledge.

Learners are often involved in mathematical activities, either suggested by the teacher or found in textbooks, such as solving routine exercises, plotting, drawing, computing, constructing and manipulating objects, making observations and calculations, discussing themes with other learners, explaining, posing problems and questions to the teacher or to other learners, attempting to solve challenging problems, giving examples or counterexamples, proving propositions or refuting them, guessing, conjecturing, etc.

1.3.3 Constrasting Scientific and Pedagogic Discourses

In this section I will explore similarities and differences between the discourses of scientific and pedagogic mathematics.

SIMILARITIES

Scientific and pedagogic processes are analogous in the sense that both have the same "structure": construction of mathematical knowledge, textual communication, and oral negotiation of meanings.

DIFFERENCES

There are three crucial differences between scientific and pedagogic discourses.

- THE QUALITY OF THE ADDRESSER AND ADDRESSEE

On the one hand, communication of scientific mathematics involves specialists or quasi-specialists. In the scientific discourse, the addresser is a mathematician, who is the creator of the mathematical message. The addressees are experts or quasi-specialists: mathematicians, other scientists and graduate students. This means that there is not much imbalance between addresser and addressees, that is, the gap in mathematical knowledge and competence, domain of language, worldview, etc., between the addresser and the addressee is relatively small. For this reason, papers in specialized journals and advanced books are written in an esoteric form, proper to the comprehension of initiates. Consequently, the consumption of scientific mathematics is restricted; no layman is qualified to read it.

On the other hand, the communication of pedagogic mathematics involves three different classes of persons: teachers, learners and authors of textbooks. In general the addresser is a teacher, who is the creator of the pedagogic message, but not of the mathematical message. The addresser, teacher or author, is a transmitter, a re-producer, a re-creator, an explanator, a performer. The addressees in general are the learners, who are by no means specialists. On the contrary, they are supposed to be more or less ignorant of the mathematical message. Then, the gulf between the addresser (teacher or author) and the addressees (learners) is relatively deep.

- THE QUALITY OF THE MESSAGE

On the one hand, scientific mathematics is original, completely new. It is still in a fluid state, that is, it is not established mathematics. Papers and advanced books are essentially vehicles of innovations; their aim is to divulge new theorems, new theories.

On the other hand, pedagogic mathematics is sedimented, solidified, well-structured mathematics. The presentation of new theorems and concepts is counter-balanced by particularizations, exemplifications, exercises.

- THE ROLE OF THE TEXT

On the one hand, the scientific text is OUTPUT; it is the result of mathematician's work. It is an instrument of communication.

On the other hand, the pedagogic text is INPUT; it is the reference for the mathematical activities within a classroom; sometimes it governs completely the process of teaching and learning.

In my view, the analogy between scientific and pedagogic processes, or between scientific and pedagogic discourse is almost destroyed by these differences. The problem is that many authors do not agree with this; they write to learners as if they were mathematicians.

MATHEMATICS TEXTBOOKS

Mathematics textbooks are still one of the most popular vehicles of communication of pedagogic mathematics. They are helpful for the teachers and learners, since they constitute a source of information and inspiration for classroom activities. However, they can also be seen as powerful cultural artefacts whereby society regulates mathematical education.

It is convenient for the purposes of discourse analysis to classify pedagogic mathematics according to the level of instruction: (i) primary school mathematics, (ii) secondary school mathematics, (iii) undergraduate mathematics, and (iv) graduate mathematics.

There should be differences between the discourses of these different categories of pedagogic mathematics. First of all, because the learners are at different stages of intellectual development. Second, the teachers have different levels of mathematical knowledge. Third, because the aims of mathematical education vary with the level of instruction. Fourth, because the organisation of teaching and learning varies according to this categorisation.

This thesis is concerned with the analysis of the discourse underlying undergraduate mathematics textbooks.

Undergraduate mathematics textbooks are in general written by mathematicians who are at the same time lecturers. It is therefore difficult if not impossible to know whether they write as mathematicians or as lecturers, that is, whether they write "scientifically" or "pedagogically".

Due to the functional duality of the authors (they are both mathematicians and lecturers) one can expect that their discourse is an amalgamation of both scientific and pedagogic discourses.

CHAPTER 2

THE DIALECTICS OF MATHEMATICAL DISCOURSE

Introduction

My aim in this chapter is to show that mathematical discourse as well as the construction of mathematical knowledge is a dialectical process.

In section 1 I will argue that the construction of mathematical knowledge is a dialectical process, in the sense that it is governed by flows and counterflows, provoked by the tensions between several contrasting forces or tendencies.

In section 2 I will discuss the two ways authors of mathematics textbooks cope with the tension between two contrasting forces or concepts: (a) monistic way (eliminating one of the concepts) or (b) complementaristic way (maintaining the tension, seeking a balance between the two concepts).

In section 3 I will describe and classify the main conflicts that affect mathematical discourse. Conflicts arise because mathematical discourse has simultaneously three different purposes: transmission of information, construction of knowledge, and negotiation of meanings.

In section 4 I will suggest that dichotomies and dualities are to some extent associated with the way the human brain works. In the last section, I will discuss how the different options faced with the several dichotomies and dualities crystallize forming great currents within the philosophy of mathematics and mathematics education.

2.1 TENSIONS

Construction of scientific knowledge is essentially dialectical: its logic, according to the philosopher of science Karl Popper, is the logic of conjectures and refutations:

> The way in which knowledge progresses, and especially our scientific knowledge, is by unjustified (and unjustifiable) anticipations, by guesses, by tentative solutions to our problems, by *conjectures*. These conjectures are controlled by criticism; that is, by attempted *refutations*, which include severely critical tests. They may survive these tests; but they can never be positively justified: they can neither be established as certainly true nor even as 'probable' (in the sense of the probability calculus).
>
> [Popper 85, preface]

Therefore, construction of mathematical knowledge is a dialectical process. According to George Polya, 'mathematics in the making, *in statu nascendi*, in the process of being invented, appears as an experimental, inductive science' [Polya 71, preface].

The philosopher of mathematics Imre Lakatos made the missing connection between Polya's and Popper's viewpoints:

> Informal, quasi-empirical, mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations.
>
> [Lakatos 87, p. 5]

This is in line with the epistemologist Jean Piaget, who says that 'the most suitable interpretation of contemporary mathematics seems to be a dialectical interpretation. The two central ideas of all dialectics are development and synthesis' [Piaget 80, p. 487].

POPPERIAN DIALECTICS

To avoid any misunderstanding, I want to make clear that I use the term *dialectics* in the sense propounded by Karl Popper:

> *Dialectic* [or dialectics] (in the modern sense, i.e., especially in the sense in which Hegel used the term) is a theory which maintains that something - more especially, human thought - develops in a way characterized by what is called the dialectic triad: *thesis*, *antithesis* and *synthesis*.
>
> [Popper 85, p. 313]

According to Popper, the dialectic process evolves in three steps:

1. There is some idea or theory or movement which may be called a *thesis*.

2. Such a thesis will often produce opposition, because it will probably be of limited value and will have its weak points. The opposing idea or movement is called the *antithesis*.

3. The struggle between the thesis and the antithesis goes on until some solution is reached which, in a certain sense, goes beyond both thesis and antithesis by recognizing their respective values and by trying to preserve the merits and to avoid the limitations of both. This solution is called the *synthesis* [*Ibidem*, p. 313].

This does not mean that the struggle finishes necessarily in one round. According to Popper, this conflict of ideas can continue:

> Once attained, the synthesis in its turn may become the first step of a new dialectic triad, and it will do so if the particular synthesis reached turns out to be one-sided or otherwise unsatisfactory. For in this case opposition will be aroused again, which means that the synthesis can then be described as a new thesis which has produced a new synthesis. The dialectic triad will thus proceed on a

higher-level, and it may reach a third level when a second synthesis has been attained.

[*Ibidem*, p. 314]

COURANT'S DIALECTICAL APPROACH

The late mathematician Richard Courant is a prominent supporter of the dialectical perspective. He wrote, in collaboration with his student H. Robbins, a book on mathematical ideas and methods entitled **What is Mathematics?**, first published in 1941, which is now a classic reference for mathematics education. In the preface to this book, they declare:

> Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality. Though different traditions may emphasize different aspects, it is only the interplay of these antithetic forces and the struggle for their synthesis that constitute the life, usefulness, and supreme value of mathematical science.
>
> [Courant & Robbins 78, preface]

Courant keeps supporting the same viewpoint in the 60's, in a debate on applied mathematics, now calling attention to the danger of one-sidedness:

> Abstraction and generalization is not more vital for mathematics than individuality of phenomena and, before all, not more than inductive intuition. Only the interplay between these forces and their synthesis can keep mathematics alive and prevent its drying out into a dead skeleton. We must fight against attempts to push the development one-sidedly towards the one pole of the life-spending antinomy.
>
> [Carrier et al 62, p. 298]

Courant reinforces the same opinion, in the centenary tribute to David Hilbert, as quoted by P. Hilton:

> Living mathematics rests on the fluctuation between the antithetical powers of intuition and logic, the individuality of grounded problems and the generality of far-reaching abstractions. We, ourselves, must prevent the development being forced to only one pole of the life-giving antithesis.
>
> [Hilton 76, p. 91]

TENSIONS IN MATHEMATICAL DISCOURSE

Construction and communication of mathematical knowledge are intimately linked. Creation of mathematical objects is of course due to individuals. However this creation is not spontaneous, it has undoubtedly a social dimension, as it is stressed by R. Hersh:

> Mathematical objects are created, not arbitrarily, but arise from activity with already existing mathematical objects, and from the needs of science and daily life.
>
> [Hersh 79, p. 22]

The products of mathematical invention (objects, theorems and theories) only acquire 'reality' (social reality) when they are shared, communicated, that is, in practical terms, when they are published in the form of papers in journals or books.

Consequently, scientific mathematical discourse should reflect the tensions which govern mathematical construction and something else.

Mathematics textbooks, as they mirror above all the tensions between scientific and pedagogic procedures, are *a fortiori* the battlefields of conflicts of varied nature, for example, philosophical, epistemological, psychological, and educational conflicts.

In other words, the discourse of mathematics textbooks is a dialectical process in the sense that its functioning is driven by several tensions or conflicts such as logic vs. intuition, general vs. particular, abstract vs. concrete, formal vs. informal.

As a matter of fact, according to the linguist V.N. Vološinov, the presence of tension is universal:

> Actually, any real utterance, in one way or another or to one degree or another, makes a statement of agreement with or a negation of something.
>
> [Vološinov 86, p. 80]

Therefore it seems important to know what kind of tensions regulate the dynamics of mathematical discourse.

2.2 DICHOTOMY OR DUALITY?

The writing of mathematics textbooks involves the solution of many practical problems. For example, the author has to decide if he or she will include applications or not; if the proof should come before or after the theorem statement; if prerequisites should be given when necessary or if they should be given at the beginning.

I expect that there exist some organising principles behind the myriad of decisions the author makes. I agree then with René Thom, who said that 'whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics' [Thom 73, p. 204].

In general authors do not declare their 'philosophy' at all. One of my aims is to expose their 'philosophy' by means of the analysis of their discourse. To do this I will use the technique of philosophical pairs, that is, pairs of contrasting concepts or ideas.

A pair of contrasting concepts can be considered either as a *dichotomy* or as a *duality*. The pair forms a *dichotomy* if the concepts are considered antagonic, competitive, mutually excluding, otherwise it forms a *duality*, that is, the concepts are considered compatible, cooperative, complementary.

There are two different ways of coping with the conflict between two "constrasting" concepts (movements, ideas, tendencies).

MONISM

The first way is to look at the pair of concepts as forming a dichotomy, that is, as if they were opposite or antagonic concepts, without possibility of compromise. The only way to resolve the dichotomy is by choosing one of the poles, at the expense of the other. This means that one of the poles of the dichotomy should be dominant, or in the extreme case, should eliminate the other. This case can be illustrated, for example, by the monistic position adopted by Morris Kline, who resolves the conflict logic vs. pedagogy within the teaching of mathematics by opting for pedagogy, at the expense of logic [Kline 70].

COMPLEMENTARISM

The other way to resolve the conflict between two concepts (ideas, movements, tendencies) is to acknowledge that the opposition between them is not so acute, that they overlap in one or more aspects, and that they can be reconciled. This is the case of the position held by Peter Hilton, who, after discussing several 'dichotomies' such as pure mathematics vs. applied mathematics and structure building vs. problem solving, opts for the complementarist point of view:

> In the case of all of these sterile and misleading antitheses, those treated in the article and many others, we need and we can have both sides of the artificial opposition ... Good education involves a kaleidoscope of qualities and attributes: its success depends on a many-faceted approach ...
>
> [Hilton 76, p. 96]

HILTON'S APPROACH

In his important paper on the spread of false dichotomies, Hilton discusses nine dichotomies: education vs. training, skill vs. understanding, useful education vs. diverting education, elitism vs. egalitarianism, art vs. science, old mathematics vs. new mathematics, pure mathematics vs. applied mathematics, structure building vs. problem solving, and axiomatics vs. constructiveness.

Hilton claims that all these nine dichotomies are false, in the sense that 'the two concepts which are set in opposition to each other do not form part of an either/or situation; that while the two concepts under scrutiny are different, they have an essential overlap, and that, when properly understood and applied, they can in fact mutually reinforce each other'.

More precisely, Hilton calls *dichotomy* an opposition between two concepts, ideas, or movements, an either/or situation, where one should select one of the poles, at the expense of the other. In set-theoretic terms, a dichotomy is analogous to a partitioning of a set S into two mutually disjoint subsets P and Q.

Hilton is very concerned about the danger of one-sided movements in mathematics education. Hilton's adversary is the movement called 'Back to Basics', whose 'primitively reactionary' enthusiasts propose 'to replace a system observed to have certain unsatisfactory features by its precise opposite', more specifically, 'to restore what they see to be the best in the old mathematics'.

Hilton's viewpoint is that we should stop thinking in terms of dichotomies, and start thinking in terms of mutual complementarity. Here complementarity has no set-theoretic connotation. It means that many dichotomies which abound in the literature of mathematics education should not be resolved by the domination of one of the poles over the other, but by the equilibrium or balance between them.

Hilton uses the term *false dichotomy* to refer to what I have called duality. He recognises then the importance of the investigation of dualities for mathematics education:

> By giving serious attention to these dichotomies [dualities], certain educational principles should emerge which should help us to improve the quality of the teaching and learning which take place in our schools and colleges.
> [*Ibidem*, p. 77]

Therefore, both Hilton and Courant seem to be on the same side. Courant has propounded that the construction of mathematical knowledge is framed by permanent clashes between opposite tendencies, and Hilton has assumed implicitly that the development of educational principles is framed by dichotomies [dualities].

In this thesis, I support the same dialectical perspective as Courant and Hilton, advocating that the discourse conveyed by mathematics textbooks, as a reflection of both scientific and pedagogic discourses, is also framed by tensions caused by conflicts between philosophical, epistemological, psychological and pedagogic tendencies, movements, and concepts.

This point of view coincides with the more generic hypothesis propounded by Diane Macdonell: 'any text is, directly or indirectly, a site of some unequal struggle between conflicting discourses and positions [Macdonell 89, p. 54].

2.3 CONFLICTS IN MATHEMATICAL DISCOURSE

Mathematical discourse has three purposes or functions: (i) transmission of information, (ii) construction of knowledge, and (iii) negotiation of meanings.

These purposes can be conflicting in the sense that the addresser gives relevance to one of these functions at the expense of the others. The main conflicts that traverse the discourse of mathematics textbooks are between the 'logics' of transmission of information and construction of knowledge, that is, LOGIC VS. HEURISTIC, and between the 'logics' of transmission of information and negotiation of meanings, that is, LOGIC VS. RHETORIC. Another major conflict is between the 'logics' of scientific process and pedagogic process, that is, LOGIC VS. INTUITION.

Therefore, there are three basic conflicts within mathematical discourse: (i) LOGIC VS. HEURISTIC, (ii) LOGIC VS. RHETORIC, and (iii) LOGIC VS. INTUITION. My suggestion is to categorize the many dichotomic or dualistic conflicts which occur in mathematics education using these three basic conflicts as headings.

The author of an undergraduate mathematics textbook can emphasize one or two of these purposes at the expense of the other(s), or he/she can seek an equilibrium between them. Therefore, a method convenient for the detection of author's tendencies (preferences, biases, prejudices) seems to be the analysis of the conflicts that govern mathematical discourse. We can investigate, for example, how the author copes with the tensions between use of logic and use of intuition, pedagogic explanation and logical justification, mathematics as process and mathematics as product.

2.3.1 Logic vs. Heuristic

The conflict logic vs. heuristic means the conflict between two 'logics': *formal logic*, the 'logic' that controls the communication of formal mathematics, and *heuristic*, the 'logic' that governs the construction of informal mathematics. In essence the conflict is between two conceptions of mathematics: mathematics as construction of knowledge vs. mathematics as transmission of information; mathematics as a process vs. mathematics as a product.

Authors of undergraduate mathematics textbooks have to tackle the following problem. How should mathematics be presented to the learners: as an intellectual activity or as a body of knowledge?

In the first case, we say that mathematics is seen as a *process*: mathematics is invention, discovery, creation, construction. This perspective is represented, for example, by George Polya who considers mathematics as problem-solving.

In the second case, we say that mathematics is seen as a *product*: mathematics is a body of knowledge, its cultural achievements should be communicated, taught, learned, consumed, stored. This perspective is represented, for example, by Jean Dieudonné, one of the members of the Bourbaki group, who supports the use of axiomatic method in the presentation of mathematics.

The influence of the dichotomy process vs. product in teaching and learning mathematics has been recognized by many mathematics educationalists. For instance, Erich Wittmann has said that there are two competing philosophic points of view: 'one orientated towards *abstract structures* and the *axiomatic-deductive style* of presenting mathematics, and a second one considering mathematics an *activity* and emphasizing *intuitive thinking* as the source of mathematical discovery' [Wittmann 81, p. 389].

These 'philosophies of mathematics presentation' are behind two antagonic pedagogic models: one emphasizing the transmission of information (lecture style), centred on the teacher, the presumed owner of information, and the other, emphasizing the construction of knowledge by means of mathematical activities, centred on the learners.

In the first model, the transmission of information model, mathematics is a structured collection of facts (theorems), names (definitions), techniques (use of formulas and algorithms), and arguments (proofs of theorems and propositions). The teacher, who is supposed to possess this mathematical knowledge, should impart it to the students, who should then memorize the facts, names and arguments, learn the techniques, and reproduce what they have learned in examinations. This model of teaching and learning mathematics is widespread: it is adopted in almost all mathematical disciplines at almost all colleges and universities in the world.

In the second model, the construction of knowledge model, mathematics should be recreated by the learners. This model is theoretically worshipped but is practised by few teachers. It is commonly used at the ends of the educational process: in the first years of primary schools and at the graduate level.

In recent years however it has assumed more prominence in secondary school mathematics in the United Kingdom through individualised learning schemes, and through some of the investigational approaches in GCSE. It is important to note that the tension between these two models is currently reflected at a political level in the attitudes towards education expressed by the British government.

The following table shows other pairs of contrasting concepts, which can be categorized under the fundamental conflict LOGIC VS. HEURISTIC.

LOGIC	HEURISTIC
transmission of information	construction of knowledge
product/fact	process/know-how
demonstration/proof	invention/creation/discovery
formal mathematics	informal mathematics
theory building	problem solving
deduction	induction/analogy
certainty	plausibility
rigour	understanding
definitive	provisory
definitions & theorems	exercises

Table 2.1 Logic vs. Heuristic

Generally speaking, authors of undergraduate mathematics textbooks are not completely monist: rare are the textbooks without exercises or without proofs of theorems. Apparently they recognize the duality between information and know-how, attempting to impart both of them.

However, some of them favour deductive logic against heuristic reasoning, putting exercises at the service of theory-building, in the form of gap-filling exercises. More details about gap-filling exercises will be given in section 6.1.

Others, on the contrary, favour problem-solving or rather routine exercise-solving, using theorems just as any other data, imitating the style of presentation of many secondary school textbooks.

Only a deep analysis of proofs of theorems and of the content of exercises can reveal how balanced are author's decisions with regard to the conflict logic vs. heuristic.

2.3.2 Logic vs. Intuition

In the first decades of this century the philosophy of mathematics was dominated by the controversies between three schools: logicism, formalism and intuitionism.

Nowadays these foundational schools have no direct influence on the development of mathematics or mathematics education at all. Perhaps the names of the schools would be attractive: we could call logicism, formalism and intuitionism the respective tendencies to emphasize logic, form and intuition in mathematical education.

Intuition is a very controversial issue in philosophy, psychology, mathematics education, and philosophy of mathematics. According to M.R. Westcott,

> Conceptions of intuition within philosophy range from the most primitive of mental functions to the most sublime; from simple awareness of existence to the apprehension of ultimate truths.
>
> [Westcott 84, p. 252]

It would be too pretentious to give here an account of all the epistemological and psychological features of intuition, that can be found, for example, in the books [Noddings & Shore 84], [Fischbein 87] and [Westcott 68]. In the following I will just present the concepts of intuition held by some renowned mathematicians.

INTUITION ACCORDING TO HILBERT

It seems paradoxical that David Hilbert, one of the chief supporters of the formalist school in the foundations of mathematics, does not discard intuition from mathematics. He mysteriously begins his remarkable book on the **Foundations of Geometry** with the following quotation from the "constructivist" Kant:

> Thus all human knowledge begins with intuitions, proceeds from thence to concepts, and ends with ideas
>
> [Kant 90, p. 569].

Hilbert & Cohn-Vossen relate the conflict logic vs. intuition to the pair abstract vs. concrete:

> In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward **abstraction** seeks to crystallize the **logical** relations inherent in the maze of the material in a systematic and orderly manner. On the other hand, the tendency toward **intuitive** understanding fosters a more immediate grasp of the objects one studies, a live rapport with them, so to speak, which stresses the **concrete** meaning of their relations.
>
> [Hilbert & Cohn-Vossen 52, preface]

INTUITION ACCORDING TO POINCARE

Few mathematicians would disagree with Henri Poincaré, who said that intuition is the instrument par excellence of mathematical invention:

> This shows that logic is not enough; that the science of demonstration is not all science and that intuition must retain its role as complement, I was about to say, as counterpoise or as antidote of logic ... Thus logic and intuition have their necessary role. Each is indispensable. Logic, which alone can give certainty, is the instrument of demonstration; intuition is the instrument of invention.
>
> [Poincaré 58, p. 21].

INTUITION ACCORDING TO FELIX KLEIN

However the divergence that occurs among mathematicians concerns the role of intuition in the communication of mathematics. Should mathematical discourse be reduced to logic? Felix Klein (1893), for example, contrary to the monistic position of M. Pasch and G. Peano, affirms the complementarity between intuition and logic:

> On one point Pasch does not agree with me, and that is as to the exact value of the axioms. He believes - and this is the traditional view - that it is possible finally to discard intuition entirely, basing the whole science on the axioms alone. I am of the opinion that, certainly, for the purposes of research it is always necessary to combine the intuition with the axioms.
>
> [Klein 73, p. 228]

INTUITION ACCORDING TO WILDER

According to R. L. Wilder, intuition and knowledge are intimately linked:

> *Intuition*, as used by the modern mathematician, means an accumulation of attitudes (including beliefs and opinions) derived from experience, both individual and cultural. It is closely associated with mathematical knowledge, which forms the basis of intuition. This knowledge contributes to the growth of intuition and is in turn increased by new conceptual materials suggested by intuition.
>
> [Wilder 67, p. 610]

TYPES OF INTUITION

According to Poincaré, there are many kinds of intuition, two of them are the most important: (i) *sensible intuition (l'intuition sensible)*, that is, the appeal to the senses and the imagination, to which the imagination is the principal contributor, and (ii) *intuition of pure number (l'intuition du nombre pur)*, that is, intuition of pure logical forms.

J.J.A. Mooij, an analyst of Poincaré's philosophy of mathematics, introduced the felicitous term *intuition of comprehension (intuition de comprehension)* to refer to Poincaré's intuition of pure number [Mooij 66, p. 115]. This is the intuition to which Poincaré refers in the following passage:

> Faced with those complex buildings constructed by the masters of Mathematics, it is not enough to verify the solidity of each part and admire their carpentry, it is necessary to understand their architecture. Now, to understand the plan of the architect, it is necessary to perceive at a glance the whole structure, and only the intuition can give us the means to embrace everything at a glance. [my translation]
>
> [Poincaré 1900, p. 125]

INTUITION ACCORDING TO DIEUDONNE

For Jean Dieudonné, mathematical intuition is the intuition of abstract objects and it is opposed to sensory intuition, the intuition of material things:

> A mathematician's talent is measured by his imagination; logic alone will never suggest a proof, it can only put it in a rigorous shape. The mathematician's imagination is based on a kind of "intuition" of the mathematical objects he is studying, although this is very different from the "intuition" of our senses, since mathematical objects are mostly the result of a long process of elaboration and abstraction which puts them completely apart from any

> correspondence with objects of the real world.
>
> [Dieudonné 75, p. 39]

According to Dieudonné, mathematical intuition grows with abstraction:

> The more abstract things are, the more they provide intuition. Why? Because abstraction eliminates all that is contingent within a theory. If you do the proper abstraction, if you are guided by a flair (an intuition if you want), you have eliminated the contingent relations. What remains? The skeleton remains, and in this skeleton, you can sometimes discover structures that otherwise you could not dream of detecting. If you have not made the effort of preliminary abstraction, you would not be able to see the wood for the trees.
>
> [Dieudonné 75, p. 54]

Dieudonné's approach is interesting for mathematics education, since it allows us to transform the conflict logic vs. intuition, which seems at first sight a conflict between things of different nature, into a conflict between kinds of intuitions: intuition of the concrete vs. intuition of the abstract. This means, for example, that the long process of learning mathematics can be characterized as a transition from the intuition of the concrete to the intuition of the abstract.

2.3.3 Logic vs. Rhetoric

Mathematics can be presented in different levels of formalization. For ease of treatment, I will distinguish only three levels: formal, quasi-formal, and informal.

Formal mathematics is developed following the so-called axiomatic method. This method allows us to only use the rules of formal logic, excluding the use of heuristic reasoning or rhetorical arguments. According to this method, the theory should start with the presentation of a finite number of undefined objects, and unproven propositions called axioms, together with the permissible rules of inference (the game rules). Every theorem should be deduced from previous theorems or axioms, by the exclusive use of the given rules of inference. Only such proofs are considered rigorous; any other type of argument should be rejected.

Examples of purely formal mathematics are those presented in two well-known books: **Foundations of Geometry** by D. Hilbert, and **Principia Mathematica** by A.N. Whitehead and B. Russell. Logical formalism is often accompanied by 'formality' in the presentation. The following excerpt from **Principia Mathematica** exemplifies what purely formal mathematics looks like.

$*213{\cdot}22.\quad \vdash :. P_{po} \in \mathrm{Ser} \,.\, \exists ! \overrightarrow{B}\text{‘}P \,.\supset :$

$QP_s R \,.\equiv .\, Q, R \in P \upharpoonright \text{“}\mathrm{sect}\text{‘}P \,.\, \dot{\exists} ! R \dot{-} Q \,.\equiv .\, Q, R \in P \upharpoonright \text{“}\mathrm{sect}\text{‘}P \,.\, Q \Subset R \,.\, Q \neq R$

$[*213{\cdot}21{\cdot}161]$

$*213{\cdot}23.\quad \vdash :. P_{po} \in \mathrm{connex} \,.\, Q, R \in C\text{‘}P_s \,.\supset : Q \Subset R \,.\vee .\, R \Subset Q$

$[*213{\cdot}1 \,.\, *211{\cdot}6{\cdot}17 \,.\, *36{\cdot}24]$

$*213{\cdot}24.\quad \vdash : \beta \in \mathrm{sect}\text{‘}P \,.\, R = P \upharpoonright \beta \,.\supset .\, \mathrm{sect}\text{‘}R = \mathrm{sect}\text{‘}P \cap \mathrm{Cl}\text{‘}C\text{‘}R$

Dem.

$\vdash .\, *36{\cdot}29 \,.\supset \vdash :. \mathrm{Hp} \,.\supset : R \Subset P :$ (1)

$[*211{\cdot}1] \quad \supset : \alpha \in \mathrm{sect}\text{‘}P \cap \mathrm{Cl}\text{‘}C\text{‘}R \,.\supset .\, \alpha \subset C\text{‘}R \,.\, R\text{“}\alpha \subset \alpha \,.$

$\supset .\, \alpha \in \mathrm{sect}\text{‘}R$ (2)

$\vdash .\, (1) \,.\, *211{\cdot}1 \,.\supset$

$\vdash :. \mathrm{Hp} \,.\supset : \alpha \in \mathrm{sect}\text{‘}R \,.\supset .\, \alpha \subset C\text{‘}R \,.\, \alpha \subset C\text{‘}P \,.\, (P \upharpoonright \beta)\text{“}\alpha \subset \alpha$ (3)

$\vdash .\, (3) \,.\, *37{\cdot}41{\cdot}413 \,.\supset$

$\vdash : \mathrm{Hp} \,.\, \alpha \in \mathrm{sect}\text{‘}R \,.\supset .\, \alpha \subset \beta \,.\, \beta \cap P\text{“}(\alpha \cap \beta) \subset \alpha \,.$

$[*22{\cdot}621 . *37{\cdot}2] \quad \supset .\, \beta \cap P\text{“}\alpha \subset \alpha \,.\, P\text{“}\alpha \subset P\text{“}\beta \,.$

$[*211{\cdot}1] \quad \supset .\, \beta \cap P\text{“}\alpha \subset \alpha \,.\, P\text{“}\alpha \subset \beta \,.$

$[*22{\cdot}621] \quad \supset .\, P\text{“}\alpha \subset \alpha$ (4)

$\vdash . (3) . (4) . \supset \vdash :. \mathrm{Hp} \,.\supset : \alpha \in \mathrm{sect}\text{‘}R \,.\supset .\, \alpha \subset C\text{‘}R \,.\, \alpha \in \mathrm{sect}\text{‘}P$ (5)

$\vdash . (2) . (5) . \supset \vdash . \mathrm{Prop}$

Figure 2.1 Russell & Whitehead's formalism

Another example of formalism in mathematical writing is the collection of graduate textbooks **The Elements of Mathematics** by N. Bourbaki. However, they are not purely formal in the sense that several 'abuses' of language are made and 'forbidden' examples are anticipated.

In contrast to the formal mathematics, there are books written informally, such as **Mathematician's Delight** (1991) by W.W. Sawyer, or **Aventuras Matematicas** (1986) by Miguel de Guzmán, which are aimed to the popularization of mathematics. Such authors wish to show the importance, or the beauty of mathematics for the laymen. They have to make a great effort in order to not bother readers with rigorous definition or deductive reasoning. In other terms, they use more rhetoric than logic.

Undergraduate mathematics textbooks are neither formal nor informal; they are written in a quasi-formal style. The actual logic that dominates such books is a mixture of formal logic and rhetoric; it is in fact a 'logic' of negotiation, similar to the 'logic' of the oral negotiation between lecturers and students.

Authors negotiate many things with the readers: the truth of theorems, the use of language, the standards of explanation and understanding, labour division, values and perspective. For the author, success in this negotiation means the continued use of his or her textbook by teachers and students.

NEGOTIATION OF PROOFS

Authors of undergraduate mathematics textbooks are not able to follow Bourbaki's commandment that proofs should be given in full (no part should be 'left to the reader' or relegated to exercises), and with the utmost precision. [Dieudonné 82, p. 619]. For example, in Complex Analysis textbooks, Jordan curve theorem is commonly presented without proof and freely used, because its proof is in fact too demanding. Sometimes, the author does not give the proof of a theorem because it is too easy, or too technical.

In short, authors should negotiate with the readers the acceptance of theorems; they have to convince the readers to accept some theorems without proofs, and other theorems with uncomplete proofs. And to do this, they have obviously to resort to rhetoric.

NEGOTIATION OF INTUITIONS

From the viewpoint of pedagogy, explanation and understanding are the crux of the question, and explanation in mathematical discourse often implies the change from logical reasoning to intuitive and heuristic reasoning, by the use of examples, figures, schemes, etc.

NEGOTIATION OF LANGUAGE

Authors should negotiate the tension between the constraining language of formal logic and the plasticity of the natural language. This negotiation takes the form of 'abuse' of mathematical language, by way of poetic licence.

NEGOTIATION OF VALUES AND PERSPECTIVE

Authors, as human beings, necessarily hold personal or ideological values and perspectives on mathematics, on learning, on teaching, on communication, on mathematical knowledge, and many other subjects, and, as human beings, they want to share them with the readers.

In my view, the ways authors attempt to communicate these values and perspectives to the readers are the most important things to be detected by discourse analysis. What is the essence of the author's message, that thing that differentiate each one from the others?

In conclusion, undergraduate mathematics textbooks are not about purely formal mathematics. Their authors give us a skeleton of formal mathematics interspersed with rhetoric, or vice-versa. It is important to observe the balance or imbalance between logic and rhetoric obtained by the authors.

In the following table, we summarize some dichotomies associated with the dichotomy logic vs. rhetoric.

LOGIC	RHETORIC
transmission of information	negotiation of meanings
formal	informal
definitions & theorems	examples & figures
formal proof	argumentation
information	expression
denotation	connotation
proving	persuading
authoritative knowledge	negotiable knowledge
objective	intersubjective
monologue	dialogue
logical reasoning	rhetorical argumentation

Table 2.2 Logic vs. Rhetoric

2.4 BRAIN DUALITY AND COMPLEMENTARITY

The dichotomies or dualities described above relate to human thinking. It is no coincidence that among the theories of brain function there is an aspect which relates physical forms (hemispheres) to function, and that the functioning relates to some of the dichotomies we have been discussing in closely analogous ways.

THE TWO HEMISPHERES OF THE BRAIN

The human brain is formed by two hemispheres, the left and right hemispheres, which are connected by nerve fibres called *corpus callosum.* It is a consensus between researchers in neuropsychology that 'there are differences in function between the two sides of the brain and that the differences are found in normal subjects as well as in patients' [Springer & Deutsch 89, p. 6].

According to the educationalist M. Raina,

> The left hemisphere of the brain is a rational-linear mind specialising in sequential processing, logical, analytical thinking and verbalisation. It is responsible for most of

> the functions that involve linguistic and numerical processes ... The right cerebral hemisphere is lateralized for multiple processing. It houses spatial perception, holistic understanding, perceptual insight, tactile sensation, musical ability, visualisation and intuitive ability. Its mode is metaphoric, analogic and holistic.
>
> [Raina 79, p. 10]

Many dichotomies are associated with the asymmetric functioning of the human brain. According to Springer and Deutsch, the dichotomic characteristics usually attributed to the processes of left hemisphere and right hemisphere can be separated into five groups, where 'the descriptions near the top of the list seem to be based on experimental evidence; the other designations appear more speculative' [Springer & Deutsch 89, p. 284].

LEFT HEMISPHERE	RIGHT HEMISPHERE
Verbal	Nonverbal, visuo-spatial
Sequential, temporal, digital	Simultaneous, spatial, analogical
Logical, analytical	Gestalt, synthetic
Rational	Intuitive
Western thought	Eastern thought

Table 2.3 Left Hemisphere vs. Right Hemisphere

Springer and Deutsch alert us about the danger of *dichotomania*, that is, the tendency of relating any dichotomy to the hemispheric specialization. However, they themselves furnished the following list of dichotomies and speculated that 'perhaps, as some have suggested, they correspond to the separate processes of the two cerebral hemispheres' [Springer & Deutsch 89, p. 286].

DICHOTOMIES

Convergent	Divergent
Intellectual	Intuitive
Deductive	Imaginative
Rational	Metaphorical
Vertical	Horizontal
Discrete	Continuous
Abstract	Concrete
Realistic	Impulsive
Directed	Free
Differential	Existential
Sequential	Multiple
Historical	Timeless
Analytical	Holistic
Explicit	Tacit
Objective	Subjective
Successive	Simultaneous

Table 2.4 Springer & Deutsch's dichotomies

Researchers are divided with respect to the degree of cooperation in which the two brain hemispheres work.

R. W. Sperry has argued that each hemisphere behaves in many respects like a separate brain [Sternberg 90, p.167].

M. Gazzaniga believes that the brain is organized into relatively independent functioning units (modules) that work in parallel, coexisting in a kind of confederation; D.E. Broadbent sees more integration between the processing of the two hemispheres, sustaining that 'the two hemispheres should be seen as performing different parts of an integrated performance rather than totally separate, parallel functions' [Sternberg 90, p. 172].

Jerre Levy, according to [Goldberg 89], declares that the hemispheres usually work in collaboration:

> In the normal individual the two hemispheres are in constant active integration and intimate collaboration. There is nothing a normal person could do that depends only on one hemisphere. Possibly, if you used an exceedingly simple task, repetitive, habitual and boring, a normal brain might show asymmetric processing, but the instant you increase the task difficulty, it would instigate bilateral hemispheric engagement.
>
> [Goldberg 89, p. 121]

Brain division is a metaphor used by some mathematics educationalists to describe the teaching of mathematics; there is a generalized opinion that teachers and authors of textbooks emphasize the working of the left hemisphere, neglecting the other hemisphere. See, for example, [Elliott 80], [Nolder 84], and [Fidelman 85].

However, I agree with Springer and Deutsch who declare that:

> Our educational system may miss training or developing half of the brain, but it probably does so by missing out the talents of both hemispheres.
>
> [Springer & Deutsch 89, p. 298]

COMPLEMENTARITY

The psychologist Norman Cook suggests an interesting theory about the functioning of the *corpus callosum.* He says that the neurophysiological role of the corpus callosum is topographic inhibition. This means that the corpus callosum reduces neural activity in the hemisphere opposite to where activity starts, operating at regional level (callosal fibres connect corresponding regions of the two hemispheres in a point-to-point or topographic manner).

More precisely, 'topographic inhibition across the corpus callosum suppresses in one hemisphere the exact same neuronal pattern of activity that originated in the other, but at the same time allows activity to develop in surrounding neurons representing *complementary*, e.g., contextual, aspects of the original information.' This means that 'equivalent areas of the two hemispheres end up active for *complementary* aspects' [Springer & Deutsch 89, p. 315].

In summary, the complementarist (or dialectical) perspective suggested by R. Courant and P. Hilton can be supported by the viewpoints of neuropsychologists such as Jerre Levy and Norman Cook.

RIGHT HEMISPHERE OR LEFT HAND?

Perhaps moved by ideological motive, the psychologist Jerome Bruner, in his book called **On Knowing: Essays for the Left Hand** (1979), prefers to use the term 'left hand' instead of right hemisphere, and 'right hand' instead of left hemisphere. He confesses that

> Since childhood, I have been enchanted by the fact and the symbolism of the right hand and the left - the one the doer, the other the dreamer. The right is order and lawfulness, *le droit* ... Of the left hand we say that it is awkward ... The French speak of the illegitimate descendant as being *a main gauche*, and though the heart is virtually at the center of the thoracic cavity, we listen for it on the left. Sentiment, intuition, bastardy.
>
> [Bruner 79, p. 2]

However, Bruner, accompanying the mathematicians Hilton and Courant, alerts us about the danger of lateralization:

> Reaching for knowledge with the right hand is science. Yet to say only that much of science is to overlook one of its excitements, for the great hypotheses of science are gifts carried in the left hand.
>
> [*Ibidem*, p. 2]

2.5 GREAT CURRENTS

In this section I will discuss some significative alternatives to the formalist philosophy dominant in the teaching and writing of mathematics. These alternatives will be seen through the works of four mathematics educationalists and philosophers: G. Polya, I. Lakatos, R. Hersh, and P. Ernest.

One of the themes of the educational and epistemological works of George Polya and Imre Lakatos is the conflict between logic and heuristic. Both of them support a heuristic approach against a deductivist approach; Polya was more interested in problem-solving (problems to find), whereas Lakatos was more akin of theory-building (problems to prove).

Reuben Hersh propounds, in line with Lakatos, that philosophers of mathematics and mathematicians should abandon the dogma that mathematical truth should possess absolute certainty, because it is incompatible with the actual practice of working mathematicians.

Paul Ernest, with his social constructivism, propounds that mathematics is essentially a social construct, establishing the connection between subjective knowledge of mathematics and objective knowledge of mathematics, and between creation and learning.

1. GEORGE POLYA

In his famous book **How to Solve It** (first edition: 1945), George Polya points out that mathematics is two-faceted:

> Yes, mathematics has two faces; it is the rigorous science of Euclid but it is also something else. Mathematics presented in the Euclidean way appears as a systematic, deductive science; but mathematics in the making appears as an experimental, inductive science.
>
> [Polya 71, preface]

Polya is a discreet critic of the Euclidean 'pedagogy':

> It is not desirable that the teacher should present many proofs in the pure Euclidean manner, although the Euclidean presentation may be very useful after a discussion in which the students guided by the teacher discover the main idea of the solution as independently as possible.
>
> [*Ibidem*, p. 71]

Polya extends his investigation on mathematical heuristic by publishing two other books: **Mathematics and Plausible Reasoning** (1954) (divided into two volumes: **Induction and Analogy in Mathematics**, and **Patterns of Plausible Inference**), and **Mathematical Discovery** (1962).

In the latter book Polya stresses the point that, in mathematics teaching, the development of know-how is as important as acquisition of information, if not more:

> Our knowledge about any subject consists of *information* and of *know-how*. If you have genuine *bona fide* experience of mathematical work on any level, elementary or advanced, there will be no doubt in your mind that, in mathematics, know-how is much more important than mere possession of information. Therefore, in the high school, as on any level, we should impart, along with a certain amount of information, a certain degree of *know-how* to the student.
>
> [Polya 81, preface]

'Know-how in mathematics', Polya says, 'is the ability to solve problems - not merely routine problems but problems requiring some degree of independence, judgement, originality, creativity' [*Ibidem]*.

Polya has substantially developed a theory of plausible reasoning or heuristic reasoning in mathematics, which can be called the logic of mathematical discovery. Heuristic is the logic of plausibility as well as formal logic is the logic of certainty:

> Heuristic reasoning is reasoning not regarded as final and strict but as provisional and plausible only, whose purpose

> is to discover the solution of the present problem. We are often obliged to use heuristic reasoning. We shall attain complete certainty when we shall have obtained the complete solution, but before obtaining certainty we must often be satisfied with a more or less plausible guess. We may need the provisional before we attain the final. We need heuristic reasoning when we construct a strict proof as we need scaffolding when we erect a building.
>
> [Polya 71, p. 113]

Therefore, heuristic reasoning should not be confounded with demonstrative reasoning:

> Heuristic reasoning is good in itself. What is bad is to mix up heuristic reasoning with rigorous proof. What is worse is to sell heuristic reasoning for rigorous proof.
>
> [*Ibidem*, p. 113]

2. IMRE LAKATOS

According to R. Hersh, the book **Proofs and Refutations** (1971) by Imre Lakatos, which is a development of his Ph.D. dissertation **Essays in the Logic of Mathematical Discovery** (1961), written under the influence of Karl Popper and George Polya, 'is the most interesting and original contribution to the philosophy of mathematics in recent decades' [Hersh 79, p. 14].

Lakatos calls *deductivist style* the Euclidean style of presenting mathematics:

> Euclidean methodology has developed a certain obligatory style of presentation. I shall refer to this as 'deductivist style'. This style starts with a painstakingly stated list of *axioms*, *lemmas* and/or *definitions*. The axioms and definitions frequently look artificial and mystifyingly complicated. One is never told how these complications arose. The list of axioms and definitions is followed by the

> carefully worded *theorems*. These are loaded with heavy-going conditions; it seems impossible that anyone should ever have guessed them. The theorem is followed by the *proof*.
>
> [Lakatos 87, p. 142]

According to Paul Ernest, 'the absolutist view of mathematical knowledge is that it consists of certain and unchallengeable truths' [Ernest 91, p. 7]. Lakatos argues that the deductivist style is linked to the absolutist philosophy of mathematics:

> In deductivist style, all propositions are true and all inferences valid. Mathematics is presented as an ever-increasing set of eternal, immutable truths. Counterexamples, refutations, criticism cannot possibly enter ... Deductivist style hides the struggle, hides the adventure. The whole story vanishes, the successive tentative formulations of the theorem in the course of the proof-procedure are doomed to oblivion while the end result is exalted into sacred infallibility.
>
> [*Ibidem*, p. 142]

Lakatos concludes his criticism of pedagogical formalism, associating it to authoritarianism:

> It has not yet been sufficiently realised that present mathematical and scientific education is a hotbed of authoritarianism and is the worst enemy of independent and critical thought.
>
> [*Ibidem*, p. 142]

There is a vast literature on Lakatos' philosophy of mathematics. In my view, the Ph.D. thesis **The Epistemological and Educational Arguments of Imre Lakatos's Proofs and Refutations** by Judith Maxwell Greig (1987) is one of the best works on it, containing a thorough investigation of Lakatos's attack on formalism [Greig 87].

3. REUBEN HERSH

In 1979 Hersh published a courageous paper on philosophy of mathematics, *Some Proposals for Reviving the Philosophy of Mathematics*, in a journal, *Advances in Mathematics*, specialized in mathematical themes.

Hersh's targets are the two major 'schools' in the philosophy of mathematics, formalism and Platonism (realism), represented, for example, by René Thom (Platonist) and Jean Dieudonné (formalist), who were opponents in a famous debate on 'modern' mathematics during 1971-1973.

According to Hersh, both formalism and Platonism are incompatible with the daily experience and practice of working mathematicians, and should be abandoned:

> The philosophical notions about mathematics commonly held by the working mathematician [Platonism and formalism] are incompatible with each other and with our actual experience and practice of mathematical work. Many practical problems and impasses confronting mathematics today have philosophical aspects. The dearth of well-founded philosophical discourse on mathematics has observable harmful consequences, in teaching, in research, and in the practical affairs of our organizations.
>
> [Hersh 79, p. 11]

For Hersh, mathematicians can abandon both philosophies, Platonism and formalism, if they abandon the dogma that mathematical truth should possess absolute certainty. Hersh completes his argument declaring that mathematical knowledge is fallible:

> We do not have absolute certainty in mathematics; we may have virtual certainty, just as in other areas of life. Mathematicians disagree, make mistakes and correct them, are uncertain whether a proof is correct or not.
>
> [*Ibidem*, p. 20]

According to Hersh, 'mathematics deals with ideas'. His credo is based on the following assumptions, which he considers not as philosophical theses but facts of experience:

(1) Mathematical objects are invented or created by humans.

(2) They are created, not arbitrarily, but arise from activity with already existing mathematical objects, and from the needs of science and daily life.

(3) Once created, mathematical objects have properties which are well-determined, which we may have great difficulty in discovering, but which are possessed independently of our knowledge of them.

The first two propositions are reactions against Platonism and formalism. The first proposition opposes the Platonist's belief in the existence of ideal entities, independent or prior to human consciousness. The second proposition opposes the formalist's opinion that 'much of all of pure mathematics is a meaningless game'.

The third proposition suggests that mathematical objects, which are created by individuals, are transformed into cultural objects by the society, more specifically by the community of mathematicians. In other words,

> Once created and communicated, mathematical objects are *there*. They become part of human culture, separate from their originator. As such, they are now objects, in the sense that they have well-determined properties of their own, which we may or may not be able to discover.
>
> [Hersh 79, p. 23]

4. PAUL ERNEST

Paul Ernest, in his recent book **The Philosophy of Mathematics Education** (1991), propounds a new philosophy of mathematics, which he calls social constructivism.

According to Ernest, social constructivism views mathematics as a social construction. In my view, the fundamental feature of Ernest's philosophy is the linking he establishes between subjective (individual, private) knowledge and objective (social, public) knowledge:

> So we have a creative cycle, with subjective knowledge creating objective knowledge, which in turn leads to the creation of subjective knowledge.
>
> [Ernest 91, p. 84]

Ernest assumes the following seven propositions about knowledge creation:

1. An individual possesses subjective knowledge of mathematics; subjective and objective knowledge are distinct.

2. Publication is necessary for subjective knowledge to become objective mathematical knowledge but not sufficient; this will depend on its acceptance.

3. Published knowledge is subject to scrutiny and criticism by others, following Lakatos' heuristic, which may result in its reformulation and acceptance as objective knowledge of mathematics.

4. This heuristic depends on objective criteria.

5. The objective criteria for criticizing published mathematical knowledge depend to a large extent on shared mathematical knowledge, but ultimately they rest on common knowledge of language, that is, on linguistic conventions.

6. A key stage in the cycle of mathematical creation is the internalization, that is the inner subjective representation, of objective mathematical and linguistic knowledge.

7. Individual contributions can add to, restructure or reproduce mathematical knowledge [Ernest 91, pp. 43-44].

The following figure 2.2 shows an illustrative diagram of this cycle.

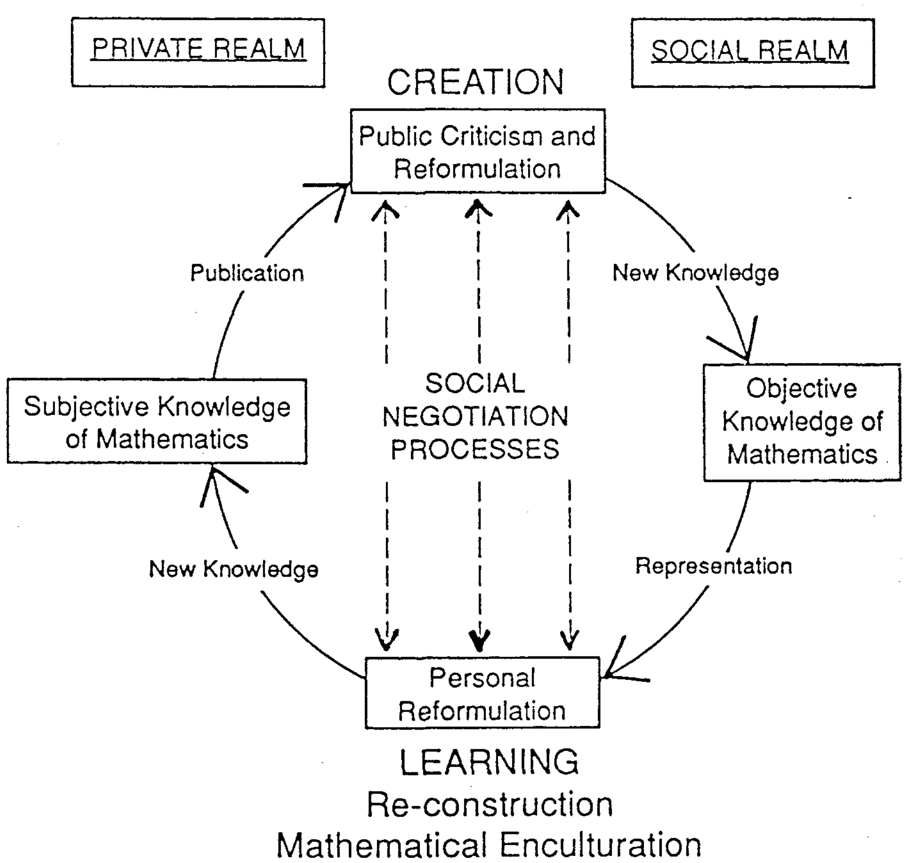


Figure 2.2 Ernest's Diagram

CONCLUSION OF THE CHAPTER

Undergraduate mathematics textbooks reflect the struggles between philosophical tendencies. In their discourses authors utilize three conflicting 'logics':

- formal logic (the logic of demonstration)
- heuristic (the logic of discovery)
- rhetoric (the logic of social negotiation)

Formal logic gives structure to the architecture of formal mathematics, organising the 'contents' of the textbook: axioms, definitions, and theorems. Formal logic also contributes partially to the organisation of mathematical argumentation: proofs.

Heuristic organises the readers' construction of mathematical knowledge by means of mathematical activities: exercises and problems.

Rhetoric organises the negotiation of meanings (motivation, persuasion, clarification, acceptance): examples, figures, applications, historical notes, metadiscourse.

CHAPTER 3

THE CONTROL OF MATHEMATICAL DISCOURSE

Introduction

My perspective is that the discourse conveyed by a mathematics textbook is the result of the compromise the author establishes between its three purposes: transmission of information, construction of knowledge, and negotiation of meanings.

I have said in section 1.3 that each component of mathematical discourse has its own 'logic': *rules of formal logic* govern transmission of mathematical information, *rules of heuristic* control the construction of mathematical knowledge, and *rules of rhetoric* dominate the negotiation of meanings.

In general authors are not involved in explaining rules of formal logic; it seems that rules of logic are 'natural', since learners are all supposed to know such rules. This is a great mistake. Textbooks that present notions of formal logic are rare. Heuristic rules are less mentioned in textbooks, and rhetorical rules are never mentioned. This situation is amazing. To learn mathematics seems the same thing as to try to improve ability in chess without learning strategies or tactics.

In the first three sections of this chapter I discuss respectively the logical, heuristic and rhetorical schemes most often used by the authors in the development of their discourse.

In the last section I give an outline of the analysis of mathematical discourse I intend to develop in the following chapters.

3.1 LOGICAL SCHEMES

Theorems and definitions are the core of formal mathematics. In this section I am interested in the logical schemes used in the presentation of theorems. There are four fundamental types of theorems: implication, equivalence, existence, and uniqueness.

1. Implication Theorem

The majority of the theorems that appear in undergraduate mathematics textbooks are explicitly stated (or can be stated), in the form of *implications*, that is, propositions of the form 'if p, then q', or symbolically $p \Rightarrow q$. In this case, p is called *antecedent* or *hypothesis* and q is called *consequent* or *conclusion.* This formal standardization is misleading, because it hides deep differences between theorems.

SUFFICIENT OR NECESSARY?

There are two ways of looking at an implication theorem of the form $(p \Rightarrow q)$.

Sometimes $(p \Rightarrow q)$ represents the fact that 'p is a *sufficient condition* for q'. This means rhetorically that the theorem's central theme is the consequent q, and that the proposition p is just a condition that guarantees that q holds; it is then a sufficient condition for the validity of q.

For example, in the context of convergence of series, it is known that some geometric series converge, and others diverge. The problem is to find a simple test whereby we can know whether a given geometric series is convergent or not, that is, to find a criterion of convergence. This problem gives rise to the theorem

'if $|q| < 1$, then the series $1 + q + q^2 + \ldots q^n + \ldots$ converges',

which gives then a sufficient condition for convergence of the geometric series.

Sometimes the implication ($p \Rightarrow q$) represents the fact that 'q is a *necessary condition* for p'. This means rhetorically that our topic of interest is the antecedent p, and that the proposition q is just a consequence of p.

For example, in the context of complex series, one problem is to know what is the asymptotic behaviour of the general term of a convergent series. This question gives rise to the theorem

'if the series $z_1 + z_2 + \ldots + z_n + \ldots$ is convergent, then
$\lim z_n = 0$',

which gives a necessary condition for the convergence of a complex series.

For many authors, any implication ($p \Rightarrow$ q) can be read either as 'p is sufficient for q' or 'q is necessary for p'. However, from the viewpoint of ordinary language, this is not convenient.

We are not allowed to interpret any implication ($p \Rightarrow q$) as 'q is a necessary condition for p'. It sounds nonsense to affirm, for example, that the convergence of the geometric series is a necessary condition for the fact that $|q| < 1$. Analogously, we are not allowed to interpret any implication ($p \Rightarrow q$) as 'p is a sufficient condition for q'. For example, it seems absurd to say that the convergence of the series $\sum z_n$ is a sufficient condition for the equality
$\lim z_n = 0$.

Therefore, 'necessary' and 'sufficient' are words that convey rhetorical meanings that are obfuscated by the use of formal logic. Discourse analysis should enhance such rhetorical differences, which are concealed by the language of formal logic.

In order to interpret a theorem given in the form of implication in terms of sufficiency or necessity, we should look for the generating problem: behind any theorem, there always exists a generating problem.

This means that an implication theorem can be read 'p is a sufficient condition for q' if the question that originates the theorem is 'under what conditions does q occur?'. Analogously, it can be read 'q is a necessary condition for p' if the question is 'what are the consequences that follow from p?

Nevertheless, there exist implication theorems that permit both interpretations. For example, the theorem 'if f is differentiable, then f is continuous' can be read in both ways: 'differentiability of a function is a sufficient condition for its continuity' as well as 'continuity is a necessary condition for differentiability'. This reading depends on the context of use.

IMPLICATION AND CLASSES

Some implication theorems can be translated to the language of classes or sets; they express that a certain class is a subclass of another one. For example, the theorem 'if the series $\sum a_n$ is absolutely convergent, then it is convergent' can be translated to class language as 'the class of absolutely convergent series is a subclass of the class of convergent series'.

However, this translation cannot be extended to all implication theorems; for example, the criterion of convergence of geometric series cited above cannot be translated to the language of set theory.

EQUIVALENT PROPOSITIONS

There are several ways of expressing the same implication. It is known for example that the implication ($p \Rightarrow q$) is logically equivalent to the following propositions:

(1) $\sim p \vee q$,

(2) $\sim q \Rightarrow \sim p$,

(3) $\sim(p \wedge \sim q)$.

Proposition (2), ($\sim q \Rightarrow \sim p$), called the *contrapositive* of the direct implication, is much used in mathematics. For example, the necessary condition for the convergence of complex series given above can be rephrased as 'if $\lim z_n \neq 0$, then the series $z_1 + z_2 + \ldots + z_n + \ldots$ is divergent',

which can be interpreted as a sufficient condition for the divergence of a series. So, by the simple use of logical equivalence, we obtain a powerful tool for classifying series.

In other words, we can, from the logical point of view, transform a 'necessary' condition into a 'sufficient' condition and vice-versa: if q is a necessary condition for p, then $\sim q$ is a sufficient condition for $\sim p$; analogously, if p is a sufficient condition for q, then $\sim p$ is a necessary condition for $\sim q$.

Readers of undergraduate mathematics textbooks are, in general, supposed to know all these equivalences, and to use them. The logical relations between statements are often given verbally, rather than symbolically, using such words as 'necessary', 'sufficient', 'if', 'then', 'only if' etc.

Discourse analysis should draw attention to the interplay between the 'calculus' of formal logic and the corresponding verbal 'calculus' of the ordinary language.

For example, consider the following three "versions" of Liouville's theorem:

1. *Priestley's version:* Let f be holomorphic and bounded in the complex plane $\mathbb{C}$. Then f is constant. [Priestley, p. 65]

2. *Derrick's version:* An entire function cannot be bounded on all of $\mathbb{C}$ unless it is a constant. [Derrick, p. 99]

3. *Depree & Oehring's version:* An entire function that is not a constant cannot be bounded in all of $\mathbb{C}$. [Depree & Oehring, p. 196]

A mathematician would say that, even though these three versions appear to be different, they are logically equivalent. A logician would recognise the equivalence pointing out however some minor technical differences. For example, Derrick's version affirms more things than the other versions.

From the viewpoint of discourse analysis, the differences are more interesting than the equivalences: even though these three definitions are logically equivalent, they appear to be different. In what sense are they different?

In my view, Depree & Oehring's version is more significant than Priestley's version, since it seems to express something about the 'behaviour' of non-constant entire functions, whereas Priestley's version is set-theoretical and then static: it essentially describes the intersection of two classes. Observation of these subtle differences should be part of discourse analysis, since it can help the uncovering of the overall discursive strategy of textbooks authors.

CONVERSE OF A THEOREM

The *converse* of the implication $(p \Rightarrow q)$ is the proposition $(q \Rightarrow p)$.
As a rule, mathematicians, when presented to an implication theorem, ask about its converse, in order to feel the 'closeness' of the relationship between hypothesis and conclusion. The maximum of 'closeness' is obviously the equivalence between them.

The converse of a theorem can be true or false. If the converse is true, we have logical equivalence between p and q; in this case, q is usually said to be a necessary and sufficient condition for p, and vice-versa.

The standard tactics to show that the converse is false is by exhibiting a counter-example.

FALLACY

Some students are tempted to 'deduce' the proposition ($\sim p \Rightarrow \sim q$) from the direct implication: this false deduction can lead to an error. For example, the ratio test of convergence of numerical series with positive terms says that 'if $\lim a_{n+1}/a_n < 1$, then the series $\sum a_n$ is convergent'. The proposition 'if $\lim a_{n+1}/a_n \geq 1$, then the series $\sum a_n$ is divergent' is false.

2. Equivalence Theorem

Equivalence theorems are those theorems in which both the direct implication and the converse implication are true. Symbolically, '$p \Leftrightarrow q$' means '$p \Rightarrow q$' and '$p \Leftarrow q$'. In words, 'p is necessary and sufficient for q', or 'p if and only if q'.

Equivalence theorem says that the antecedent and the consequent are two different ways of expressing the same thing. Sometimes such theorems are called *characterization theorem*, since they characterize some mathematical object.

For example, A.J. White gives the following theorem of characterization of closed sets [White, p. 57]:

> Let (X,d) be a metric space, and let F be a subset of X. The following statements are equivalent:
> (i) F is closed,
> (ii) $X \sim F$ is open,
> (iii) if $\{x_n\}$ is a sequence in F converging to x then $x \in F$.

The existence of characterization theorems prove that mathematicians look at mathematical objects from different viewpoints. Through equivalence theorems we can therefore change perspectives. The choice of perspectives is part of authors' discourse.

3. Existence Theorem

There are two types of proofs of existence theorem: (i) ostensive proof (one exhibits the object or at least gives a construction of it), (ii) non-ostensive proof (one does not exhibit the object).

- **Ostensive Proof**

One can prove the existence of a mathematical object, by exhibiting it. For example, to prove that every non-zero complex number has an inverse, it is enough to show its inverse:

If $z = (x,y) \neq (0,0)$, then $z^{-1} = (x/(x^2 + y^2), -y/(x^2 + y^2))$.

- **Non-ostensive Proof**

Sometimes one proves the existence of a mathematical object, without exhibiting it. For example, one proves the Fundamental Theorem of Algebra, which asserts that every polynomial equation has at least a root, without exhibiting any root. The proof is by contradiction: the non-existence is absurd!

Another example is the proof of the existence of an infinity of prime numbers. This theorem is also proved by *reductio ad absurdum*. There is no way of exhibiting all the prime numbers.

It is a sad fact of life in mathematics that sometimes we have to prove the existence of objects, without knowing precisely what they are.

4. Uniqueness Theorem

Proofs of uniqueness theorems are standard: one supposes that there are two objects that satisfy the property and then show that they coincide.

Sometimes the uniqueness refers to representation. For example, when we represent a complex number in the form $z = x + iy$, we say that x is THE real part and y is THE imaginary part, because we prove or accept that the representation is unique. In other words, if $z = x_1 + iy_1$ and $z = x_2 + iy_2$, then $x_1 = x_2$ and $y_1 = y_2$.

3.2 HEURISTIC SCHEMES

Mathematicians use logical and heuristic schemes to construct mathematical theorems and theories, and to solve mathematical problems. These same schemes are also used in the communication of mathematical knowledge.

Heuristic reasoning in mathematics works in two directions: horizontal and vertical.

By horizontal movement we compare mathematical objects and classes that are at the same 'ontological' level, establishing similarities and differences, analogies and disanalogies of structure and behaviour.

By vertical movement we pass from one 'ontological' level to another one: by generalization, we pass from objects to classes, and by specification or particularization, we pass from classes to subclasses, or to particular objects.

One of the difficulties of understanding mathematical discourse is due to the fact that we have to deal with both movements at the same time. For example, the set of complex numbers is presented as an extension of the set of real numbers, that is, we pass from a class to a superclass. On the other hand, complex numbers are shown to be 'analogous' to real numbers because both are fields, metric spaces, and vector spaces.

In the following I will discuss two schemes of heuristic reasoning: analogy (horizontal movements) and generalization (vertical movements).

3.2.1 Analogy

According to the psychologists S. Vosniadou and A. Ortony,

> The ability to perceive similarities and analogies is one of the most fundamental aspects of human cognition. It is

> crucial for recognition, classification, and learning, and it plays an important role in scientific discovery and creativity.
>
> [Vosniadou & Ortony 89, p. 1]

According to G. Polya, analogy is a sort of similarity: 'similar objects agree with each other in some respect, analogous objects *agree in certain relations* of their respective parts' [Polya 71, p. 37].

This concept of analogy is formalized by the psychologist D. Gentner:

> An analogy is a mapping of knowledge from one domain (the base) into another (the target), which conveys that a system of relations that holds among the base objects also holds among the target objects. Thus analogy is a way of focusing on relational commonalties independently of the objects in which those relations are embedded.
>
> [Gentner 89, p. 201]

There are many kinds of analogies: analogy of objects, classes, functions, operations, theories, definitions, theorems, proofs, etc.

SIMILARITY OF OBJECTS

Mathematical objects are similar (or analogous) if they share attributes, that is, if they have common properties. In the language of classes, they are similar if they belong to the same class.

Sometimes mathematicians identify analogous objects. For example, fractions are identified with recurrent decimals. This identification comes from the need to regard fractions as particular instances of real numbers; it is then said that real numbers are a generalisation of rational numbers. Another way to say this is to blur the distinction between objects and signs: we say that fractions and recurrent decimals are different 'representations' of rational numbers.

ANALOGY OF CLASSES OF OBJECTS

Two classes of objects are analogous if they have the same mathematical structure. For example, complex numbers are analogous (similar) to real numbers because they have the same algebraic structure: both are fields. However this analogy is not too strong: the field of real numbers is ordered, whereas the field of complex numbers is not ordered. Complex numbers are also analogous to real numbers in a geometric sense: both are real vector spaces. However, this analogy is weak since they have different dimensions.

Authors of mathematics textbooks often use analogy as a rhetorical device, hiding the fact that analogous classes or objects also present disanalogies.

ANALOGY OF FUNCTIONS

Two functions are analogous if they have the same 'behaviour'. This means that they are instances of the same high-order concept. For example, continuous complex functions are analogous to continuous real functions, because complex numbers and real numbers have a common structure (they are topological spaces) that is preserved by continuous functions.

ANALOGY OF OPERATIONS

I will deal with this type of analogy in section 9.1, where the *principle of the permanence of the formal law* that guides the extension of operations is discussed.

ANALOGY OF THEORIES

Sometimes a complete mathematical theory is constructed based on analogy with known theories. Examples: (1) complex analysis is 'analogous' to Real analysis; (2) the theory of difference equations is 'analogous' to the theory of differential equations, (3) calculus of variations is 'analogous' to the theory of maxima and minima of functions.

ANALOGY OF DEFINITIONS

Analogy of definitions is important for the transference of techniques or intuitions. For example, the derivative of a complex function is formally analogous to the derivative of a real function. This implies that many algebraic theorems can be easily transferred from one area to another. However, the geometric interpretation is not transferable from real case to the complex case.

ANALOGY OF THEOREMS

The important thing about analogous theorems is that they have in general the same type of proof. An example, is the necessary condition for the minima or maxima of functionals in Calculus of Variations: Euler's differential equation is analogous to the equation $f'(x) = 0$ found in Calculus of Real Functions.

ANALOGY OF PROOFS

Authors of undergraduate mathematics textbooks often use analogy of proofs as an excuse for the omission of proofs.

ANALOGY OF PROBLEMS

George Polya's trilogy (**How To Solve It** (1971), **Mathematical Discovery** (1981), and **Mathematics and Plausible Reasoning** (1954)) is the best reference for this subject.

EXAMPLE: USE OF ANALOGY IN STEWART & TALL'S TEXTBOOK

Stewart & Tall assume that they base their discourse on two organising principles: analogy and geometric intuition. They declare in the preface that

> ... we have organised the material around two basic principles: (1) generalize from the real case; (2) when that reveals new phenomena, use the rich geometry of the plane to understand them.
>
> [Stewart & Tall, preface]

That generalization from the real case means use of analogy becomes clear from the following:

> Unlike the gradual emergence of the complex number concept, the development of complex analysis seems to have been the direct result of the mathematician's urge to generalize. It was sought deliberately, by ANALOGY with real analysis.
> [Stewart & Tall, p. 5]

The following collection of explicit references to analogy found in Stewart & Tall's textbook demonstrates how frequent is the use of heuristic schemes in their discourse.

ANALOGY OF DEFINITIONS

'The notion of a limit $\lim_{z \to z_0} f(z)$ is ANALOGOUS to the real case and its properties follow by similar arguments.' [Stewart & Tall, p. 25]

'By ANALOGY with the real case, we say that a series $\sum z_r$ is absolutely convergent if ... [Stewart & Tall, p. 54]

'If f is a complex function defined on an open set S, then, by ANALOGY with the real case, f is said to be differentiable at a point $z_0 \in S$ with derivative $f'(z_0) \in \mathbb{C}$ if ...' [Stewart & Tall, p. 64]

'The next part of the grand plan is to define complex integration by ANALOGY with the real case and establish the inverse relation between differentiation and integration.' [Stewart & Tall, p. 95]

'Note that this [definition of limit of a sequence of complex numbers] is IDENTICAL with the usual definition for real sequences.' [Stewart & Tall, p. 49] (Here they use the term 'identity' instead of 'analogy'.)

ANALOGY OF GEOMETRICAL REPRESENTATIONS

'There is a natural geometric representation of complex numbers as a plane, ANALOGOUS to that of the reals as a line.' [Stewart & Tall, p. 10]

ANALOGY OF PROPERTIES

'In this chapter we shall develop some elementary properties of sequences and series of complex numbers, mostly by direct ANALOGY with the real case, and then specialize to a deeper study of power series.' [Stewart & Tall, p. 48]

'In many ways results concerning complex differentiation follow naturally by ANALOGY with the real case.' [Stewart & Tall, p. 65]

ANALOGY OF IDEAS

'A SIMILAR idea gives us the complex version of the General Principle of Convergence' [Stewart & Tall, p. 50] (Here they use the term 'similarity' instead of 'analogy'.)

ANALOGY OF PROOFS

'Standard properties of complex limits may be proved by using methods ANALOGOUS to the real case' [Stewart & Tall, p. 26]

ANALOGY OF TECHNIQUES OF COMPUTATION

'The computations are ANALOGOUS to the real case; ...' [Stewart & Tall, p. 66]

DISANALOGY

'It is therefore not possible to use inequalities, ANALOGOUS to those for reals, when discussing complex numbers.' [Stewart & Tall, p. 18]

'We apply these [Cauchy-Riemann equations] to prove that if a function with connected domain has zero derivative, it must be constant. While the result is ANALOGOUS to the real case, the proof is not.' [Stewart & Tall, p. 64]

3.2.2 Generalization and Particularization

According to Polya, '*generalization* is passing from the consideration of one object to the consideration of a set containing that object; or passing from the consideration of a restricted set to that of a more comprehensive set containing the restricted one' [Polya 71, p. 108].

HOW THEOREMS ARE GENERALIZED

We consider implication theorems T of the form $H \Rightarrow C$, where H means hypothesis (or hypotheses) and C means conclusion. There are two basic ways of generalizing theorems: (i) by 'weakening' the hypotheses, or (ii) by 'strengthening' the conclusion.

- WEAKENING HYPOTHESES

One way of generalizing theorems is by 'weakening' hypotheses: the theorem T_2 is more general than theorem T_1 if the hypotheses of T_1 imply those of T_2 , but not vice-versa, while the conclusion is the same [Davis & Hersh 86, p. 134]. In this case, it is usual to say that we have 'weakened' the hypotheses of the theorem T_1; this means that the hypotheses of T_1 were super-restrictive, that there were more restrictions than necessary for the validity of the conclusion.

Davis & Hersh gave some examples of this type of generalization. I will give the possible 'logic' behind each generalization.

EXAMPLE 1

Theorem: Every number that ends in 0 is divisible by 2.

Generalization: Every number that ends in 0, 2, 4, 6, or 8 is divisible by 2.

Logic of generalization: There is a 'natural' partition of the set of natural numbers: numbers that end in 0, in 1, ..., in 9. The theorem affirms something about the first class. What about other classes?

EXAMPLE 2

Theorem: If ABC is an equilateral triangle, then the angle at A equals the angle at B.

Generalization: In an isosceles triangle, the base angles are equal.

Logic of generalization: the theorem affirms something about equilateral triangles. Triangles are classified in three classes: equilateral, isosceles or scalen. What about other classes of triangles?

EXAMPLE 3

Theorem: In a right triangle $c^2 = a^2 + b^2$

Generalization: In any triangle $c^2 = a^2 + b^2 - 2ab \cos C$

Logic of generalization: Pythagoras's theorem affirms something about the sides of a right triangle. Triangles can be right-angled or not. What about the other class?

• STRENGTHENING CONCLUSIONS

The second way of generalizing theorems is by 'strengthening' the conclusion, that is, the generalized theorem has the same hypotheses, while its conclusion implies the conclusion of the first theorem, but not vice-versa. This means that the first theorem has deduced less conclusions than it is possible to deduce.

EXAMPLE 4

Theorem: the complex exponential function is continuous.

Generalization: the complex exponential function is entire.

There is a third way to generalize theorems: by changing its setting or environment. For example, we can pass from the tri-dimensional environment $\mathbb{R}^3$ to the four-dimensional $\mathbb{R}^4$, or more generally, to the n-dimensional environment $\mathbb{R}^n$. Another example: we can pass from metric space to topological space, or from normed space to topological vector space, or from the field of real numbers $\mathbb{R}$ to the field of complex numbers $\mathbb{C}$. Despite the fact that this kind of generalization can be included into the first case of generalization, it is worthwhile to give it a particular relevance.

EXAMPLE 5

Theorem: if a 3-dimensional box has edges x, y, z, then its diagonal d is given by

$$d = \sqrt{x^2 + y^2 + z^2}$$

Generalization: if a n-dimensional box has edges $x_1, x_2, \ldots, x_n$, then its diagonal d is given by

$$d = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

This particular example shows how important is this change in the setting of theorems: we can transfer 'Euclidean' concepts to n-dimensional vector spaces, and more generally, to infinite-dimensional vector spaces.

3.2.3 Setting and Perspective

Mathematical discourse is dynamic; authors are constantly changing either the setting (scenery, environment) of the subject or the perspective (vantage point, viewpoint) or both.

For example, authors of Complex Analysis textbooks ordinarily introduce complex numbers, setting the subject 'complex numbers' in the algebraic context of generalization of real numbers: the problem is to construct a field $\mathbb{C}$ that extends $\mathbb{R}$, and allows a solution to the equation $x^2 + 1 = 0$.

Afterwards, by way of geometric interpretation, they change the perspective: they regard $\mathbb{C}$ as a vector space. And then, by way of the introduction of modulus, they change again the perspective: they regard $\mathbb{C}$ as a normed space. Finally, by way of introduction of distance, they regard $\mathbb{C}$ as a metric space.

By hindsight, the reader should understand the general plan of the author: to set the complex numbers as a set of numbers possessing three different structures: algebraic, geometric and topological structures.

Generalization and particularization participate in both setting and change of perspective. For instance, by exemplification, the author changes the level of generality and a fortiori changes the perspective, passing from general to particular, from the class to the object, from abstract to concrete. By referring to mother structures (fields, groups, vector spaces, topological spaces, etc.), the author rises the level of generality, in a converse movement.

In order to perceive the heuristic reasoning of setting and selecting perspectives which supports mathematical discourse, it is necessary to perform a movement like the movement of binoculars: from general to particular, and from particular to general.

3.3 RHETORICAL SCHEMES

The discourse of undergraduate mathematics textbooks is a mixture of formal and informal discourse. On the one hand, there is the discourse of formal mathematics, which is characterized by the discourse of logical explanation and argumentation; on the other hand, there is the discourse of informal mathematics, which is characterized by the discourse of pedagogic explanation and argumentation.

The discourse of informal mathematics is essentially a rhetorical discourse, a persuasive discourse, whereby meanings and values are negotiated between author and reader. It is driven by the author's educational and psychological conceptions (prejudices, bias, preferences), which often seem to be unconscious.

In this section I will discuss the most neglected of the functions of mathematical discourse: the rhetorical function. How does the author explain concepts without (or as well as) defining them? How does the author give evidence for the truth of propositions without (or as well as) proving them? What else does the author negotiate other than mathematics?

USE OF INTUITION

Intuition is necessary in mathematical discourse, because formal logic is not clear enough to explain concepts, and often has insufficient power to convince the student about the truth of propositions. I agree, together with Polya, with the statement of Hadamard about the main function of logic in mathematics:

> I think that everybody prefers intuitive insight to formal logical arguments, including professional mathematicians. Jacques Hadamard, an eminent French mathematician of our times, expressed it so: "The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there never was any other object for it." Yet, if we exclude professional mathematicians, almost nobody remains who would be in a position to properly appreciate formal arguments.
> [Polya 81, p. 129]

Intuition is a rhetorical tool; the author utilizes different strategies of intuitive explanation such as imagery, exemplification, metaphors, analogies, particularizations, models, etc. to convince the reader about the validity of a proposition.

In mathematical discourse it is convenient to distinguish two modes of intuitive explanation: EXAMPLES, and FIGURES. In other words, we have to investigate (i) the rhetorical function of examples, particular cases, models, etc., and (ii) the rhetorical function of figures, diagrams, tables, figures of speech, etc.

- NEGOTIATING INTUITION

Authors often negotiate intuition explicitly. As an illustration, I present below a collection of quotations from the textbook **Real Analysis** by A.J. White, where he discusses or negotiates intuition.

We take an entirely NAIVE AND INTUITIVE VIEW of set theory. [White, p. 1]

... INTUITIVE JUDGEMENTS used in more complex situations which involve conclusions REMOTE FROM OUR INTUITIONS may well lead to errors which are less easy to detect. [White, p. 16]

We aim at a standard of proof which eliminates any reliance on INTUITION, and in which each statement is a simple logical consequence of its predecessors. However, it should be clearly realized that this BANISHMENT OF INTUITION applies only to formal proofs. Obviously our axiom scheme will be chosen so as to capture as many as possible of our INTUITIVE IDEAS about the real numbers, and in this, as in any other mathematical study, a clear INTUITIVE UNDERSTANDING is essential to a proper grasp of the subject and as a means of conjecturing new results and suggesting methods of proof. [White, p. 18]

Our last axiom for ℝ [upper bound axiom] is in a different class from the others. Its role in the real number system was only perceived and crystallized towards the end of the nineteenth century, and so it cannot realistically be regarded as INTUITIVE in content. [White, p. 30]

The definition of metric space given above is an abstraction based largely on experience with the real numbers, and so it is natural, particularly since we live in a world which is (at any rate approximately) Euclidean, that our INTUITIONS about metric spaces should be Euclidean in character. [White, p. 45]

The word 'compact' itself probably gives the best INTUITIVE DESCRIPTION of the notion of compactness. [White, p. 64]

On the other hand, it can be shown that if

$$x_n = \sum_{r=1}^{n} 1/r^2$$

then $\{x_n\}$ converges to $\pi^2/6$; but it would require an extraordinary INTUITION to conjecture this result and remarkable technique to prove it by elementary methods. [White, p. 89]

It should be noticed that, although the notion of finite sum has a substantial INTUITIVE BACKGROUND, it still requires a formal definition to introduce it into our mathematical system. The idea of assigning a sum to an infinite subset of ℝ has a more meagre INTUITIVE CONTENT, and indeed INTUITION too freely applied can lead to serious error. [White, p. 187]

DISCOURSE OF OMISSION (CONCEALMENT, TACIT ASSUMPTION)

I call discourse of omission what the discourse analyst Teun A. van Dijk has called the ideology of the unsaid:

> The ideological nature of discourse in general, and of news discourse in particular, is often defined by the unsaid.
> [Dijk 86, p. 178]

This perfectly applies to mathematical discourse as I will try to show in many places.

GAPS

Authors of mathematics textbooks do not tell all the truth; mathematical discourse has in general many gaps. The author omits details intentionally. This omission of details is an important aspect of mathematical discourse.

It seems paradoxical that authors support the axiomatic treatment of mathematics, but give an exposition with several gaps. Some gaps are explicit; they are presented as exercises. Other facts are hidden, mainly because they can disturb in one way or another their discourse.

NEGOTIATION OF VALUES

The author negotiates other things besides mathematics: values, epistemological points of views, cognition, etc.

- NEGOTIATING IMPORTANCE

One way to negotiate the values of concepts, techniques and theorems, is by the phrase 'it is important' or similar phrases.

EXAMPLE. In order to illustrate how authors negotiate the importance of concepts and theorems, I present some quotations of the textbook **Real Analysis: An Introduction** by A.J. White.

We now introduce some operations on sets. It is IMPORTANT to obtain facility in handling these operations ...[White, p. 2]

We are now ready to make what is possibly THE MOST IMPORTANT DEFINITION IN MATHEMATICS - that of function. [White, p. 5]

It is IMPORTANT, particularly in formal arguments, to adhere strictly to the terms of the above definition. [White, p. 6]

It is frequently convenient, and often increases our understanding, to have alternative formulations of an IMPORTANT result or concept. [White, p. 31]

In this chapter we concentrate on another IMPORTANT aspect of the real numbers - the concept of distance. [White, p. 45]

Our final example concerns a simple but IMPORTANT METHOD of constructing a whole family of metric spaces from a given metric space. [White, p. 47]

The sequence $\{x_n\}$ is said to converge to x if ... This is ONE OF THE MOST IMPORTANT DEFINITIONS in analysis. [White, p. 49]

It is IMPORTANT to notice that the terms 'open' and 'closed', as they have been defined here, do not carry the overtones of their everyday meaning. [White, p. 56]

It is A FACT OF GREAT IMPORTANCE that the converse of Theorem 2-10 [Every compact subset of a metric space is closed and bounded] is true for the Euclidean metric spaces. [White, p. 66]

There is another IMPORTANT THEOREM concerning continuous functions on a compact metric space. [White, p. 68]

As a first IMPORTANT APPLICATION of the concept of uniform convergence, we prove that the uniform limit of a sequence of continuous functions is continuous. [White, p. 102]

• NEGOTIATING COGNITION

Authors negotiate cognition by using phrases such as 'IT IS EASY' or 'IT IS DIFFICULT', and similar phrases. Authors use these phrases as rhetorical devices to control the boundary between triviality and significance. I continue exhibiting examples from the same textbook **Real Analysis** by A.J. White.

According to White's definition, a metric space (X,d) is compact if every infinite subset of X has a cluster point in X. White states the following proposition, saying that it is true: A metric space (X,d) is compact if and only if every open cover of X has a finite subcover. [White, p. 67]

He begins the proof stating that 'half of this proposition is EASY', and then he proves this easy part. Afterwards, he says 'the other half of the proposition is more TROUBLESOME, and since we do not use the result anywhere the proof is omitted. [White, p. 68]

The phrase 'it is easy to see' means rhetorically that 'the reader should fill the details'. White is particularly fertile in devising new modes to say that 'the reader should fill the details':

IT IS EASY TO VERIFY ... [White, p. 2]

IT IS CLEAR THAT ... [White, p. 2]

... and IT FOLLOWS EASILY THAT ... [White, p. 29]

IT IS EASY TO PROVE (by induction) that ... [White, p. 85]

... and the sequence $\{\lambda a_n\}$ converges TRIVIALLY to 0. [White, p. 86]

IT IS EASILY VERIFIED THAT ... [White, p. 86]

IT IS NOT HARD to conjecture and prove that ... [White, p. 89]

EVIDENTLY, the sequence $\{f(x_n)\}$ in f[X] converges to 1 ... [White, p. 99]

IT IS TRIVIAL THAT ... [White, p. 100]

The proofs of these results are left as an exercise, which should present NO DIFFICULTY if the proofs of Theorems ... have been understood. [White, p. 107]

On the other hand, the phrase 'it is difficult' is a sign that something complicated is near to be presented.

This [inequalities] is an extensive and sometimes DIFFICULT SUBJECT. [White, p. 39]

... *ab initio* discussion of the integrability of the various functions commonly used in analysis is likely to be both TEDIOUS AND DIFFICULT. This reflection may make the rather dry and technical nature of the next few theorems a little more palatable. [White, p. 143]

The following result substitutes monotone convergence for uniform convergence; however, since the proof is rather STRENUOUS, we give only a sketch. [White, p. 166]

3.4 ANALYSIS OF MATHEMATICAL DISCOURSE: AN OUTLINE

Mathematical discourse is a discourse through which meanings, truth, intuitions, values, behaviour, philosophical and ideological viewpoints, etc. are negotiated between author and reader. Discourse analysis of mathematics textbooks is the investigation of such negotiations.

My approach is to regard mathematics textbooks as the battlefields of the struggle between logic, heuristic and rhetoric. Therefore, this approach gives relevance to the discursive schemes used by the author throughout the textbook. In these negotiations authors use logical, heuristic and rhetorical schemes. Logical schemes are used to make the presentation **rigorous,** heuristic schemes are used to make it **comprehensible**, and rhetorical schemes are used to make it **acceptable.**

In the following I present an outline of what will be developed in the following chapters.

Mathematical discourse has an informational function. The organisation of mathematical information involves description, explanation, and argumentation, that is, the informational function is a sum of *descriptive function*, *explanatory function*, and *argumentative function.*

NEGOTIATION OF MATHEMATICAL OBJECTS

Mathematical discourse is about mathematical objects or concepts and properties and relations between concepts. Authors of undergraduate mathematics textbooks have a bunch of concepts (ideas, notions) to impart to their readers. Among these concepts there are few fundamental concepts that deserve deep analysis. Concepts are in general presented through formal DEFINITIONS. Therefore, analysis of the negotiation of concepts is synonymous to analysis of definitions. This theme will be developed in chapter 4.

NEGOTIATION OF TRUTH

Properties of concepts or relations between concepts are presented in the form of THEOREMS, LEMMAS, PROPOSITIONS, COROLLARIES, etc., whose truth is negotiated through PROOFS or RHETORICAL ARGUMENTS. This theme will developed in chapter 5.

NEGOTIATION OF COMPREHENSION

Logical arguments are not enough to make readers to understand the text. Mathematical discourse is also the discourse of pedagogic explanation and argumentation. Therefore the authors make use of illustrative devices: EXAMPLES and FIGURES. This theme will also developed in chapter 5.

NEGOTIATION OF ACTIVITIES

Mathematical discourse has a heuristic function. Authors of mathematics textbooks organise readers' activities through lists of EXERCISES or PROBLEMS. Moreover, authors leave several GAPS in the text to be filled by the readers. This means that the reader is periodically invited by the author to construct his/her own mathematical knowledge. The discourse analyst should then evaluate what kind of heuristic schemes the reader has to use in order to solve the exercises, and fill the gaps. This will be theme of chapter 6.

OTHER NEGOTIATIONS

Authors negotiate other things such as prerequisites, applications, and contextualization. These topics will be developed in the same chapter 6. Negotiation of ambiguities in nomenclature and notation will be discussed in chapter 7.

ORGANISATION OF DISCOURSE

Authors organise the 'content' and the 'form' of presentation in different ways. I will call ARCHITECTURE OF MATHEMATICS the way the author organises the network of DEFINITIONS and THEOREMS. Naturally I will call STYLE the way the author organises his or her discourse. These topics will be developed in chapter 8.

PART II

NEGOTIATION

OF

MATHEMATICAL KNOWLEDGE

CHAPTER 4

NEGOTIATION OF MATHEMATICAL OBJECTS

Introduction

Authors of undergraduate mathematics textbooks display their discourse basically in terms of *definitions*, *theorems* and *proofs.* In this chapter I will discuss the nature of mathematical definition, its types and functions.

In the first section, I will discuss the conceptions of definition held by three divergent schools: realism (Plato), nominalism (John Stuart Mill, Robinson) and conceptualism (Kant).

In section 2, I will deal with mathematical definition, focusing on the problem of the conception of mathematical object. In this case, I will follow Kant's conceptualism, accepting that mathematical objects are invented and not discovered. Furthermore, I will stress the distinction between formal and informal definitions.

In section 3, I will describe some functions of mathematical definition. By means of analysis of definitions, it is possible to accompany how the theory develops driven by one of the four principles: specification, generalization, classification and extension by analogy.

4.1 GENERAL THEORY OF DEFINITION

A definition involves two expressions: the term to be defined called *definiendum*, and the defining expression called *definiens*.

For example, in the mathematical definition 'if f is differentiable everywhere in a domain S except for points at which f has poles, then f is said to be *meromorphic*', the *definiendum* is 'meromorphic function' and the *definiens is* 'function which is differentiable everywhere in a domain except for points at which it has poles'.

NOMINAL AND REAL DEFINITIONS

In traditional logic two kinds of definitions are recognized: real and nominal definitions.

• HEMPEL'S THEORY OF DEFINITION

According to the philosopher of science Carl G. Hempel, a *real definition* is a statement of the "essential nature" or the "essential characteristics" of some entity. For example, 'man is a rational animal' is a real definition because rationality characterizes the essence of the human species [Hempel 52, p.2].

On the other hand, a *nominal definition* is a convention which introduces an alternative - and usually abbreviatory - notation for a given linguistic expression. It may be characterized as a stipulation to the effect that a specified expression, the *definiendum*, is to be synonymous with a certain other expression, the *definiens*, whose meaning is already determined. Therefore, a nominal definition introduces a "new" expression and gives it meaning by stipulation [*Ibidem*, p. 2].

Hempel soon replaces this 'nominalistic definition' of nominal definition by a more 'realistic definition':

> We may say that a nominal definition singles out a certain *concept*, i.e., a non-linguistic entity such a property, a class, a relation, a function, or the like, and, for

convenient reference, lays down a special name for it. [Hempel 52, p. 4]

Hempel criticizes the "definition" of real definition, because the notion of essential nature is too vague. He replaces it by other two types of definitions: (i) empirical analysis, and (ii) meaning analysis.

Empirical analysis is not related to mathematics because 'it states characteristics which are, as a matter of empirical fact, both necessary and sufficient for the realization of the phenomenon under analysis'.

Meaning analysis or *analytic definition* is concerned with an expression which is already in use and makes its meaning explicit by providing a synonymous expression, which has to be previously understood. Dictionaries for a natural language are intended to provide analytic definitions for the words of that language.

- COHEN & NAGEL'S THEORY OF DEFINITION

According to the logicians M.R. Cohen and E. Nagel, a *nominal definition* is an agreement or resolution concerning the use of verbal symbols: a new symbol is to be used for an already known group of words or symbols. The definiendum is thus to have no meaning other than the definiens [Cohen & Nagel 36, p. 228].

In a real definition, they say, the definiens is an analysis of the idea, form, type, or universal symbolized by the definiendum. Both the definiens and the definiendum possess a meaning independently of the process of definition which equates them. 'A real definition, therefore, is a genuine proposition, which may be either true or false' [Cohen & Nagel 36, p. 230]. This means that they interpret real definitions as if they were Hempel's analytic definitions.

MATHEMATICAL DEFINITIONS

I am interested in discussing the nature of mathematical definitions: Are they nominal or real definitions?

Pascal, for example, in his **L'Esprit de la Geometrie** does not recognize definitions in geometry other than nominal definitions (*definition de nom*). The mathematician G. Peano has generalised this opinion, saying categorically that 'in mathematics all definitions are nominal' [Peano 73, p.237]. We can call them nominalists with respect to definitions.

Cohen & Nagel, in opposition to Peano, believe in the existence of real definitions in mathematics. They say, for example, that the following definition is a real definition:

Figure A is similar to figure A' .=. The ratio of the distance between any two points P, Q, on A and the distance between the corresponding points P', Q', on A' is constant. Df.

According to Cohen & Nagel, this is a true definition of what is ordinarily meant by similar figures, because (i) the right-hand side means precisely what the left-hand side does, and at the same time (ii) the right-hand side offers an analysis of the structure of that which both sides symbolize.

Following Cohen & Nagel, we could say by the same token that the mathematical definitions of continuity of a function, stability of a dynamical system, and probability of an event are real definitions as well.

This dispute between 'realists' and 'nominalists' within mathematics is just a reflection of a more general philosophical divergence. According to R. Robinson, author of a monograph on definition, the major philosophical problem in the theory of definition is precisely the following one:

> Do we define things, or words, or concepts? In other words, are we to be realists, or nominalists, or conceptualists, about definition?
>
> [Robinson 54, p. 7]

Translating this question to mathematics: are mathematical objects things, words or concepts?

As it is expected, there are three philosophical doctrines (schools) about definitions: realism, nominalism and conceptualism.

REALISM

Realists support that definitions should reveal the essence or nature of the things defined. The Greek philosophers Socrates, Plato and Aristotle are representatives of realism. 'Definition is in fact, according to Plato, the end of the process of getting to know the most real things there are, which he called Forms or Ideas' [Robinson 54, p. 8].

NOMINALISM

Plato's belief in the objectivity of knowledge was undermined by the nominalism of Sophists, Skeptics and Cynics. A more recent nominalist was John Stuart Mill:

> The simplest and most correct notion of a Definition is a proposition declaratory of the meaning of a word: namely, either the meaning which it bears in common acceptance, or that which the speaker or writer ... intends to annex to it.
>
> [Mill 1879, p. 86]

A contemporary nominalist is Robinson:

> I propose that by *definition* we always mean a process concerning symbols, a process either of equating two symbols or of reporting or proposing a meaning for a symbol; and that we never use *definition* as a name for a process that is not about symbols, because in that usage it is ambiguous and should be replaced by more specific terms.
>
> [Robinson 54, p. 191]

Robinson, despite his nominalism, points out the occurrence in mathematics of the following five kinds of "definitions":

1. abbreviations,
2. the nominal definition of the symbols of the system,
3. the analysis of concepts,
4. the analysis of concepts into specified concepts of the system,
5. the improvement of concepts.

According to Robinson, the first two activities are nominal definitions, whereas the others are not genuine definitions.

Robinson recognises then that mathematicians, besides giving nominal definitions, are also involved with analysis and improvement of concepts. Immanuel Kant has concentrated his attention on these latter activities.

CONCEPTUALISM

Kant is the great supporter of conceptualism: 'to define, as the word itself indicates, really only means to present the complete, original concept of a thing within the limits of its concept' [Kant 90, p. 586].

According to Kant, there are three kinds of concepts: empirical, a priori, and invented concepts. Empirical concepts cannot be defined but only be *made explicit*. Moreover, he asks, what useful purpose could be served by defining an empirical concept, such as, for instance, that of water?

Concepts given a priori such as cause, substance, right, equity, etc., cannot be defined as well, but only be *exposed*. Therefore 'the only remaining kind of concepts, upon which this mental operation [definition] can be tried, are arbitrarily invented concepts':

> A concept which I have invented I can always define; for since it is not given to me either by the nature of understanding or by experience, but is such as I have myself deliberately made it to be, I must know what I have intended

to think in using it.

[Kant 90, p. 587]

Consequently,

> ... *mathematics is the only science that has definitions.* For the object which it thinks it exhibits a priori in intuition, and this object certainly cannot contain either more or less than the concept, since *it is through the definition that the concept of the object is given* - and given originally, that is, without its being necessary to derive the definition from any other source.
>
> [*Ibidem*, p. 587]

Kant can also be considered as a forerunner of constructivism: 'while philosophical definitions are never more than expositions of given concepts, *mathematical definitions are constructions of concepts*, originally framed by the mind itself' [*Ibidem*, p. 587].

It is interesting that in the theory of definition we came across the same or at least analogous philosophical currents which pervade mathematical education, namely realism or Platonism, nominalism or formalism, and conceptualism or constructivism!

4.2 MATHEMATICAL DEFINITION

Discussion about mathematical definition falls ultimately to the notion of mathematical concept or object. Are there mathematical objects? What is their nature?

According to formalists, there are no mathematical objects. Mathematics just consists of axioms, definitions and theorems - in other words, formulas [Davis & Hersh 86, p. 319].

According to Platonists, mathematical objects are real. Their existence is an objective fact, quite independent of our knowledge. They are immutable - they were not created, and they will not change or disappear [Davis & Hersh 86, p. 318].

The analysis of mathematical discourse I propound is based on a distinct perspective.

Contrary to formalism, I believe that mathematical objects exist, that mathematical activity is not a meaningless game. Contrary to Platonism, I see *mathematical objects* as products of human creation. They exist, they are objective in so far as they are cultural objects. In other words, mathematicians, through the act of definition, reify *concepts*, that is, transform a diffuse and vague concept into a mathematical object, that is, a cultural object, an object of social use and manipulation.

This perspective is very akin to the one exposed by Davis & Hersh in the book **The Mathematical Experience**. They reconcile the two seemingly contradictory propositions, that (1) mathematics is a human invention or creation, and that (2) mathematics is an objective reality, in the sense that mathematical objects have definite properties. In short, I make mine their dictum:

> Mathematics does have a subject matter, and its statements are meaningful. The meaning, however, is to be found in the shared understanding of human beings, not in an external nonhuman reality. In this respect, mathematics is similar to an ideology, a religion, or an art form; it deals with human meanings, and is intelligible only within the context of culture. In other words, mathematics is a humanistic study. It is one of the humanities.
>
> [Davis & Hersh 86, p. 410]

This viewpoint is the reverse of Platonism, in the sense that mathematical objects do not exist a priori as Perfect Forms, waiting to be discovered by humankind. On the contrary, they are cultural objects created by mathematicians, acquiring significance through the social exchange of meanings. Like any other scientific theories, mathematical theories are developed, corrected, refuted, forgotten, or rejected. We could say that the Platonic world of mathematical objects exists, but as a result of human action.

FORMAL AND INFORMAL DEFINITIONS

The job of mathematicians is to produce (create or invent) new mathematical objects, and investigate their properties and mutual relations. These new objects are introduced into the mathematical community through mathematical definitions, whether formal or informal.

Generally speaking, I will say that an author is giving a **formal definition** of a concept when he/she is defining it in terms of previously accepted mathematical concepts. This mathematical 'formalism' should not be confounded with symbolic or linguistic formalism. For example, a definition of simply connected region in terms of 'holes' is informal. In fact, it cannot be transformed into a formal definition, whatever the linguistic form it takes, because 'hole' is not an acceptable mathematical concept.

I will call informal definitions *explications* or *explanations*.

Authors of mathematics textbooks introduce new mathematical concepts in several ways: (1) solely by means of a formal definition, (2) by means of a formal definition together an explication, (3) solely by means of an explication, (3) by means of a particular case, (4) by means of an example.

EXAMPLE

In [Derrick, p. 24] we can read the following formal definition: 'a set is *open* if all its points are interior points'. In this definition, 'open set' is the definiendum, and 'a set such that all its points are interior points' is the definiens.

Derrick defines open set in terms of interior points. The definition of interior point of a set arises from the analysis of the topological relation between a generic point of the complex plane and the given set. In other words, given a set S, the points of the complex plane can be classified in three categories: a point z_o of the complex plane is an *interior point* (some ε-neighborhood of z_o is contained entirely in S), an *exterior point* (some ε-neighborhood of z_o is contained entirely in the complement of S), or a *boundary point* of S (every ε-neighborhood of z_o contains points in S and points not in S).

Observe that the words 'interior', 'exterior' and 'boundary' are descriptive words; they convey intuitive meanings to the mathematical objects. Because of this, Derrick's definitions do not require further explication.

Other authors prefer to first introduce an 'analytic' definition. Marsden, for example, gives the following formal definition: 'a set A is open if, for each point $z_o \in A$, there is a real number ε, $\varepsilon > 0$, such that if $|z - z_o| < \varepsilon$, then $z \in A$.'

Marsden knows that this formal definition is elusive, and hence that the readers need a complementary intuitive explication. He explicates the formal definition, saying that:

"This means that some small disk around z_o lies in A. Note that the value of ε depends on z_o; as z_o gets close to the 'edge' of A, ε gets smaller. Intuitively, a set is open if it does not contain any of its 'boundary' or 'edge' points." [Marsden, p. 41]

Another author, Priestley, after giving a definition equivalent to Marsden's, explicates openness in a different way:

"Informally, S is open if, from any given point in S, there is room to move some fixed positive distance in any direction without straying outside S; how large this distance can be will vary from one point to another. It is a need for such 'elbow room' that dictates that the sets in so many of our later theorems be open." [Priestley, p.6]

These examples illustrate the fact that in general formal definitions are not clear-cut; supplementary explication are often necessary for their understanding.

Formal and informal definitions have opposite features. On the one hand, formal definition tends to be concise, objective, precise, rigorous, and logical. On the other hand, informal definition tends to be detailed, value-laden, diffuse, motivating, and intuitive.

4.3 FUNCTIONS OF DEFINITIONS

According to the philosopher Mario Bunge, the chief functions of definition in science are the following [Bunge 67, p. 136]:

(1) *formation (introduction) of new signs.* This may be done mainly with the aim of abbreviating or simplifying expressions.

(2) *formal introduction of new concepts* on the basis of old concepts.

(3) *specification of meaning*: defining is a way of determining the meaning of terms which may have been in pre-systematic usage before.

(4) *interrelation of concepts*: by linking concepts, definition contributes to organization or systemicity.

(5) *identification of objects*: together with descriptions, definitions supply criteria for the recognition of objects.

(6) *logical hygiene*: ambiguity and vagueness can be reduced if the terms are defined.

(7) *precise symbolization* of certain concepts, and consequently exact analysis of them.

In my view, the most important functions of definition in mathematical discourse are those Bunge has mentioned in (2) and (4): mathematical definitions contribute to development of the theory, introducing new concepts, and to organization and systemicity of the theory, linking concepts. This is in line with Cohen and Nagel who state the following logical functions of definition:

> Logically, definitions aim to lay bare the principal features or structure of a concept, partly in order to make it definite, to delimit it from other concepts, and partly in order to make possible a systematic exploration of the subject matter with which it deals.
>
> [Cohen & Nagel 36, p. 232]

How do definitions arise in mathematics? To answer this question it is necessary to know the strategies and methods used by mathematicians to develop new concepts. In the following I will discuss some of these strategies: (1) specification, (2) generalization, (3) classification, and (4) extension by analogy.

4.3.1 Specification

The development of mathematics is driven by two opposite forces: a tendency for generalization, and a tendency for specification (or particularization). The latter tendency is responsible for the constant use in mathematics of definitions by specification.

Definition by specification or definition by genus and difference contains two components, the *genus* and the *differentia*. For example, in the definition 'circle is the plane figure having all its points equidistant from a fixed point', the genus of "circle" is "plane figure" and the differentia is "having all its points equidistant from a fixed point".

In other terms, definition by specification means that a subclass (species) of a given class (genus) is being determined.

Mathematicians do not define mathematical concepts without a purpose. The important thing in the analysis of definitions is to know the reasons for such definitions, why such a subclass or such a property is being chosen.

For example, why do we define absolutely convergent series in the study of series of complex numbers?

These series are interesting for many reasons. One of them is practical: we reduce the investigation of convergence of a complex series to a known subject which is simpler, the investigation of the convergence of a series of positive real terms. Another one is theoretical: to separate convergent series in 'strongly' convergent and 'weakly' convergent, where 'strong' means convergence independently of the signs of the terms. Another one is operational: we can multiply absolutely convergent series, whereas multiplication of two convergent series is not always possible. Whatever the reason, mathematicians have created the species (absolutely convergent series) from the genus (convergent series).

COMPLEMENTARY DEFINITION

A definition by specification implies the automatic definition of the *complementary definition*, that is, the definition of the negative of the concept. For example, 'divergent series is a series that is not convergent'.

Generally speaking, the primary concept is simpler than the complementary concept, because there is a natural tendency to conceptualize simple things instead of complex things. However, the analysis of the complementary concept can be important to show the limits of the primary concept. For instance, continuity becomes clearer when the concept of discontinuity is made more explicit. The concept of convergent series becomes clearer after exemplification of some divergent series.

Nonexamples, anti-examples or counterexamples have the explicative function of delimiting the range of the concept, by showing what the concept cannot be. Few authors are aware of the importance of complementary definitions.

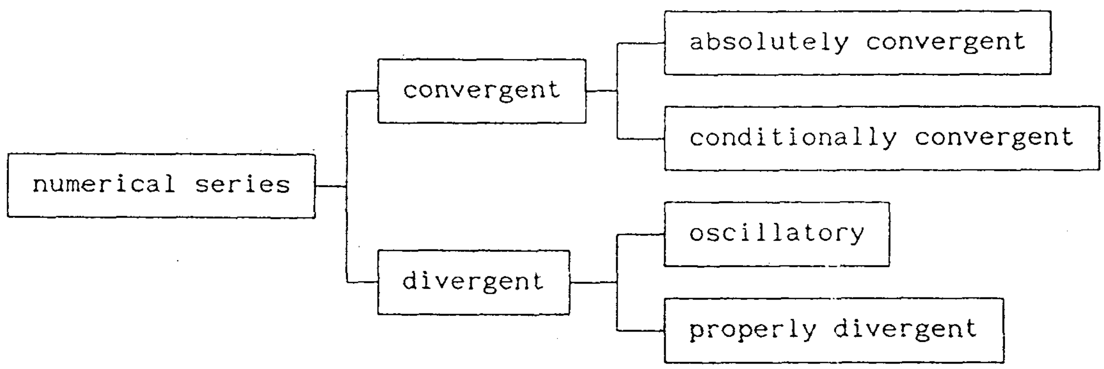
Note 1: definition by genus and difference requires proof of existence, otherwise it can be void or completely full. This is usually provided by means of examples and anti-examples.

Note 2: a specification can be regarded as a particular kind of classification (the species and the complement).

TREE OF DEFINITIONS

Successive definitions by specification create a structure that can be called *a tree* of definitions. For example, if we start with numerical series, the definition of convergence divides the class of numerical series into two subclasses: convergent and divergent series. Afterwards, the concept of absolute convergence divides the subclass of convergent series into two sub-subclasses: absolutely convergent and conditionally convergent series. The subclass of divergent series in its turn can be divided into oscillatory and properly divergent series.

Diagrammatically:



4.3.2 Generalization

There exist at least two different notions of generalization:

(i) the creation of a super-class (as opposite to specification that creates a sub-class): semicontinuity is a generalization of continuity, in the sense that continuous functions are automatically semicontinuous, i.e., the class of semicontinuous functions contains the class of continuous functions.

(ii) generalization by abstraction of properties. For example, topological space is a generalization of metric space, in the sense that metric space is an instance of topological space. The change in this case is deeper than in the first case.

4.3.3 Classification

Classification is more global than specification. Specification means dichotomous classification: we select a property, and then we divide a class into exactly two classes: the class of the members that satisfy the property, and the complementary class.

In classification, other criteria can be present. For example, isolated singularities of complex functions can be classified, according to the 'behaviour' of the function in their neighborhoods, into removable singularities, poles, and essential singularities.

4.3.4 Extension by Analogy

The majority of definitions in Complex Analysis are of this type. I will illustrate with an example how this type of definition appears.

Consider the definition of complex exponential function. The problem is: how to define a 'good' extension of the real exponential function, that is, a function $f\colon \mathbb{C} \to \mathbb{C}$ that satisfies the equation

$$f(x) = e^{x}, \text{ for all } x \in \mathbb{R},$$

and inherits many of the 'good' properties of the real exponential function.

There are at least three possible solutions to this problem:

1. Ahlfors defines the complex exponential function as the solution of the differential equation

$$f'(z) = f(z)$$

with the initial condition $f(0) = 1$. See [Ahlfors, p. 43].

2. Marsden defines the complex exponential function by means of a formula involving its real and imaginary parts: if $z = x + iy$, then

$$e^z = e^x(\cos y + i \sin y)$$

See [Marsden, p. 25].

3. Priestley defines the complex exponential function by the power series

$$e^z = \sum z^n/_{n!}$$

See [Priestley, p. 22].

An extension by analogy needs explanation and justification. Marsden, for example, uses two pages to persuade the reader that his definition is acceptable.

The definition of complex exponential function requires a reconceptualization of real exponential function. Ahlfors's definition is intrinsic; it mimics the real case, but it does not depend on it. Marsden's definition is operational, he gives a formula. It is perhaps the easiest one: easy to memorize and to manipulate. However, it lacks motivation. Priestley's definition is the classical one, but it requires proof of convergence of a power series.

MULTIPLICITY OF DEFINITIONS

As we saw in the case of complex exponential function, the same concept can be seen from different points of view. This is reflected, for example, in the theorems of equivalence type, in which several propositions or properties are shown to be equivalent. Therefore, in this case one can choose one of these properties as the defining property of the involved concept.

For example, there are two equivalent definitions of 'compact set' in $\mathbb{C}$:

(1) A set in $\mathbb{C}$ is compact if it is closed and bounded,

(2) A set X in $\mathbb{C}$ is compact if every family of open sets that covers X has a finite subfamily that covers X.

Authors of textbooks on Complex Analysis will choose (1) or (2) according to their purposes and personal inclinations. Observe that (1) is problem-solving oriented, it is more operational than (2), for it works as a practical criterium of compactness, whereas definition (2) is theory-building oriented, it is more convenient than (1) to be used in the proofs of theorems.

Another difference occurs at the level of generality: definition (2) is more general than definition (1) in the sense that it can be applied for general topological spaces, where the first definition has no meaning in this context.

In this case, the choice can reveal author's preference for one of the poles of the following two dilemmas: problem-solving versus theory-building, and generality versus particularity, discussed respectively in section 2.3 and section 3.2.

Note I will discuss more on definitions in section 9.1, which deals with the introduction of the concept of complex number.

By way of conclusion, in my view, the analysis of definitions in mathematical discourse should be orientated in order to detect authors' strategies in the choice of alternative definitions.

In short, this analysis should allows us to know what the philosopher G. Granger calls *mathematical style*:

> Style can be regarded, on the one hand, as a certain manner of introducing the concepts of a theory, of linking them, of unifying them; on the other hand, as a certain manner of delimiting the intuitive content in the determination of these concepts.'
>
> [Granger 74, p. 30]

CHAPTER 5

NEGOTIATION OF TRUTH AND COMPREHENSION

Introduction

In section 1, I am concerned with how authors of undergraduate mathematics textbooks negotiate truth by means of proofs or rhetorical arguments of the theorems.

Examples and figures are usually considered as pedagogic tools that aid the understanding of mathematical definitions and theorems. They are illustrative. They are rhetorical tools.

In section 2, I will discuss the role of examples in mathematical discourse. Exemplification should be contrasted to setting. Exemplification lowers the level of generality and abstractness, whereas setting goes in the opposite direction.

In section 3, I will discuss the role of figures in mathematical discourse. The use of figures should be contrasted with the use of words and symbols. According to the psychologist A. Paivio, imagery and verbal language are interconnected.

5.1 NEGOTIATION OF TRUTH: PROOF OR RHETORIC?

Authors of undergraduate mathematics textbooks do not prove all the theorems. For example, when the proof is too 'easy', authors usually do not prove the theorem, leaving it to the reader. In the case the proof is too 'difficult', authors do not prove it as well, sending the reader to more advanced book.

'Ease' and 'difficulty' are not objective, but subjective matters. This implies that proving or not proving is essentially a subjective matter, a choice of each author. In other terms, the truth of the theorems is a matter of negotiation between author and reader.

Even when the authors give a proof, such proof can be a partial proof in the sense that some details are missing. This means that each author negotiates the truth of theorems in a particular way, depending on the characteristics of the theorems.

MODES OF NEGOTIATING PROOFS

Authors negotiate the truth of a theorem in different ways:

1. giving a complete formal proof, where 'complete' means that there are no important points missing in the proof, and 'formal' means that the reasoning is based only on formal logic,
2. giving a partial proof, omitting parts or details of the proof,
3. giving an outline of the proof,
4. giving the proof in a particular case,
5. giving an informal proof,
6. giving a wrong proof,
7. omitting the proof, but indicating how to prove (for example saying that the proof follows by induction, analogy, etc.),
8. omitting the proof, but giving examples for which the theorem holds,
9. omitting the proof, but leaving it to the reader,
10. omitting the proof, but sending readers to read references,
11. omitting the proof, and saying nothing else,
12. omitting both the formal statement and the proof of the theorem.

Note: This classification of kinds of negotiation of truth is neither exhaustive nor mutually exclusive. There can be for example a combination of these cases.

In the following example I show how a particular pair of authors make use of these rhetorical devices.

EXAMPLE 1: STEWART & TALL

In their textbook **Complex Analysis** (1988), I. Stewart and D. Tall negotiate the proofs of theorems with the readers in different ways. I have found instances of all cases, except cases 4 and 12.

Case 1: Many of the theorems, as is expected in a textbook, have complete formal proofs. Among them I cite the Fundamental Theorem of Algebra, which is provided with two complete formal proofs. [Stewart & Tall, p. 185, p. 233]

Case 2: Many parts of the proof of the first theorem, Theorem 1.1, are left to the reader. [Stewart & Tall, p. 11]

Case 3: Stewart & Tall leave the proof of Schwarz's Reflection Principle as an exercise, giving a hint, which can be considered a sketch of the proof. [Stewart & Tall, p. 286]

Case 5: All the treatment of analytic continuation and Riemann surface is informal. [Stewart & Tall, chapter 14]

Case 6: The proofs of proposition 4.3 and proposition 5.2 are wrong, due to misprints. [Stewart & Tall, p. 66, p. 89]

Case 7: Parts of the proof of proposition 2.2 are left, because they are analogous to the real case. [Stewart & Tall, p. 26]

Case 8: They state Picard's theorem, give two examples, however they do not give the proof, because its proof 'requires machinery considerably beyond the reach of this text'. [Stewart & Tall, p. 205]

Case 9: De Moivre's Theorem is left as an exercise with an indication (proof by induction). [Stewart & Tall, p. 20]

Case 10: The reader is sent to another textbook to see the properties of Joukowski transformation. [Stewart & Tall, p. 256]

Case 11: Jordan Contour Theorem is just mentioned. [Stewart & Tall, p. 157]

In the following example I show that different authors negotiate the truth of the same theorem in different ways. This example, the Generalized Associative Law, is interesting because, besides negotiating the proof of the theorem, authors are involved in negotiating its statement.

EXAMPLE 2: GENERALIZED ASSOCIATIVE LAW

The following five authors omit the proof of the theorem, however they justify this omission in different ways.

First author. Birkhoff & Bartee acknowledges the difficulty of proving and even stating the generalized associative law:

> One learns in high-school algebra the following somewhat imprecisely stated principle: *the result of performing any associative operation on a sequence of n terms depends only on the order in which they are listed and not on the order in which they are combined.* To actually prove this principle, called the *generalised associative law*, from the simple three-term associative law, or indeed to even formulate it precisely, REQUIRES SOME RATHER SOPHISTICATED REASONING.
>
> [Birkhoff & Bartee, p. 25]

After this paragraph, Birkhoff & Bartee illustrate the theorem for the case of five elements, however they omit the proof of this theorem and the statement as well.

Second author. Blyth & Robertson are extremely pragmatic:

> There is, in fact, a general result which say that, for an associative law, the way in which compound expressions are bracketed does not affect the result. THE PROOF OF THIS IS SOMEWHAT INVOLVED, SO WE SHALL SIMPLY ACCEPT IT.
>
> [Blyth & Robertson, p. 2]

Third author. F.M. Hall uses the rhetoric of the obviousness of the theorem, to justify the omission of the proof:

> It is possible to give a formal proof by induction, but the result is so obvious that TO DO SO WOULD RAISE UNNECESSARY COMPLICATIONS.
>
> [Hall, p. 201]

Notice the contrast between Hall and Birkhoff & Bartee: for Hall the result is obvious, however for Birkhoff & Bartee, the result is difficult even to formulate.

Fourth author. S. Lang avoids the statement and the proof of the theorem, due to the complicated notations involved in it:

> We omit the proof in the general case (done by induction), because IT INVOLVES SLIGHT NOTATIONAL COMPLICATIONS WHICH WE DON'T WANT TO GO INTO.
>
> [Lang, p. 14]

Fifth author. G.D. Mostow omit the proof because there are more interesting things to do:

> The assertion of this theorem is quite easily proved in any given special case where the number of elements involved is relatively small, and the truth of the theorem soon become rather obvious. A complete proof of the theorem, however,

make use of mathematical induction, which will not be treated until the next chapter. We shall omit the proof of Theorem 4.2 **in order to get on quickly with more interesting matters.**

[Mostow, p. 9]

The following two authors omit the proof, leaving it to the reader as an exercise.

Sixth author. J.T. Moore indicates the method of proof:

Problem. Use the Second Principle of Induction to prove the "generalized" associative law: Any product $a_1.a_2.a_3 \dots a_n$ is independent of the position of parentheses.

[Moore, p. 27]

Seventh author. I. Herstein gives the method of proof and an example:

Problem. If S is a set closed under an associative operation, prove that no matter how you bracket a_1, a_2, ..., a_n, retaining the order of the elements, you get the same element in S (e.g. $(a_1.a_2).(a_3.a_4) = a_1.(a_2.(a_3.a_4))$; use induction on n.)

[Herstein, p. 32]

Eighth author. J. Gilbert and L. Gilbert give the "wrong" statement. They call generalized associative law what is a lemma.

Generalized Associative Law. Let $n > 2$ be a positive integer, and let a_1, a_2, ..., a_n denote elements of a group G. For any positive integer m such that $1 \leq m < n$,

$$(a_1 a_2 \dots a_m)(a_{m+1} \dots a_n) = a_1 a_2 \dots a_n$$

[Gilbert & Gilbert]

The following authors prove the theorem in different ways.

Ninth author. S. Perlis uses the above theorem as a lemma to prove the Generalized Associative Law [Perlis, p. 20].

Tenth author. N. Jacobson is another writer that introduces a lemma in order to make the proof "easier". Nevertheless, he does not give the formal statement of the theorem. [Jacobson, p. 39]

Eleventh author. Bourbaki' Associativity Theorem is probably the most general of the Generalized Associative Laws. His statement is the following:

Let E be an associative magma whose law is denoted by T. Let A be a totally ordered non-empty finite set, which is the union of an ordered sequence of non-empty subsets $(B_i)_{i\in I}$ such that the relations $\alpha\in B_i$, $\beta\in B_j$, $i < j$ imply $\alpha < \beta$; let $(x_\alpha)_{\alpha\in A}$ be an ordered sequence of elements in E with A as indexing set. Then

$$T_{\alpha\in A}\, x_\alpha = T_{i\in I}\, (T_{\alpha\in B_i}\, x_\alpha)$$

[Bourbaki, p. 4]

Twelfth author. J. Lelong-Ferrand & J.M. Arnaudies essentially repeat in their textbook the same theorem as Bourbaki. It should be noted that Bourbaki's Treatise is not intended for undergraduate students, whereas Lelong-Ferrand & Arnaudies' textbook is [Lelong & Arnaudies, p. 45].

I have shown how different perspectives are held by the authors of textbooks on the same theorem: the generalized associative law. This proves an important point of my approach to mathematical discourse: that authors are involved in negotiating the truth of theorems, and this negotiation involves not only logical devices, but rhetorical devices.

5.2 THE ROLE OF EXAMPLES IN MATHEMATICAL DISCOURSE

There are two kinds of examples in mathematics textbooks: examples associated with definitions (examples serve to clarify concepts) and examples associated with theorems (examples serve to clarify facts, situations or relations).

5.2.1 Exemplifying Definitions

In mathematics we can define either particular or general objects. Definition of a particular mathematical object obviously does not require examples, since the defined object is its own example. In the sequel we are interested in general mathematical objects.

Authors should give examples of a definition otherwise the definition could be empty or superabundant. For instance, together the definition of convergence of series, one should give an example of convergent series, and also an anti-example, that is, an example of divergent series.

LEVELS OF GENERALITY

Exemplification is related to levels of generality and levels of abstraction.

Mathematical discourse occurs at several levels of abstraction. In undergraduate mathematics it is usual to work consecutively or simultaneously at different levels of generalization. Exemplification means lowering the level of generalization or abstraction. Examples serve to make the things more concrete, specific, particular.

Examples and definitions go in opposite directions: exemplification is a movement from the general to the particular or specific, whereas definition is a movement from the individual to the universal, from concrete to abstract, from particular to general, from specific to generic. The goal of exemplification is comprehensibility, whereas the goal of definition is universality.

Sometimes it is convenient to consider a subclass, instead of a particular object. This means that, in the process of understanding an abstract concept associated to a general class, it is convenient to think first in terms of a particular subclass, and afterwards in terms of a particular member.

For instance, the complex plane is an example of topological space. Between complex plane and topological space there is the notion of metric space. This is why many authors prefer to look at complex plane as a metric space,

PEDAGOGICAL FUNCTION OF EXAMPLES

Examples have also a pedagogical function of facilitating the comprehension of a definition. For instance, some definitions are so complicated (for example, the definition of Riemann surface) that the authors prefer to present the definition through an example. Stewart & Tall approach Riemann surfaces investigating the logarithmic function of complex numbers; Silverman introduces the Riemann surface through the function $\sqrt[n]{z}$.

HEURISTIC FUNCTION OF EXAMPLES: EXAMPLES ARE EXERCISES

In mathematics it is not sufficient to declare that a certain object is an example of a given definition: sometimes a proof is necessary! So, some examples are camouflaged exercises. For instance, it is not enough to say that the series

$$1 - 1/2^2 + 1/3^2 - 1/4^2 + \dots$$

is absolutely convergent; it is necessary to prove it.

5.2.2 Theorems: Examples and Counterexamples

Examples of theorems are their applications in particular cases. Its function is to show how the formula or the technique works.

COUNTEREXAMPLES

After the proof of a theorem of implication type ($p \Rightarrow q$), mathematicians ask automatically about the converse: what about ($q \Rightarrow p$)? If the converse is false, it is necessary to give a counterexample. Counterexamples are the antidotes to false conjectures.

For example, Cauchy-Riemann equations are necessary conditions for analyticity, however they are not sufficient. The counter-example is the following function defined by $f(z) = z^5/|z^4|$, $z \neq 0$, $f(0) = 0$, for which the Cauchy-Riemann equations hold, but it is not analytic at $z = 0$.

RHETORICAL FUNCTION OF EXAMPLES (EXAMPLES AS RHETORICAL DEVICES)
Sometimes the author states a general theorem, but instead of giving a proof, he or she uses an example to 'illustrate' the validity of the theorem.

Some authors [Derrick, for example] calls examples what usually are called solved exercises.

5.3 THE ROLE OF FIGURES IN MATHEMATICAL DISCOURSE

Figures, together with examples, are commonly used in undergraduate mathematics textbooks as tools of explanation, justification and illustration.

Authors are divided as to the use of figures, even authors of textbooks on the same subject. Consider for example the discipline Complex Analysis: Stewart & Tall utilize nearly 140 figures in their book **Complex Analysis**. On the other hand, in W.J. Thron's textbook **Introduction to The Theory of Functions of A Complex Variable** there is no figure at all.

Is there any ground for the non-use of figures in mathematics textbooks? The answer is no, according to the psychologist Allan Paivio. He proposes a dual- coding approach to imagery and verbal language:

> The two processes represent the activity of independent but interconnected systems that are specialized for picking up, storing, organizing, retrieving, and manipulating stimulus

information. The imagery system deals with information concerning concrete objects and events, the verbal system with linguistic information.

[Paivio 79, p. 163]

5.3.1 Classification of Figures

The mathematics educationalist Stuart P.O. Plunkett classifies figures in pictures and diagrams. A figure is a *picture* if it represents spatial relations; a figure is *diagram* if it represents non-spatial relations [Plunkett 79].

Pictures use spatial relations to represent spatial relations. Examples of pictures are geometrical drawings, three-dimensional models, maps, pictures (in the ordinary sense), and drawings.

Diagrams use spatial relations to represent non-spatial relations. Examples of diagrams are Venn diagrams, tables, flow-charts, graphs.

In certain cases, for example Cartesian graphs, it is difficult to distinguish between pictures and diagrams. Is the graph of the function $y = x^2$ a picture or a diagram? On the one hand, it is a diagram, because it represents an 'algebraic' relation between x and y. On the other hand, the equation $y = x^2$ is the algebraic representation of a parabola, and then, the Cartesian graph is a geometric representation of a geometric object, that is, it is a picture.

Nevertheless, this classification can be helpful to discourse analysis because it can provide a general idea of the uses authors make of figures.

5.3.2 Functions of Figures

My intention here is to consider figures according to their three functions or dimensions: informational, heuristic or rhetorical.

INFORMATIONAL FUNCTION

Figures as well as sentences and formulae serve to convey information.

Figures (pictures, according to Plunkett) are used in mathematics writing to describe spatial situations or relations, specially for visualization, for example, by representing tri-dimensional figures in the plane.

Figures are also used as diagrams, to represent non-spatial relations. Diagrams such as Venn diagrams, or the commutative diagrams in Algebra are so important that now they belong to mathematical language.

HEURISTIC FUNCTION

Figures help construction of knowledge. To say the minimum: 'Draw a figure' is one of the Polya's commandments to solve problems.

RHETORICAL FUNCTION

Figures enter in the negotiation of truth between author and reader. They are used for convincing the readers of some truth, whose verbal proof is complicated or non-convincing; they are sometimes used as a part of the proof. Sometimes figures replace words completely in the so-called 'proofs without words'.

Another rhetorical function of figures is to negotiate intuition. Figure is an aid for comprehension, and memorization.

EXAMPLE

Let us examine the figures in the textbook **Real Analysis** (1968) by A.J. White, which contains only 13 figures. I will classify figures in pictures and diagrams, and according to what they illustrate: definitions, examples or proofs.

Chapter 0 on Set Theory (Notation and terminology) contains five figures, all of them are diagrams in Plunkett's sense: figures 0.1 and 0.2 are Venn diagrams illustrating the definitions of union, intersection, complement, and symmetric difference of two sets; figure 0.3 shows the graph of $y = x^2$ as an example of graph; figure 0.4 is a pictorial representation of function; figure 0.5 shows an example of Cartesian product.

Chapter 2 on Metric Spaces contains two figures: figure 2.1 is a picture: it illustrates the proof that 'every open ball is an open set'; figure 2.2 is a diagram: it illustrates the definition of continuous function.

Chapter 3 on Real Functions contains five figures, all of them are diagrams: figure 3.1 illustrates the proof of the Intermediate Value Theorem; figures 3.2 and 3.3 illustrate examples; figure 3.4 illustrates the definition of uniform continuity; figure 3.5 illustrates the proof of a theorem.

In chapter 5 on the Riemann Integral, the only figure is a diagram: figure 5.1 illustrates the proof of a theorem.

It is noteworthy that White expresses in his text his opinion about the functions of figures. For example, he comments on the heuristic power of figures, such as figure 2.1, of suggesting methods of proof:

> Every open ball is an open set ... The proof is illustrated by Figure 2.1. Although obstinately Euclidean, and to a certain extent misleading, **this sort of picture can be very helpful in clarifying complicated situations and suggesting methods of proof**. The reader is recommended to supply suitable diagrams to illustrate the results which follow ...
>
> [White, p. 53]

White acknowledges that figures such as figure 3.5 are helpful to give counter-examples:

> A sequence of continuous functions may fail to converge uniformly and still have a continuous poitwise limit. However, complications like this can often be resolved with the aid of a diagram.
>
> [White, p. 103].

Figure 3.1 has a rhetorical function: it participates in the intuitive proof of the Intermediate Value Theorem given by White, before the formal proof.

White uses figure 5.1 to negotiate intuition. White uses a figure showing a CONTINUOUS function in order to illustrate the general definition of Riemann integral of a BOUNDED function. Moreover, he bases his argument on the figure:

> A glance at Figure 5.1 shows that, INTUITIVELY, $\underline{\int}_a^b f(x)\,dx$
>
> is the closest we can get to the "area" under the graph of f by approximating from inside with finite families of rectangles.
>
> [White, p. 141]

I will continue the discussion on figures in section 9.2, where I consider the illustrations of Riemann surfaces.

CHAPTER 6

NEGOTIATION OF EXERCISES AND APPLICATIONS

Introduction

In section 1, I will discuss the discursive functions of exercises: heuristic function (exercise as a tool of teaching know-how), rhetorical function (exercise as a tool of negotiating truth), informational function (exercise as a tool of transmitting new information).

In section 2, I will propound a classification of exercises in three types: manipulative, definitional and inferential.

In section 3, I will deal with interactions between disciplines: prerequisites, application and contextualization.

6.1 EXERCISES

One of the purposes of a textbook is to provide activities for the readers themselves to construct their knowledge. In undergraduate mathematics textbooks this is materialized by lists of exercises or problems, and by means of gaps intentionally left by the authors in the expository text.

Ranging from the problem books of the Schaum's Outline Series to the advanced books of *The Elements of Mathematics* by N. Bourbaki, almost every mathematics textbook contains a set of exercises at the end of chapters or sections.

There is then an apparent consensus among the authors of mathematics textbooks about the educational value of problem-solving or exercise-solving. However, a finer analysis shows that exercises have sometimes purposes different from helping learners's construction of mathematical knowledge.

FUNCTIONS OF EXERCISES

According to J.D. Depree and C.C. Oehring, authors of a Complex Analysis textbook, exercises have many purposes:

> In addition to their usual purpose of stimulating the reader to interact with the mathematics, the exercises serve as places to outline (through the hints) propositions which are needed in later developments but whose proofs are cumbersome, or similar to others presented, or otherwise unsuited for detailed discussion in the text.
>
> [Depree & Oehring, preface]

• HEURISTIC FUNCTION

'To stimulate the reader to interact with the mathematics' is the *heuristic function* of exercises. Exercises are the text's complement. Text provides theoretical information to the readers, while exercises give them practice, know-how. Text means product, while exercises mean process.

• INFORMATIONAL FUNCTION
'To serve as places to outline propositions which are needed in later developments' is the *informational function* of exercises.

Some exercises convey supplementary information. They are camouflaged theorems. According to E.L. Lima, author of a Real Analysis textbook, 'they inform the reader about further results, some of which do not appear in the text just for a matter of author's taste' [Lima, preface]. I call them *information-supplementing exercises.*

• RHETORICAL FUNCTION
Other exercises have a *rhetorical function*: they participate in the author's negotiation of truth. Authors leave to the readers the task of proving propositions 'whose proofs are (i) cumbersome, or (ii) similar to others presented, or (iii) otherwise unsuited for detailed discussion in the text'. I call them *gap-filling exercises.*

6.1.1 Information-Supplementing Exercises

Theoretically orientated authors are used to give additional information to the readers in the form of exercises.

Sometimes this information is later used as a fact. For example, Derrick propounds the proof of an extension of Morera's theorem as an exercise (exercise 23, section 2.3). Derrick's readers will have to accept this theorem as true, even when they are not able to prove it, because the author will use this result as a fact in the proof of Weierstrass's theorem on the sum of a uniformly convergent series of analytic functions (section 3.2).

EXAMPLE. One striking example of abuser of information-supplementing exercises is A.J. White, in his book **Real Analysis**. He divides the problem material into exercises (that belong to the text), and problems (given at the end of the chapters).

The purpose of exercises is to provide practice:

> The exercises in the text are intended mainly to provide practice in the manipulation of the definitions and the use of the theorems. They form an integral part of the discussion, and a conscientious attempt to work them all is an essential step to understanding the text.
>
> [White, preface]

However, problems have different functions:

> The problems at the end of each chapter are of a different kind. With one or two exceptions, the text does not depend on them. For the most part, they are problems of the "project" type, that is to say they invite the student to fill out the details in a sequence of propositions which provides further information on the subject matter of the chapter or indicates the direction of some further development. The hints are usually very copious, and the details that remain are well within the capacity of any student who has properly understood the text.
>
> [*Ibidem*, preface]

This discourse sounds reasonable, however the practice does not correspond to the discourse. For example, at the end of the chapter on Metric Spaces, White demands the proofs of theorems that practically cover a complete course on metric spaces, as we can see by the titles of the problems: (a) alternative conditions for a metric, (b) equivalent metrics, (c) a family of metrics for $\mathbb{R}^2$, (d) further examples of metric spaces, (e) continuous functions, (f) interior points and open sets, (g) cluster points and closure points, (h) distance from a subset, (i) compactness, (j) dense sets and separable spaces, (k) connectedness, (l) products of metric spaces, (m) infinite dimensional spaces, (n) the completeness axiom, (o) topological spaces and generalized sequences, (p) uniformly continuous functions, (q) characterization of compact spaces, (r) open sets in $\mathbb{R}$, (s) reflexive relations in a metric space.

This set of 'problems' occupies nearly thirteen pages, while the corresponding text occupies nearly twenty six pages, including the exercises. This demonstrates clearly that White's intent is to use problems or exercises as a means to convey more theoretical information.

6.1.2 Gap-Filling Exercises

Authors of undergraduate mathematics textbooks as a rule do not prove all the theorems they use, and they often omit parts of the proofs of many theorems.

So, they have to negotiate the truth of propositions with the readers. Such negotiation is sometimes unfair: the authors simply give to the readers the trouble to fill the gaps without any help. As a consequence of this 'negotiation', the reader becomes responsible for the validation of the proposition. In case the reader is able to fill out the gap, it's OK, the reader has done his or her part. In the case the reader fails to provide the proof, then, he or she has no alternative: he or she should accept unconditionally the truth of the proposition on faith of the writer's authority.

Sometimes the author negotiates in advance the truth of propositions given as exercises. For instance, E. Fischer says in the preface to his textbook:

> Even if the reader does not solve all the problems, we expect him to read each one and to understand the result contained in it. **In many cases the results cited in the problems are used as proofs of later theorems** and constitute a part of the development. When the reader is asked, in a problem, to prove a result which is used later, this usually involves paralleling work already done in the text.
>
> [Fischer, preface]

Another author, K. Ross, considers that the learners should be gradually prepared to cope with the lack of details in proofs, because in advanced textbooks this economy in argumentation will be standard:

> In the early sections, the proofs are very detailed with careful references for even the most elementary facts. Most sophisticated readers find excessive details and references a hindrance (they break the flow of the proof and tend to obscure the main ideas) and would prefer to check the items mentally as they proceed. Accordingly, in later chapters the proofs will be somewhat less detailed and references for the simplest facts will often be omitted. This should help prepare the reader for more advanced books which frequently give very brief arguments.
>
> [Ross, preface]

Some authors decide to leave some 'dirty' job to the readers. For example, W.R. Derrick, in his **Complex Analysis and Applications** says at the end of the proof of a theorem:

> The remaining proofs are immediate consequences of the definition of a line integral in Section 2.1. Their straightforward and SOMEWHAT TEDIOUS verification will be left for the exercises.
>
> [Derrick, p. 89]

6.2 CLASSIFICATION OF EXERCISES

There are many ways of classifying mathematical problems or exercises.

PURE AND APPLIED

The principal aim of an *applied exercise is* the *application* of the mathematical theory to another field such as physics, biology, engineering. To solve applied exercises learners should have some previous knowledge of the field in which mathematics is being applied. Applied exercises are intended to improve learners' ability to cope with contextualized mathematics.

PROBLEMS TO FIND AND PROBLEMS TO PROVE

One of the best known classification of problems is Polya's classification, who distinguishes two kinds of problems: 'problems to find', and 'problems to prove'.

> The aim of a "problem to find" is to find a certain object, the unknown of the problem ... 'Problems to find' may be theoretical or practical, abstract or concrete, serious problems or mere puzzles. We may seek all sorts of unknown; we may try to find, to obtain, to acquire, to produce, or to construct all imaginable kinds of objects ... The principal parts of such problems are the *unknown*, the *data* and the *condition.*
>
> [Polya 71, p. 154]

> The aim of a "problem to prove" is to show conclusively that a certain clearly stated assertion is true, or else to show that it is false ... If a "problem to prove" is a mathematical problem of the usual kind, its principal parts are the *hypothesis* and the *conclusion* of the theorem which has to be proved or disproved.
>
> [*Ibidem*, p. 155]

ANALYSIS OF EXERCISES

I will now refine Polya's classification of problems in the context of undergraduate mathematics textbooks.

• MANIPULATIVE EXERCISES

In undergraduate mathematics textbooks 'problems to find' correspond to *manipulative exercises*, where the learners are asked to perform some mathematical operations introduced in the text.

An exercise is *manipulative (operational, technical, instrumental)* if it demands the use of some particular mathematical operation or technique developed in the text, or techniques assumed to be previously known such as arithmetic, algebraic manipulation, techniques of calculus or linear algebra, etc. Exercises that ask for drawing some figure or plotting some graph will also be included in this category.

Manipulative exercises' purpose is to improve learners' technical abilities of manipulating mathematical objects.

- INFERENTIAL EXERCISES

I will call *inferential exercises* what Polya has called 'problems to prove'. An *inferential exercise* demands the proof of a proposition.

It is interesting to investigate what kind of propositions the readers are asked to prove. I distinguish the following types of inferential exercises:

(i) *application of a theorem* (it is asked to apply a theorem to a particular context or for a particular object),

(ii) *completion of a theorem* (gap-filling exercise, it is asked to give the details of a proof left incomplete in the text),

(iii) *corollary of a theorem* (it is asked to prove a consequence of a theorem),

(iv) *converse of a theorem* (it is asked to prove the converse of a theorem, or to give a counter-example)

(iv) *criticism of a theorem* (it is asked to criticize a theorem, exploring the limits of its validity)

(v) *extension of a theorem* (it is asked to prove an extension of a theorem).

The aim of inferential exercises is to improve readers' capacity of proving theorems.

Besides manipulative exercises (problems to find) and inferential exercises (problems to prove) there is an intermediate category: the definitional exercises.

• DEFINITIONAL EXERCISES

An exercise of mathematics is *definitional* if it refers to a particular definition (to check whether a particular object is an instance of a general definition or not, to prove some property of the concept, etc.).

Examples of definitional exercises are (i) to prove or disprove that a given function is continuous; (ii) to show that a given set is compact; (iii) to give an example of conditionally convergent series.

Some exercises introduce new definitions, mainly definitions of particular objects such as gamma function, Legendre polynomial, Bessel functions, etc.

The aim of a definitional exercise is to improve the understanding of a definition.

Note: This classification takes into account that the majority of exercises in undergraduate mathematics textbooks have as principal clues some definite technique, concept or theorem.

Many authors of textbooks emphasize manipulative and definitional exercises rather than inferential exercises. However there are authors that do the opposite. In my view, learners should perform mathematical activities in the following sequence:

(i) the first step should be *manipulative*: the learners should operate with mathematical objects, before reflecting too much on their essence;

(ii) the second step is *ontological*: once the learners have been familiarized with mathematical objects through their manipulation, they should try to understand their essence, investigating their properties;

(iii) the third step is *relational*: the learners should try to understand the relations between mathematical objects, investigating how they are articulated forming a kind of architecture.

EVALUATION OF EXERCISES

Exercises can be analysed also in terms of techniques of solution. The solution of mathematical exercises depends on two things: (i) understanding of the problem and (ii) identification of the clue.

- UNDERSTANDING

Understanding of mathematical exercises requires ability to understand mathematical language, use of logic and intuition, ability of interpretation, and of course factual knowledge, provided by the text, such as the knowledge of definitions, theorems, proofs and examples.

- IDENTIFICATION OF THE CLUE

To identify the clue to solving the exercise, that is, to know what is expected and what tools are to be used, the learners should use their cognitive functions such as logical schemes, heuristic schemes, intuition, visualization, ability to use symbols, ability to cope with generality and abstraction, knowledge of facts and techniques, and of course their experience in the solution of mathematical exercises.

The scheme of exercise analysis I propound is then, first, to observe what the exercise is demanding: to perform operations, to instantiate definitions, or to provide arguments; second, to find the clue to solving the exercise: what to do and what tools are necessary.

EXAMPLE. Here I will analyse some of the exercises propounded by K. A. Ross (1980) on Properties of Continuous Functions in his textbook **Elementary Analysis: The Theory of Calculus** [Ross, 99-100], in order to show how the scheme works.

> Exercise 18.1: Let f be as in Theorem 18.1. Show that if $-f$ assumes its maximum at $x_0 \in [a,b]$, then f assumes its minimum at x_0.

Classification: inferential exercise. Gap-filling exercise: it asks for the completion of the proof of Theorem 18.1.

Exercise 18.2: Reread the proof of Theorem 18.1 with $[a,b]$ replaced by (a,b). Where does it break down ? Discuss.

Classification: criticism of a theorem. It explores the limits of validity of the Theorem 18.1.
Evaluation: this type of exercise is invariably difficult. However, the author himself destroys the difficulty of this exercise revealing in the text that the function $f(x) = 1/x$ is continuous but unbounded on $(0,1)$, and that the function x^2 is continuous and bounded on $(-1,1)$ but it does not have a maximum value on $(-1,1)$.

Exercise 18.3: Use calculus to find the maximum and minimum of $f(x) = x^3 - 6x^2 + 9x + 1$ on $[0,5)$.

Classification: operational exercise.
Evaluation: it has no relation with the text; it only requires the remembering of prerequisites.

Exercise 18.6: Prove that $x = \cos x$ for some x in $(0, \pi/2)$.

Classification: inferential exercise. Application of a theorem.
Evaluation: the difficulty of the exercise lies in perceiving that the Intermediate Value Theorem should be applied: the reader should transform the equation $x = \cos x$ into the equation $x - \cos x = 0$, and then to look at the left side of this equation.

Exercise 18.7: Prove that $x\, 2^x = 1$ for some x in $(0,1)$.

Classification: repetitive exercise; it is analogous to the previous exercise.

Exercise 18.8: Suppose that f is a real-valued continuous function on $\mathbb{R}$ and that $f(a)f(b) < 0$ for some $a,b \in \mathbb{R}$. Prove that there exists x between a and b such that $f(x) = 0$.

Classification: inferential exercise. it is a corollary of the Intermediate Value Theorem.

> Exercise 18.9: Prove that a polynomial function f of odd degree has at least one real root.

Classification: inferential exercise. This is an application of the Intermediate Value Theorem.
Evaluation: it is a very demanding exercise. The author gives an heuristic hint: to consider first the case of a cubic. Fortunately the author provides the solution at the end of the textbook.

> Exercise 18.10: Suppose that f is continuous on [0,2] and that $f(0) = f(2)$. Prove that there exist x,y in [0,2] such that $|y - x| = 1$ and $f(x) = f(y)$.

Classification: It demands the application of the Intermediate Value Theorem.
Evaluation: the main difficulty is the presence of two variables; in order to apply the theorems given in the section, it is necessary to eliminate one of them. It is crucial to observe that we can assume without loss of generality that x belongs to the interval [0,1]. Then y should be $x + 1$. Therefore, the question is to prove that there exists x in [0,1] such that $f(x) = f(x + 1)$. Now it is enough to repeat what was done in exercise 18.10: consider the left side of the equation $f(x) - f(x + 1) = 0$, or equivalently, the left side of the equation $f(x + 1) - f(x) = 0$. This is a difficult exercise. Unfortunately the author destroys the pedagogical value and the beauty of the exercise, by giving the following hint: consider the function $g(x) = f(x + 1) - f(x)$ on [0,1].

6.3 INTERACTION BETWEEN DISCIPLINES

In most of the countries, if not all, the teaching of undergraduate mathematics is hierarchically organised as a sequence of disciplines, where each discipline is in general dependent on disciplines of the previous year, and parallel disciplines are quite independent.

This curricular division of mathematics in disciplines is practically universal. For example, Calculus, Real Analysis, and Complex Analysis are taught as disciplines in different countries such as Great Britain, France, Brazil, Iran.

Since undergraduate mathematics textbooks are pedagogic tools, they should reflect this curricular organisation: the subject matter of a textbook is a particular discipline. Sometimes it is difficult for the readers to understand how such discipline is related with other disciplines, because of authors' discourse: authors like to say their textbooks are self-contained, independent of other disciplines.

There are two types of interaction between disciplines:

(a) *internal interaction*, that is, interaction between the given discipline and other mathematical disciplines (prerequisite, application), and

(b) *external interaction*, that is, interaction between the given discipline and non-mathematical disciplines (motivation, application).

For example, Complex Analysis has the following characteristics:

(i) internal interaction: it has Calculus or Real Analysis as prerequisites, and it can be applied to other mathematical disciplines such as Partial Differential Equations or Stochastic Processes;

(ii) external interaction: it can be applied to non-mathematical matters such as Electricity, and Fluid Dynamics, and some of its topics can be motivated by non-mathematical disciplines such as Heat Theory.

In other terms, a discipline is articulated with other disciplines through the relations of dependence (prerequisites), application and motivation.

6.3.1 Prerequisites

As textbooks refer to disciplines which belong to a hierarchical system, it is expected that their contents are to some extent dependent on other subjects.

However, some authors seem to have an actual compulsion to declare that their texts are autonomous, self-contained, attempting to negate any kind of dependence on other disciplines or at least minimize this dependence. For example, A.J. White begins the preface to his textbook on Real Analysis uttering that:

> The only absolute prerequisites for reading this book are an interest in mathematics and willingness occasionally to suspend disbelief when a familiar idea occurs in an unfamiliar guise.
>
> [White, preface]

White himself knows that this affirmation is just rhetorical:

> But only an exceptional student would profit from reading the book unless he has previously acquired a fair working knowledge of the processes of elementary calculus.
>
> [*Ibidem,* preface]

This is not original. Bourbaki says:

> In principle, it [the series of books called **Elements of Mathematics**] requires no particular knowledge of mathematics on the reader's part, but only a certain familiarity with mathematical reasoning and a certain capacity for abstract thought.
>
> [Bourbaki, preface]

Nicolas Bourbaki as well as White corrects his rhetoric:

> Nevertheless, it is directed especially to those who have a good knowledge of at least the content of the first year or two of a university mathematics course.
>
> [*Ibidem*, preface]

White's and Bourbaki's discourses are identical: they are expressions of the ideological discourse of the compartmentalization of mathematical disciplines. This tendency should be opposed. According to P. Hilton:

> We must insist far less on the autonomy and (apparent) independence of the various mathematical disciplines and emphasize their (real) interdependence.
>
> [Hilton 82, p. 160]

Authors try to solve the problem of prerequisites in the first chapter. They use it as a reminder, a refresher, or as a sort of foundations for the theory. Their discourse sometimes sounds strange: This chapter is not to be read ... The reader can skip this chapter ...

In the following I will show that authors' tendency to isolate mathematical disciplines from non-mathematical disciplines is still stronger.

6.3.2 Applications

There are basically three types of authors of undergraduate mathematics textbooks: (i) those who practically ignore any kind of applications, (ii) those who give application only to other areas of mathematics, and (iii) those who give applications of the theory to non-mathematical disciplines.

I will illustrate this divergence within Complex Analysis textbooks authors.

NO APPLICATIONS

G.J.O. Jameson is an example of a purist author: his textbook practically has no application whatsoever. He only propounds two applied exercises, one about finite-dimensional division algebras and one about a particular method of evaluating the probability integral, topics that he himself considers 'slightly esoteric' [Jameson, preface].

INTERNAL APPLICATIONS

Ahlfors is an example of internal applicationist. He applies complex analysis to Dirichlet's problem and to linear differential equations.

PHYSICAL APPLICATIONS

Churchill & Brown develop physical applications in their textbook: applications to Fluid Dynamics, Electrostatic Potential, and Heat Conduction.

APPLICATIONS TO ENGINEERING

Some authors give applications; others simply discourse on applications.

In the book [Weltner, K., et al, p. 220], the authors allude to internal and external applications:

Internal applications, that is, applications within mathematics: "Complex numbers are important in the solution of differential equations."

External applications, that is, applications in other areas: "They [the complex numbers] are also a useful concept in electrical engineering, and they are indispensable in the study of quantum physics."

In the book [Spencer, A.J.M. et al], it is noted down some applications of complex analysis in engineering:

Complex numbers arose from the study of algebraic equations, and the theory which subsequently developed, called the theory of complex variables, has found wide applications in theoretical studies in many branches of engineering and science. For example, complex impedance are used in electrical engineering, complex potentials are used in fluid dynamics and elasticity, and a complex time dependence is used extensively whenever vibration and wave problems are studied.

[Spencer et al]

Kreyszig classifies engineering problems, which may be treated and solved by complex analysis in two types:

a) the first type is constituted by elementary problems, for which it is sufficient the knowledge of complex numbers acquired in college algebra and calculus. Examples: many applications in connection with models of electric circuits and mechanical vibrating systems.

b) the second type of problems requires a deep knowledge of the theory of complex functions. Examples: some problems in the theory of heat, in fluid dynamics, and in electrostatics [Kreyszig].

Kreyszig, besides discoursing, shows applications: its book has a nice chapter on applications, the chapter 17 on complex analysis applied to potential theory.

MOTIVATION

Bak & Newman are very eclectic: they give applications to other mathematical disciplines, applications to non-mathematical disciplines, and they also use physical ideas to motivate mathematical theorems [Bak & Newman].

6.3.3 Contextualization

Another aspect of mathematical discourse is contextualization: (a) historical contextualization, and (b) socio-cultural contextualization.

Few authors contextualize the subject matter of their discourse.

HISTORICAL CONTEXTUALIZATION

Some authors contextualize their subject through reference to historical events. Stewart & Tall is a rare example in terms of Complex Analysis textbooks (see chapter 9.3).

SOCIO-CULTURAL CONTEXTUALIZATION

Socio-cultural contextualization is non-existent (no author is interested to situate the topics they explore in their texts with social or cultural reality; ethnomathematics is yet unthinkable in undergraduate mathematics textbooks).

CHAPTER 7

NEGOTIATION OF NOMENCLATURE AND NOTATION

Introduction

In this chapter I am interested on the ways authors of undergraduate mathematics textbooks negotiate the question of ambiguity in mathematical nomenclature and symbolism.

In the first section I will discuss two problems of ambiguity in mathematical nomenclature: synonymy, where the same mathematical object is designated by different names, and homonymy, where different objects receive the same name.

Afterwards I call attention on another important point, the relation between naming and meaning. Should names convey meanings?

In section 2 I will deal with synonymy and homonymy in mathematical notation.

7.1 MATHEMATICAL NOMENCLATURE

Mathematicians, apart from creating mathematical objects, create their own vocabulary. Mathematicians are likely to think that they are in the same position as Humpty Dumpty, see [Carroll 62, p. 274]:

> 'When I use a word', Humpty Dumpty said in a rather scornful tone, 'it means just what I choose it to mean - neither more nor less'.
>
> 'The question is', said Alice, 'whether you *can* make words mean so many different things'.
>
> 'The question is', said Humpty Dumpty, 'which is to be master - that's all'.

According to the historian of mathematics Phillip S. Jones, as quoted in [Greene 62, p. 488], 'mathematicians are inclined to think that they are the complete masters; that, irrespective of what they may mean elsewhere, in mathematics, words do as they are told, mean exactly and only that which they are defined to mean'.

NAMING

In order to designate new mathematical objects, mathematicians either invent new words (and then these words will be unique to mathematics) or adapt words from ordinary language.

A new word has the advantage that it does not introduce ambiguity: it denotes exactly the defined mathematical object and does not bear any connotation at all. In this case the correspondences OBJECT - NAME and NAME - MEANING are one-to-one. For example, 'hypotenuse' denotes 'the side of a right-angled triangle opposite the right angle' and means nothing more.

In the second case, where ordinary words are used as technical terms, a clash of meanings always happens, namely, the clash between technical meaning and ordinary meaning(s). In this case, *mathematical terms* bring connotations from the ordinary language, which sometimes can be an educational hindrance.

For example, adjectives such as *imaginary* and *irrational* convey negative connotations, as E. Hille remarks:

> There is no other branch of mathematics where the terminology shows such a marked distrust of the objects named as the field of numbers. The terms *radical*, *surd*, *negative*, *irrational*, and *imaginary* all have noncomplimentary connotations, and in most cases these terms indicate the opposition which once upon a time met these revolutionary innovations.
>
> [Hille, p. 19]

When mathematicians adapt words from the ordinary language to mathematics, the technical meaning is either (i) subtly different from the ordinary meaning, or (ii) very different from the ordinary meaning. See [Marcus 80, p. 361], or [Orton 87, p. 127].

Words belonging to the class (i) are potential trouble-makers. For example, the word *similar* as applied to geometrical figures belongs to this class. Mathematical similarity of figures means more than the likeness of their forms; it is a metric notion, involving proportionality between their segments. Two mathematically similar figures are 'similar' in the ordinary sense but not vice-versa. In this case, learners tend to confound technical and ordinary meanings.

Topological terms such as *open* and *closed* are also sources of confusion for the beginners, since in Topology 'closed' does not mean 'not-open'. It is hard, for example, to accept the existence of sets which are simultaneously open and closed.

Perhaps words of the class (ii) provoke some perplexity among the beginners but are not confusing. For example, *square root* of a number has a technical meaning, which of course has no relation with the ordinary botanical meaning.

7.1.1 Synonymy and Homonymy

Terminology of mathematics changes with time: some terms seem stable (for example, *analytic* function), others become obsolete and disappear (for example, *synectic* function). It changes with geography as well. For example, *σ-algebra* in Great Britain and U.S.A. is equivalent to what in France is called *tribu* (tribe).

The following xenophobic opinion of J. Dieudonné illustrates how sociological is the issue of mathematical terminology:

> Concerning terminology, Bourbaki's attitude, as expressed in his *Directions for use*, is to accept or at least tolerate traditional terminology, unless it is ambiguous, or ungrammatical, or incompatible with the normal use of language. It turns out that in many fields of recent origin, a number of mathematicians, mainly of Anglo-Saxon or German origin, were guilty of particularly acute carelessness in showing a total lack of imagination and a complete contempt for their languages.
>
> [Dieudonné 82, p. 621]

As a result of these historical and nationalistic clashes, mathematical nomenclature is not standard. Each mathematical object obviously receives a name. Problems of ambiguity arise when the correspondence between objects and names is not one-to-one: (i) the same mathematical object receives two or more names (*synonymy*), or (ii) the same name is attributed to two or more mathematical objects (*homonymy*).

SYNONYMY: MULTIPLICITY OF NAMES

Synonymy occurs mainly because different mathematicians observe and give relevance to different aspects of the same mathematical object.

Textbook authors are not supposed to create names, but to select them from among the existing ones. They react differently with respect to the problem of the multiplicity of names. We can distinguish three classes of authors:

(i) those who use only one name, and do not mention any other names (intolerant of ambiguity)
(ii) those who use only one but mention several names (tolerant, but restrictive in use), and
(iii) those who mention and use several names (tolerant, and eclectic).

EXAMPLE: DIFFERENTIABLE COMPLEX FUNCTION
Authors of Complex Analysis textbooks diverge about the naming of differentiable complex function.

Authors who do not tolerate synonymy, that is, members of the class (i) above are rare. G. J. O. Jameson is one of them: he only uses the term 'differentiable function'; consequently, his readers do not learn what an analytic function is!

In my view, Jameson's attitude should be criticized. Authors should not conceal the existence of terminological dispute among mathematicians. We have to cope with the multiplicity of names in mathematics because names are products of convention, and convention depends on the will of the majority of the members of the mathematical community.

I. Stewart and D. Tall intend to be rigorous: they only use the term 'differentiable function' until page 183, where they allow themselves to use the term 'analytic function', after proving that every differentiable function on an open disc is the sum of a power series:

> Note that a complex function f is differentiable if and only if it is analytic. The two words just emphasize different points of view, and may be used interchangeably.
>
> [Stewart & Tall, p. 183]

The great majority of the authors belong to the class (ii): in general authors mention other names for 'differentiable function'. W. R. Derrick is an example: he defines analytic (or holomorphic) function, uses only the term 'analytic function', but, at the end of the chapter, he gives the following synonyms for analytic: holomorphic (again!), monogenic, and regular.

The class (iii) is as small as class (i). E. Hille is an interesting representative of this class: he is extremely tolerant and eclectic. His book is called Analytic Function Theory, one of its chapters is called 'Holomorphic Functions', and he defines 'differentiable' function. Moreover, Hille mentions six synonyms for 'differentiable': (1) holomorphic, (2) analytic, (3) monogenic, (4) regular, (5) regular-analytic, and (6) synectic.

HOMONYMY: MULTIPLICITY OF OBJECTS

Contrary to synonymy, homonymy, that is, the attribution of the same name to different objects, fortunately is not so frequent in mathematical language; this proves that mathematicians are fertile creators of names. On the other hand, as it will be shown in the next section, homonymy of symbols is a genuine problem.

One intriguing example of geographical homonymy is the word *billion*, which means a thousand millions in the United States and France, and a million millions in Great Britain and Germany, although the British media have now adopted the American usage.

In Complex Analysis, the term *curve* is confusing, because sometimes it means the map, sometimes it means the path. It is amazing that curves in mathematics can be straight! Another ambiguous term is *region*: some authors use the term in the sense of an open connected set, whereas others use it as an open connected set plus points of its boundary.

7.1.2 Description and Eponymy

According to Dieudonné, one of the principles orientating the introduction of new technical terms in Bourbaki's textbooks is 'to find names evocative of the notions they designate or of their originators' [Dieudonné 82, p. 621].

Thus, in mathematical nomenclature there are evocative and non-evocative names, where the evocative names can be either descriptive or eponymous. More precisely,

(i) *descriptive* names are names intended to evoke the notions by describing their characteristic properties,
(ii) *eponymous* names are names intended to evoke their originators, and
(iii) *non-evocative* names are neutral; they are just names, they do not intentionally evoke anything at all.

The main purpose of using descriptive names is quite obvious: to facilitate intuitive understanding. Examples of good descriptive names are 'continuous' function, 'linear' function, ball, boundary, 'connected' set. A bad name is *tribe* to denote σ-algebra, because sigma connotes enumerability, while tribe has no evocative connotation.

Because of the need of frequent reference in their discourse, mathematicians give names not only to mathematical objects, but also to 'discursive' elements, such as theorems, axioms, equations, formulae, etc. Examples of such descriptive names are: triangle inequality, parallelogram law of vector addition, commutative property of multiplication, fixed point theorem, chain rule, comparison test, axiom of choice.

In undergraduate mathematics the learner becomes familiar with eponymous terms such as Banach space, Hilbert space, Lebesgue measure, Riemann integral, Cauchy's sequence, Taylor's series, Bessel function, Peano's curve, etc.

Eponymous names of theorems, lemmas, axioms, etc. are even more frequent; eponymy seems to be mathematicians' mania. For example, we can find in Complex Analysis names such as Cauchy-Riemann equations, Cauchy's theorem, Cauchy's integral formula, Schwarz's lemma, Schwarz's reflection principle, Lipschitz condition, Parseval's inequality, Dirichlet problem.

Non-evocative names are not so frequent in mathematics. Examples are *field, group, ring, martingale, functor, syzygy.*

Eponynous names, whenever possible, should be substituted or supplemented by descriptive names, since they 'are ineffective and inefficient means of conveying meaning in mathematical discourse' [Henwood & Rival 80, p. 205].

NAMING OR NUMBERING?

Some textbooks authors prefer to use numbers, instead of names, for the identification of theorems, axioms, etc. This type of referencing gives a bureaucratic tone to mathematical style. See for example the following proof of the Eilenberg criterion given by J. Dieudonné:

> Suppose a and b are in the same connected component A of $\mathbb{C} - H$. As $\mathbb{C} - H$ is open in $\mathbb{C}$ and $\mathbb{C}$ is locally connected ((3.19.1) and (3.20.16)), A is open in $\mathbb{C}$ (3.19.5). By (9.7.2) there is a path $t \to \gamma(t)$ in A, defined in $I = [0,1]$, such that $\gamma(0) = a$, $\gamma(1) = b$. As $\gamma(t) \notin H$ for any value of t, the mapping $(z,t) \to f(z,t) = s_{a,\gamma(t)}(z)/|s_{a,\gamma(t)}|$ is continuous in $H \times I$, and $f(z,0) = 1$, $f(z,1) = s_{a,b}(z)/|s_{a,b}(z)|$; the result follows from (Ap.2.5).
>
> [Dieudonné, p. 250]

In my view, the system of reference by names utilized for example by Stewart & Tall is more helpful to the readers than Dieudonné's numerical system of reference, because names such as Paving Lemma, Estimation Lemma, and Identity Theorem are evocative.

By way of conclusion, discourse analysis should take into account the following aspects of author's policy on nomenclature:

(a) multiplicity of names (mention the existing names or not),

(b) system of reference (naming theorems or numbering them),

(c) etymology (the author explains the linguistic meaning of the name or not).

7.2 MATHEMATICAL NOTATION

Symbols are undoubtedly the core of mathematical language. They are used either to designate mathematical objects, classes of objects, and functions or to designate operations, and relations.

In general Latin and Greek letters (lower-case and capital letters) are used as symbols of mathematical objects, classes and functions. Therefore, the learner of undergraduate mathematics is supposed to know Latin and Greek alphabets. Some authors also use Gothic letters.

On the other hand, the class of operational and relational signs is relatively small. In this case ambiguity is intentional. The particular meaning is given by the context in which the symbol is been used; this implies that ambiguity should be resolved by the context, or by a suitable convention.

7.2.1 Homonymy of Symbols

Mathematicians use the same symbol to designate different things, for many reasons. One reason is economy of symbols: they do not want to overload their memory with a great arsenal of symbols. Another reason is that they desire to stress analogy.

EXAMPLE: MEANINGS OF THE SIGN (+)

A striking example is the symbol (+) often used in the definition of vector space to denote two different operations: addition of scalars and addition of vectors. These operations are 'analogous' in the sense that both satisfy the properties of a group.

The meaning of the sign (+) is given by convention: if the author makes the convention that Greek lower-case letters designate vectors and Latin lower-case letters designate scalars, then $\alpha + \beta$ is a sum of vectors, and not of scalars, while $a + b$ is a sum of scalars, and not of vectors.

Some authors show that tolerance to ambiguity is the price to be paid by the elimination of complicated notations. For example, in [Finkbeiner, p. 26], we can see what could be the 'rigorous' formal definition of vector space:

Definition 2.1. A system $\mathcal{V} = \{V, F; +, \cdot, \oplus, \odot\}$ is called a *vector space over the field* $\mathfrak{F}$ if and only if

(a) $\{F; +, \cdot\}$ is a field $\mathfrak{F}$ whose identity elements are denoted by 0 and 1,

(b) $\{V; \oplus\}$ is a commutative group whose identity element is denoted θ,

(c) for all $a, b \in F$ and all $\alpha, \beta \in V$, $a \odot \alpha \in V$ and

(i) $(a + b) \odot \alpha = (a \odot \alpha) \oplus (b \odot \alpha)$,

(ii) $a \odot (\alpha \oplus \beta) = (a \odot \alpha) \oplus (a \odot \beta)$,

(iii) $(ab) \odot \alpha = a \odot (b \odot \alpha)$,

(iv) $1 \odot \alpha = \alpha$.

We can not cope with this 'rigorous' definition, because it is too complicated. The following example shows how different authors cope with sign ambiguity.

EXAMPLE: AMBIGUITY OF THE SIGN OF EQUALITY

A great obstacle for the authors of texts on complex numbers is to explain the meaning of the 'equality' between the argument of the product zw of two complex numbers and the sum of the arguments of z and w. Authors attempt to overcome this obstacle in different ways. In the following I present three different approaches.

First approach: For some authors, an equality between two mathematical objects does not imply that the left side is the same thing as the right side. For example, W. R. Derrick, in the context of arguments of complex numbers, uses the sign of equality with a particular meaning. According to Derrick, argument of a complex number is a number determined up to a multiple of 2 π:

> The angle of inclination of the vector z, determined except for a multiple of 2π, is called the *argument* of z and is denoted by arg z.
>
> [Derrick, p. 12]

For Derrick, the equality (or identity)

$$\arg zw = \arg z + \arg w$$

has the following particular meaning:

> ... the polar angle of the vector zw is the sum of the polar angles of the vectors z and w. Since the argument is determined up to a multiple of 2π, the equation
>
> $$\arg zw = \arg z \quad + \quad \arg \quad w$$
>
> is interpreted to mean that if particular values are assigned to any two of the terms, then there is a value of the third term for which equality holds.
>
> [*Ibidem*, p. 14]

Second approach: Other authors attempt to solve the problem of ambiguity, by restricting the range of the argument. For example, W. Ledermann first observes that:

> Its value [The value of the angle θ] is, however, not completely determined by the equations (1) $x = r \cos \theta$, $y = r \sin \theta$, since arbitrary whole multiples of 2π can evidently be added or subtracted from it.
>
> [Ledermann, p. 22]

His solution is:

> In order to obviate this ambiguity, we impose the further condition that
>
> $$-\pi < \theta \leq \pi. \qquad (2)$$
>
> For a given non-zero number z there exists one and only one value of θ which satisfies (1) and (2).
>
> [*Ibidem*, p. 22]

However, Ledermann's restriction will not be very helpful to solve the problem of the argument of product of two complex numbers:

> ... it would not be correct to say that arg $z_1 z_2$ is equal to $\theta_1 + \theta_2$, because this number might fall outside the range (2). All we can assert is that
>
> $$\arg (z_1 z_2) = \arg z_1 + \arg z_2 + 2k\pi,$$
>
> where k = 0 or 1 or -1, and it is only on examining condition (2) that we can decide which is the correct value of k.
>
> [Ledermann, p. 23]

Third approach: Other authors solve the ambiguity problem, by the use of set theory notations, defining argument of a complex number as a class. For example, H.A. Priestley defines the argument of a non-zero complex number *z* as

$$[\arg z] = \{ \theta \in \mathbb{R}: z = |z| e^{i\theta} \}.$$

He remarks that:

> The bracket notation [arg z] is designed to emphasize that the argument of z is a set of numbers, not a single number. In fact, [arg z] is an infinite set, of the form
>
> $$\{ \theta + 2k\pi: k \in \mathbb{Z}\},$$
>
> where θ is any fixed number such that
>
> $$e^{i\theta} = z/|z|.$$
>
> [Priestley, p. 24]

We have now an actual equality, an equality between sets:

$$[\arg zw] = \{ \theta + \phi : \theta \in [\arg z],\ \phi \in [\arg w]\}$$

7.2.2 Synonymy of Symbols

Synonymy of symbols means that different symbols are used to designate the same thing. For example, there exist at least four notations for the derivative of a function:

(i) $f'(x)$ (Lagrange's notation),

(ii) $D f$ (Cauchy's notation),

(iii) dy/dx (Leibniz's notation), and

(iv) $\dot{x}$ (Newton's notation).

In this case, the choice of notation is not just a question of personal taste. Each of them is convenient for some purpose: ease of manipulation, clarity, association with other concepts, etc.

EXAMPLE: DETERMINANT OF UNITARY MATRIX

A. Graham, in his textbook **Nonnegative Matrices and Applicable Topics in Linear Algebra** (1987), states and 'proves' that the determinant of any unitary matrix equals 1, which is a false theorem.

His "proof" runs like this. Let U be a unitary matrix, that is, a matrix having complex elements which satisfies the equation

$$U^* U = I = U U^* \qquad (1)$$

where $U^* = \bar{U}'$ is the conjugate transpose of U.

To prove that $|U| = 1$, where $|U|$ denotes the determinant of U, we use (1) and the result established in a previous section that

$$|U U^*| = |U|\ |U^*|$$

Hence $|I| = 1 = |U|\ |U^*| = |U|^2$. The result follows.

The above argument presents three errors.

First error: the determinant of a unitary matrix is not necessarily a real number. To see this we calculate the determinant of the matrix given as an example of unitary matrix at the top of the same page:

$$(1/5).\begin{bmatrix} -1 + 2i & -4 -2i \\ 2 - 4i & -2 -i \end{bmatrix}$$

The determinant equals $(4 - 3i)/5$, which is not a real number.
Second error: the author uses the wrong equation

$$|U^*| = |U| = |\overline{U}|, \qquad (2)$$

instead of

$$\det U^* = \overline{\det U},$$

which is a direct consequence of two facts: (i) the determinant of the transpose of U equals the determinant of U, and (ii) the determinant of the conjugate of U is the conjugate of the determinant of U.

Third error: we can not infer from the equation $|U|^2 = 1$ that $|U| = 1$, even if determinant of U is a real number. In this case, it would be equal to +1 or -1.

What are the causes of such errors? I claim that these errors are due to the bad choice of notation for the determinant of a matrix. In fact, it seems that the author was induced in error by the similarity between the symbol for the determinant of a matrix and the usual symbol for the absolute value of a complex number.

Probably the error in the equation (2) was caused by a false analogy with the equality between the absolute value of the conjugate of a complex number z and the absolute value of z : $|\overline{z}| = |z|$.

The second error, that $|U|^2 = 1$ implies $|U| = 1$, seems to be caused by the association of the symbol $|U|$ with the absolute value of a real or complex number. Actually, if z is a complex number, the following implication is true : $|z|^2 = 1$ implies $|z| = 1$.

All this mess could be avoided by the simple use of the more suitable notation det U instead of U, as it is done for example in [Kreyszig].

In the discourse analysis of a textbook one should pay attention on the way the author copes with ambiguity: does he/she conceal the problem of ambiguity? Does he or she conceal the multiplicity of names and symbols?

CHAPTER 8

DISCOURSE ORGANISATION

Introduction

In section 1 I will argue that the notion of architecture of mathematics is important for the discourse analysis of undergraduate mathematics textbooks.

In section 2 I extend the discussion to the more broad concept of style, which comprehends all the discursive strategies of textbook authors.

In section 3 I will analyse the discourse of axiomatization of Complex Analysis propounded by W.J. Thron.

8.1 ARCHITECTURE OF MATHEMATICS

METAPHOR: MATHEMATICS AS ARCHITECTURE

Mathematicians sometimes use metaphors in order to express his philosophical conceptions about the nature of mathematics. For instance, N. Bourbaki in his paper **The Architecture of Mathematics** (1950) conceives mathematics as a big city,

> whose outlying districts and suburbs encroach incessantly, and in a somewhat chaotic manner, on the surrounding country, while the center is rebuilt from time to time, each time in accordance with a more clearly conceived plan and a more majestic order, tearing down the old sections with their labyrinths of alleys, and projecting towards the periphery new avenues, more direct, broader and more commodious.
>
> [Bourbaki 50, p. 230]

This metaphor reinforces the architectural features of mathematics. In fact, Bourbaki uses this metaphor as a guideline for his conception of mathematics as a hierarchy of structures. 'The organising principle', Bourbaki says, 'will be the concept of a hierarchy of structures, going from the simple to the complex, from the general to the particular' [Bourbaki 50, p. 228].

According to Bourbaki, there is a central nucleus formed by the *mother-structures* (algebraic structures, structures of order, topological structures). Beyond this first nucleus, appear the *multiple structures*, which are combinations of mother-structures, such as topological algebra and algebraic topology. Further along, in the periphery, there are the *particular theories*, such as theory of numbers, real analysis, complex analysis, differential geometry. These classical theories, however, have no longer their autonomy: they are crossroads, 'where several more general mathematical structures meet and react upon one another' [Bourbaki 50, p. 229].

The mathematics educationalist E. Begle (1979) gives a more 'mathematical' picture of mathematics, that is, as a graph:

> We can think of mathematics as an immense three-dimensional linear graph, with facts and concepts as the nodes, and operations and principles as the connecting arcs ... Thus we can consider that part of the task of mathematics education is to assist our students to construct in their own minds selected parts of this mathematical network.
>
> [Begle 79, p. 7]

The mathematician P. Halmos completes Begle's and Bourbaki's metaphors, saying simply that mathematics is architecture:

> Mathematics is security. Certainty. Truth. Beauty. Insight. Structure. Architecture.
>
> [Halmos 85, p. 127]

In my view, 'mathematics as architecture' is a metaphor which is convenient for formalists. Despite my dislike for formalism, I believe that the notion of architecture of mathematics as a way of organising formal mathematics is important for discourse analysis.

ORGANISATION OF FORMAL MATHEMATICS

Architecture of mathematics is synonymous with organisation of formal mathematics, that is, the network of its definitions and theorems. It is through formal definitions that some authors of undergraduate mathematics textbooks introduce mathematical concepts. It is through theorems that they present properties and relations between concepts.

FLOW CHART OF THEOREMS

The core of the formal mathematics contained in a textbook is the set of its theorems. This core is an organised set of theorems. It is important to perceive it as a network, rather than as a mere sequence of theorems.

It is inevitable that theorems are written one after another, however this order of writing does not mean that the set of theorems is 'linearly ordered', where each theorem has the same significance as another one. On the contrary, theorems differ in importance: some of them are more important than others, whatever the meaning of "importance".

This collection of theorems is actually a network, a graph, where each element is linked to other elements by the relation of dependence. Analysis of the flow chart of dependence detect which theorems are architecturally important, in the sense that they enter into the proofs of many theorems; in a chart of dependence, such theorems are the nodes from which a great number of arrows originates.

EXAMPLE

Few authors present a general overview of chapters showing how theorems are linked. The author of the textbook **Introduction to Complex Analysis** (1990), H.A. Priestley, is a rare exception. For example, he shows the organisation of the theorems in chapter five through the flow chart in figure 8.1.

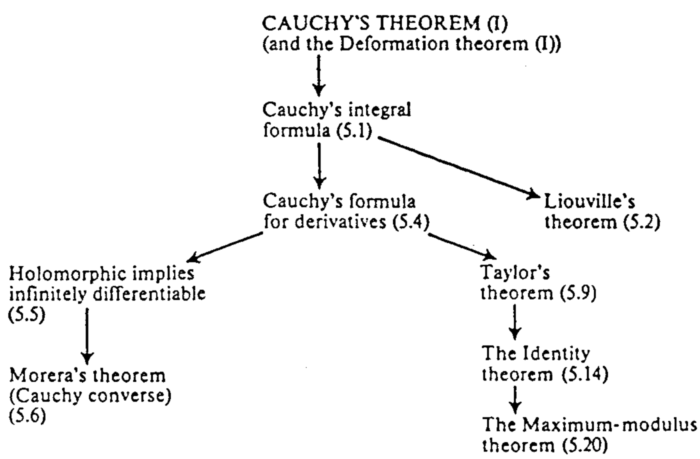


Figure 8.1 Priestley's flow chart

This flow chart indicates clearly the architectural importance of Cauchy's integral formula and of Cauchy's formula for derivatives; they are used as tools in the proofs of many other theorems.

The flow chart of dependence gives then a general idea about the way the network of theorems is constructed by the author.

For example, the flow chart presented by Priestley shows that the terminal theorems are Morera's, Maximum modulus, and Liouville's theorems. As a matter of fact, Morera's theorem is a kind of converse of Cauchy's theorem, and the maximum modulus and Liouville's theorems are theorems about the 'behaviour' of analytic functions.

In order to illustrate how complicated can be a network of theorems, I present in the figure 8.2 the flow chart of the theorems of the volume I of Euclid's **Elements of Geometry**, according to [Granger 74, p. 46].

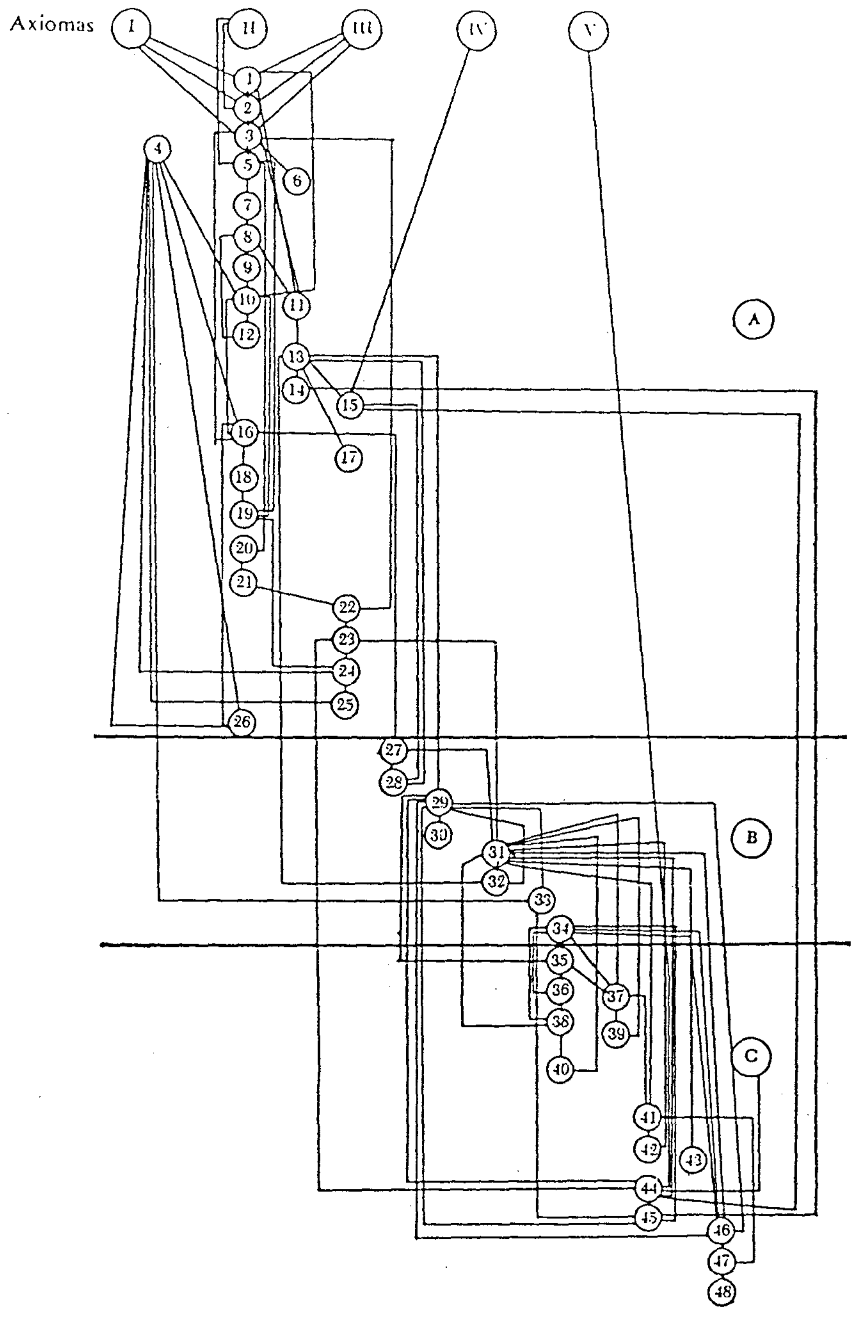


Figure 8.2 *Elements of Geometry*'s dependence flow chart

8.2 MATHEMATICAL STYLE

I understand by *style* the way the author organises his or her discourse. More precisely, there are three kinds of organisation in the discourse of undergraduate mathematics textbooks:

(a) *organisation of mathematical information*, that is, architecture of mathematics;

(b) *organisation of the negotiation author-reader:* negotiation of truth, intuitions, perspectives, values, etc.

(c) *organisation of readers' activities*, organisation of exercises (text as a source of activities),

Style will mean the way the author copes with this treble organisation.

HEURISTIC STYLE VS. FORMALISTIC STYLE

There exist two clearly conflicting styles of writing mathematics, which, for simplicity, I call heuristic style and formalistic style. The heuristic style emphasizes intuition, colloquial language, problem-solving, whereas the formalistic style emphasizes rigorous proof, precise language, theory-building.

HEURISTIC STYLE

G. Polya in the preface to his joint textbook with G. Latta, **Complex Variables** (1974), propounds the following guidelines of what I call the heuristic style:

1. Start from something that is familiar, useful, or challenging - from some connection with the world around us, from the prospect of some application, or from an intuitive idea.

2. Do not be afraid of using colloquial language when it is more suggestive than the conventional precise terminology. In fact, do not introduce technical terms before the student can understand the need for them.

3. Do not enter too early or too far into the heavy details of a proof. First, give a general idea or just the intuitive germ of the proof.

FORMALISTIC STYLE

N. Bourbaki and J. Dieudonné (one of the founders of Bourbaki's group) support opposite principles:

1. From the axiomatic point of view, mathematics appears thus as a storehouse of abstract forms - the mathematical structures ... Of course, it can not be denied that most of these forms had originally a very definite intuitive content; but, it is exactly by **deliberately throwing out this content**, that it has been possible to give these forms all the power which they were capable of displaying and to prepare them for new interpretations and for the development of their full power'. [Bourbaki 50, p. 231]

2. 'The only thing that we believe indispensable is to write our proofs so that the starting point (axiom or theorem) be clearly precise as well as each reasoning step; for this it is necessary to have a common language without ambiguity and not under the domination of the diversity of "intuitions"; it is this language that the axiomatic method and the "formalisation" of mathematics provide us'. [Dieudonné 68, p. 249]

3. 'Returning to the overall conception of Bourbaki, proofs were to be given in full (no part was ever "left to the reader" or relegated to exercises), and (barring accidental mistakes, of which there were a few) with the utmost precision' [Dieudonné 82, p. 619].

Bourbaki's treatise is not really intended for the use of undergraduate students, however their principles are followed by formalist authors of undergraduate textbooks.

There are two important variants of the formalistic style: Bourbaki's style and Landau's style.

BOURBAKI'S STYLE

The characteristics of Bourbaki's style are:

(a) emphasis on the architecture of mathematics,

(b) use of axiomatic method,

(c) maximum of generality,

(d) from general to particular,

(e) proofs of all theorems.

Despite the logical formalism, Bourbaki's texts have many examples, exercises, metadiscourse and historical notes. Figures are few.

LANDAU'S STYLE

Edmund Landau wrote two textbooks in the early 30's in the so-called Euclidean style, Satz-Beweis (theorem-proof) style or Landau's style, translated as **Foundations of Analysis** (1951) and **Differential and Integral Calculus** (1951).

Landau says in the preface to his textbook **Grundlagen der Analysis** (1930):

> My book is written, as befits such easy material, in merciless telegram style ("*Axiom*", "*Definition*", "*Theorem*", "*Proof*", occasionally "*Preliminary Remark*", rarely words which do not belong to one of these five categories).
> [Landau 51a, preface]

This textbook has no examples, and no figures. Motivation, intuitions, explanations, and metadiscourse are given in the 'preliminary remarks'.

How does he negotiate axiomatization with the students? He demands from the readers knowledge of the language and good 'mental' qualities:

> I will ask of you only the ability to read English and to think logically - no high school mathematics, and certainly no higher mathematics. [*Ibidem*, preface for the student]

However, as axiomatization means starting from the very beginning, the author should make *tabula rasa* of readers' knowledge; knowledge is not required:

> Please forget everything you have learned in school; for you haven't learned it.
>
> [*Ibidem*, preface for the student]

In his second textbook **Einfuhrung in die Differentialrechnung und Integralrechnung** (1934), Landau continues to write in Satz-Beweis style. Despite the absence of figures, he now gives some examples. There are no applications and no exercises.

In the next section, I will investigate Thron's discourse, who is a follower of Landau's style.

8.3 THRON'S DISCOURSE

In this section I will analyse the discourse of W.J. Thron in his textbook **Introduction to The Theory of Functions of A Complex Variable** (1953), focusing on three aspects: architecture, activities, and negotiation.

The scheme of discourse analysis I will use here is the following:

1. ARCHITECTURE OF MATHEMATICS: how mathematics is organised (author's construction of mathematics). Two aspects are relevant:
(a) SETTING OF THE THEORY: foundations or pre-requisites,
(b) DEVELOPMENT OF THE THEORY: organisation of the contents; the network of definitions and theorems.

2. ACTIVITIES: how readers' activities are organised (readers' construction of mathematics). Analysis of exercises.

3. NEGOTIATION: how the author interacts with the readers.
(a) NEGOTIATION OF TRUTH: proofs or rhetorical arguments.

(b) NEGOTIATION OF COMPREHENSION: figures, examples, appeals to intuition, analogies, metaphors.
(c) NEGOTIATION OF LANGUAGE: nomenclature, notation, metadiscourse.
(d) OTHER KINDS OF NEGOTIATION: historical aspects, applications.

Note As it will be shown below, Thron's discourse is not a very common discourse in the field of Complex Analysis; I have selected it because it is a deviant case, in the hope that its analysis can help the understanding of more "standard" discourses.

8.3.1 Architecture

How does Thron organise the mathematics in his text?

A. SETTING OF THE THEORY

Analysis of the preface and a scanning of the introductory part of his text reveal that Thron is determined to give an axiomatic presentation of Complex Analysis. He outlines his approach in the first paragraph of the preface, which I will quote emphasizing some features with bold letters.

> A student of mathematics who is taking a course in the theory of functions of a complex variable should in general be mature enough to appreciate a **rigorous treatment** of the subject. The better student will even have become **quite impatient with the phrase "it can be shown"** which he is likely to have encountered only too frequently in the years before. A definite need thus exists for an introduction to function of complex variables in which **all results are derived from a simple set of axioms**. Since no such book is available, I have attempted to fill the gap by writing a text in which **occur neither "intuitive proofs" nor theorems for whose proofs the reader is refered to other sources.**
>
> [Thron, preface]

From this passage I presume that Thron's project is guided by the following four principles:

1. The theory should start from its foundations (all results are

derived from a simple set of axioms);

2. Every theorem should be proved (no phrase "it can be shown");

3. Proofs should be rigorous (no "intuitive proofs");

4. The text should be self-contained (no theorems for whose proofs the reader is referred to other sources).

I will show that Thron will not be able to carry out these four principles in the construction of his text; in fact, he will violate all of them.

The first principle is violated because the set of axioms of set theory chosen by Thron is incomplete. For example, both the axiom of union (if X is a set, there is a set consisting of all elements of all elements of X), and the axiom of power set (if X is a set, there is a set consisting of all subsets of X) are used but neither given as axioms nor proved.

Therefore, NOT ALL RESULTS IN THRON'S TEXTBOOK ARE DERIVED FROM A SIMPLE SET OF AXIOMS.

Thron himself declares in the preface that the second principle will be violated:

> The only intentional exception to this rule is the **omission of those proofs that seemed to be simple enough to be left as exercises for the reader**. In certain borderline cases outlines of proofs are given.
>
> [*Ibidem*, preface]

Later on he will "negotiate" with the reader the truth of the theorems by means of the following commandment, which is certainly incompatible with rigorous treatment, or with axiomatic method of exposition:

> Remark 2.1 If no proof is given for a theorem, it means that the proof is left to the reader as an exercise.
>
> [Thron, p. 8]

Therefore, NOT ALL THEOREMS IN THRON'S TEXTBOOK ARE PROVED BY THE AUTHOR.

The third principle is violated in many places, for example in section 28, where he confesses that he was not rigorous:

> Remark 28.5 Throughout this section we have been fairly careless as far as the difficulties that will arise from the values ∞ and 0 are concerned. It is hoped that the reader noticed this and provided the missing arguments.
>
> [Thron, p. 201]

Therefore, NOT ALL PROOFS IN THRON'S TEXTBOOK ARE RIGOROUS.

The fourth principle is violated for example by the following

> Remark 2.7 It thus follows that apart from isomorphism there exists at most one complete ordered field. The question remains whether there can exist a set satisfying all the requirements of a complete ordered field. A partial answer to this is that, if there exists a set satisfying the Peano axioms, then there exists a set that forms a complete ordered field. This is shown by constructing real numbers from the natural numbers. This process is carried out in detail by E. Landau in his **Grundlagen der Analysis**.
>
> [Thron, p. 15]

Therefore, THRON'S TEXTBOOK IS NOT SELF-CONTAINED.

INTUITION VS. RIGOR

In the third paragraph of the preface Thron refers to the conflict between intuition and rigor. Thron "justifies" de-emphatization of intuition in his text with the following argument:

> Intuition and rigor are fundamental in mathematics. That the former appears only in occasional remarks in the following pages should not be taken to mean that I underrate its

> importance. However intuition is usually overemphasized in the early years of a student's mathematical training, and it was part of my job, as I saw it, to rectify this.. In addition by its very nature intuition is subjective and as such is best developed by the student himself to suit his own needs.
>
> [*Ibidem*, preface]

Thron's discourse is purely rhetorical. If Thron acknowledges the importance of both intuition and logic, one expects that he endeavours to give a balance between intuitive and logical reasoning in the text. A scanning of his text shows that there are no figures, no examples, and no applications in his text. This fact clearly implies that in Thron's discourse practically there is no room for intuition.

Thron justifies the paucity of intuition in his text, declaring that intuition is a subjective, personal matter, and not an interpersonal matter: intuition is best developed by the student himself. This means that, according to him, intuition is not negotiable between author and reader.

In my view, this is an educational error, since author's intuitions can help the development of readers' intuitions. Thron himself, despite his adherence to logical formalism, makes appeal to intuition in case logic does not work. For example, after a complicated definition of orientable surface, Thron negotiates its intuitive understanding:

> The essential content of the definition **in intuitive language** is that a surface is orientable if it is possible to introduce a positive direction for a sufficiently large family of curves on the surface and if this direction is preserved under every continuous deformation, on the given surface, of one curve of the family into another curve.
>
> [Thron, p. 221]

B. ORGANISATION OF THE CONTENTS

Thron's project is the axiomatization of Complex Analysis.

Thron starts with the axioms of set theory (section 1), the theory of real numbers (section 2), and the theory of cardinal numbers (section 3), as a preparation for the section 4, where he introduces complex numbers.

Thron uses section 5 for the proof of the general associative and commutative laws of the sums and products, which seems to be out of context.

After this, as a preparation for the section 8 on the plane of complex numbers, Thron deals with the theory of topological spaces (section 6), and the theory of metric spaces (section 7).

In section 9 he introduces differentiability of complex functions. In section 10 he goes back to foundations, developing the theory of real functions of real variables.

In section 11 he introduces curves and regions in the plane of complex numbers.

As a preparation for the section 15 on integration, Thron presents more foundational matters: some combinatorial topology (section 12), Jordan curves (section 13), rectifiable and directed curves (section 14).

After this section the course is quite standard. However it should be noticed that Thron displays some theorems of Complex Analysis in a *sui generis* order. For example, the theorem on Cauchy-Riemann equations, usually one of the first theorems of Complex Analysis, appears near the end of the book, in section 27 on conformal maps (p. 183).

8.3.2 Activities

In order to know what kind of activities Thron propound to his readers, I made the classification of exercises of Thron's textbook in six categories:

1. Proof of a theorem stated in the text, but unproven by the author;
2. Completion of a proof left uncomplete by the author;
3. Exercises to prove, that is, new theorems to prove;
4. Extension of the theory, where a new concept is introduced;
5. Exemplification;
6. Exercises to find, that is, routine exercises whose purpose is just to apply the theory to particular objects or situations.

The table 9.1 shows the results of this classification.

CLASSIFICATION OF EXERCISES

PROOF OF A THEOREM STATED IN THE TEXT:	116
COMPLETION OF A PROOF:	24
EXERCISES TO PROVE:	22
EXTENSION:	2
EXEMPLIFICATION:	6
EXERCISES TO FIND:	33
OTHER TYPES:	23
TOTAL:	223

Table 9.1 Classification of exercises in Thron's textbook

DATA ANALYSIS

Exercises of type 1 and 2 are what I call gap-filling exercises. The table shows that more than 60% of the exercises are gap-filling exercises. This implies two things:

(i) Thron's text is very porous, in the sense that it has many gaps; readers are often asked to do the author's job, that is, to prove theorems.

(ii) Thron uses the section of exercises, neither to train the students, nor to open new perspectives to the readers, but to make readers go back to the text. The text is all; exercises are then at the service of the text.

8.3.3 Negotiation

(a) NEGOTIATION OF TRUTH

Thron negotiates the truth of the theorems in three ways:

(i) proving the theorem;

(ii) outlining the proof;

(iii) leaving the proof to the reader.

It was already shown that Thron leaves many theorems (to be exact, 140) to the readers to prove or to complete the proofs.

The question is: how does Thron prove theorems or outline proofs?

Thron is an adept of Satz-Beweis style forged by Edmund Landau, which I have discussed in section 3.5. His textbook is a sequence of axioms, definitions, and theorems, interspersed by rare remarks. No figures, no examples, no motivations, no applications.

As a principle, Thron is economic with the use of arguments. Some proofs are telegrams like the following "proof" of the theorem 19.10:

> *Proof*. Th. 19.8 and Th. 19.7.

Some outlines of proof are rhetorical rather than logical; they look like more a comment or a order than an argument. For example, the outline of proof of the theorem 28.3 is an educated order:

> *Outline of proof* A rather involved computation which the student can carry through for himself ... [Thron, p. 192]

The outline of proof of the theorem 13.10 is a comment:

Outline of proof A careful examination of the possible configurations leads directly to this result. [Thron, p. 81]

Why does Thron use Satz-Beweis style?

One reason is practical: the foundations of the theory occupy much space, and then it is necessary to reduce the space for argumentation and explanation.

Another reason is authoritarianism: the author can control the whole discourse, without paying attention to the learners' needs.

(b) NEGOTIATION OF COMPREHENSION

As I observed earlier, there are neither figures, nor examples in it. Thron declares in the preface that he is a follower of Edmund Landau, whose textbook **Grundlagen der Analysis** (1930) has neither figures nor examples as well. However, Landau was more friendly with his readers than Thron in the textbook **Differential and Integral Calculus** (1951), where some examples appear, even though figures are still absent.

Absence of figures means that Thron has opted for verbalism and symbolism; absence of examples means that Thron has opted for generality and abstraction; the absence of both figures and examples indicate that Thron will minimize other appeals to intuition.

APPEALS TO INTUITION

I mentioned above one example of appeal to intuition. Another remark provides the intuition of Riemann surface:

> Remark 31.3 An **intuitive picture** of what a Riemann surface is can be obtained by the following consideration. Since every analytic function can be generated by its power series expansions, we provide ourselves with circular pieces of paper, one for each element at each point of the domain of existence of the function ... Starting with some element we glue to its piece of paper (with the right amount of overlapping) all pieces that belong to elements that are

> direct continuations of the original element and proceed with this process until all pieces have been glued together ...
>
> [Thron, p. 222]

Even in this case Thron rejects the use of illustrations.

(c) NEGOTIATION OF LANGUAGE

Definitions, names and notations are all established without motivation, or any other type of negotiation. No alternative names, no alternative notations.

Metadiscourse is made through the rare remarks. Sections have no introduction.

(d) OTHER KINDS OF NEGOTIATION

Nothing about historical aspects of Complex Analysis. No applications.

8.3.4 Evaluation

Thron's axiomatization of Complex Analysis is incomplete and non-rigorous, contrary to what he says in the preface. Thron's construction of mathematics is not rigorous but his discourse is "rigorousist". Thron is essentially a monist: he opts for rigor at the expense of intuition.

Thron's basic error of architecture is his foundationalism: to construct Complex Analysis on top of set theory, real numbers, topology, and combinatorial topology, instead of using them as a language, or as a tool, is an absurd project, from the viewpoint of teaching and learning mathematics.

Thron adopts the Satz-Beweis style, which means maximum of information, and minimum of explanation, and argumentation. Its use in textbooks is pedagogically disastrous.

Thron's textbook contains only axioms, definitions, theorems, proofs, and occasional remarks. No figures, no examples, no applications, no motivations. It is clear that Thron is not intended to negotiate comprehension.

Moreover, the presence of so many logical gaps (lack of proofs, lack of details in proofs) indicates that Thron's discourse is the discourse of the imposition of truth.

PART III

CASE STUDIES

CHAPTER 9

CASE STUDIES

Introduction

In this chapter I will investigate, as case studies, some aspects of the discourse of Complex Analysis.

In section 1, I will analyse the discourse of the introduction of complex numbers in Complex Analysis texbooks.

In section 2, I will analyse the discourse of the introduction of Riemann surfaces, by which the authors open to the readers the perspective of a new concept that is too complicated to be treated in introductory textbooks.

In section 3 I will discuss the discourse of Stewart & Tall's textbook on Complex Analysis.

9.1 DISCOURSE ANALYSIS OF A CONCEPT: COMPLEX NUMBERS

Mathematicians have created several definitions of complex numbers. In this section I am interested in investigating two things: What definition of complex numbers do the authors of Complex Analysis textbooks negotiate with the readers? What is their discourse on the definition of complex number?

9.1.1 The Conflict Between Two Algebraic Definitions

First of all I will discuss the tension between two of the most used definitions: (i) complex number as an expression of the form a + bi, where a and b are real numbers and (ii) complex number as an ordered pair of real numbers (a,b).

Historically speaking, the first definition came first. Mathematicians have formally manipulated expressions such as $5 + \sqrt{(-15)}$ as if they possessed the same algebraic properties of real numbers, without attributing 'meaning' to them, since the times of Cardan (**Ars Magna**, 1545) until Wessel (1797), who introduced the geometric interpretation of complex number as a point in the plane. It is known that Gauss was the first to use the notation a + bi or a + ib and to call it *complex number*; therefore, I will refer to this definition as *Gauss' definition*.

The second definition, complex number as an ordered pair of real numbers, was first given by Hamilton (1837); therefore, this definition will be called *Hamilton's definition*.

- COMPLEX NUMBER AS AN EXPRESSION

Some authors of Complex Analysis textbooks like L.V. Ahlfors introduce complex numbers as symbolic expressions. I will show that this approach is supported by a heuristic principle, *the principle of the permanence of the formal law*.

AHLFORS' DISCOURSE

L.V. Ahlfors introduces complex numbers in the following way:

> It is fundamental that real and complex numbers obey the same basic laws of arithmetic. We begin our study of complex function theory by stressing and implementing this analogy ... From elementary algebra the reader is acquainted with the *imaginary unit* i with the property $i^2 = -1$. If the imaginary unit is combined with two real numbers α, β by the processes of addition and multiplication, we obtain a *complex number* $\alpha + i\beta$.
>
> [Ahlfors, p. 1]

Notice that Ahlfors emphasizes that complex numbers are expected to obey the same arithmetical laws that govern the real numbers.

• THE PRINCIPLE OF THE PERMANENCE OF THE FORMAL LAW

This way of introducing complex number is based on a heuristic principle called the *principle of the permanence of the formal law,* which establishes the following:

> Every time a newly introduced concept depends upon operations previously employed, the propositions holding for these operations are assumed to be valid still when they are applied to the new concepts.
>
> [Durège 1896, p. 9]

This principle is in general attributed to H. Hankel (1867), even though it was previously formulated in other terms by M. Ohm (1822) and G. Peacock (1833).

Hankel's principle can be invoked for example to justify the definition of the power a^n, where n is a negative integer. The only reason to define a^{-1} as $1/a$ is that the new concept should obey the law

$$a^m/a^n = a^{m-n}$$

which is known to be valid when m,n are positive integers, $m > n$.

In the case of complex numbers, the principle says that the laws of arithmetic (the properties that characterize real numbers as a field) will remain valid for them. It is this principle that justifies the use of the notation a + bi or a + ib.

Hankel's principle is just a heuristic principle, not a logical principle. We have to be careful in using such a principle, because in every extension of operations, some laws remain valid, while other laws are broken.

For example, properties such as associativity and commutativity that are valid for natural numbers remain valid for the addition of integers, however the law of least element (every subset has a least element) is broken. The properties of a ring possessed by the integers remain valid for the addition and multiplication of rational numbers, however, the property of the successor (each number has a successor) is not valid any more. In the case of complex numbers, the law of trichotomy (any two numbers are comparable) which is valid within real numbers is broken in the case of complex numbers: it is not possible to define an order relation compatible with the arithmetic operations in the same way as for the real numbers.

It is educationally important to call attention not only to the gains in the extension of an operation but also to its losses.

In summary, the choice of the definition of complex number as an expression of the form a + bi or a + ib is grounded on a heuristic principle.

- COMPLEX NUMBER AS A PAIR OF REAL NUMBERS

The definition of a complex number as a pair of real numbers is usually considered as a formal definition, whereas the previous definition is considered informal.

Many authors adopt it mainly because they consider it a 'rigorous' definition: complex numbers are defined in terms of real numbers, and not in terms of symbols. Nevertheless, they invariably get rid of the notation of complex number as a pair of real numbers in favour of the Gauss' notation as soon as possible.

CHURCHILL & BROWN'S DISCOURSE

Churchill & Brown introduce complex numbers as Hamilton did:

> Complex numbers z can be defined as ordered pairs $z = (x,y)$ of real numbers x and y, with operations of addition and multiplication to be specified below ... The *sum* $z_1 + z_2$ and *product* $z_1 z_2$ of two complex numbers $z_1 = (x_1,y_1)$ and $z_2 = (x_2,y_2)$ are defined by the equations
>
> $$(x_1,y_1) + (x_2,y_2) = (x_1 + x_2, y_1 + y_2),$$
>
> $$(x_1,y_1)\,(x_2,y_2) = (x_1x_2 - y_1y_2,\ y_1x_2 + x_1y_2)$$
>
> [Churchill & Brown, p. 1]

The pedagogic problem of Hamilton's definition is that multiplication of complex numbers appears arbitrary, artificial, out of the blue.

Churchill & Brown present their definition without any introductory discourse; the reader has just to swallow it.

STEWART & TALL'S DISCOURSE

Stewart & Tall also adopt Hamilton's definition:

> We define a *complex number* to be an ordered pair (x,y) of real numbers. Addition and multiplication of complex numbers are defined by:
>
> $$(x_1,y_1) + (x_2,y_2) = (x_1 + x_2, y_1 + y_2),$$
>
> $$(x_1,y_1)\,(x_2,y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2)$$
>
> [Stewart & Tall, p. 10]

After the definition, Stewart & Tall declare explicitly that they have begun at the end of the historical process:

> This definition is the culmination of several centuries of struggle to understand complex numbers, and it shows how elusive a simple idea can be.
>
> [Stewart & Tall, p. 10]

Contrary to Churchill & Brown, Stewart & Tall prepare the reader to accept Hamilton's definition through an introductory chapter on the historical development of complex number and complex analysis. However, they make clear that they will not slavishly copy history:

> ... we feel that cultural changes often affect the status of conceptual problems: what was once an important difficulty can become a triviality when viewed with hindsight. It is not always necessary to drag today's students through yesterday's hang-ups.
>
> [Stewart & Tall, preface]

This reversal of history is reinforced in another context:

> Though sometimes it is useful to see the development of the theory in its historical context, it is not always necessary to fight the historical battles again.
>
> [Stewart & Tall, p. 8]

As Stewart & Tall's policy is to reverse history, they 'recover' the notation $a + ib$ as a logical consequence of field-isomorphism between the set of complex numbers of the form $(x,0)$ and the set of real numbers. In other words, as $x = (x,0)$, $y = (y,0)$, and $i = (0,1)$, we can write

$$x + iy = (x,0) + (0,1)(y,0).$$

Therefore, Stewart & Tall replace Hankel's heuristic principle used implicitly by Ahlfors by a logical reasoning.

In my view, both discourses, Stewart & Tall's and Ahlfors' discourses, illustrates how distinctly dialectical authors' discourse can be.

On the one hand, Ahlfors's discourse (the way the definition is formulated, the language used, and the overall context of logical development of a theory) gives the impression that logic will be the guide. The heuristic nature of the assumption is concealed. This can disorient the learner, who thinks that logic is being used but cannot understand or reconstruct any underlying logic for the principle. Few authors are conscious enough to admit and declare they are doing heuristic rather than logic.

On the other hand, Stewart & Tall's discourse (historical introduction, discussion of controversies in the past) suggests that complex numbers will be treated in a historico-critical perspective. However, they opt for the formal definition, 'which satisfies our modern standards of rigour' [Stewart & Tall, p. 1]. Paradoxically, they invoke history in order to give support to an anti-historical approach to complex numbers.

- THE MIDDLE WAY: NEHARI'S DISCOURSE

Z. Nehari solves the tension between Gauss' definition and Hamilton's definition by the compromise, considering both of them as alternative definitions. First he defines complex number as a symbol of the form $a + bi$, alleging convenience. Afterwards, by way of contenting critics, he presents Hamilton's definition as well.

It is interesting to observe how Nehari uses rhetorical and heuristic schemes in support of his position.

Nehari starts his discourse by posing the following problem: a quadratic equation with real coefficients may, or may not, have real solutions. Is there a more general class of numbers in which such equation always has solutions?

In order to solve this problem, he draws an analogy with negative numbers:

... let us suppose that we have no conception of negative numbers and that we are called upon to solve the equation $x + a = b$, where a and b are positive numbers. We would then have to distinguish between two cases. If $b \geq a$, the equation has a solution, and if $b < a$ it has not. As we know, this distinction is made unnecessary by the introduction of negative numbers, i.e., BY POSTULATING THAT the equation $x + 1 = 0$ has a solution and by denoting this solution by -1. This procedure may initially look rather arbitrary, but it is then JUSTIFIED by showing that the usual laws of arithmetic can safely be applied to negative numbers, provided a few simple rules - such as, for example, $(-a)(-b) = ab$ - are observed.

[Nehari, p. 1]

Nehari does not justify the existence of negative numbers logically, for example, by claiming that they can be defined as pairs of natural numbers. He justifies the existence of negative integers by hindsight, because they work arithmetically well. In other words, he uses Hankel's heuristic principle.

He solves the problem of roots for the equation $x^2 + 1 = 0$ by analogy:

The problem of finding a number whose square is a given negative number can be treated in a similar way. We *define* a number i by the equation

(1) $$i^2 = -1$$

and we then show that the introduction of this number and the use of numbers of the form

(2) $$\alpha = a + bi$$

where a and b are real numbers, are compatible with the usual laws of arithmetic if certain elementary rules are observed.

[*Ibidem*, p. 2]

Central to his argument is then the appeal to the analogy between negative numbers and complex numbers (heuristic device). He takes for granted that the reader has already accepted the 'existence' of negative integers (rhetorical device).

According to Nehari, the use of the expression a + bi is convenient:

> The use of the symbol *i* makes it possible to condense all the laws of operation involving numbers of the form a + bi into the following simple rule: Apply the usual rules of algebra, and whenever i^2 appears, replace it by -1.
>
> [Nehari, p. 2]

Nehari then imagines a possible criticism of this heuristic presentation:

> At this point, one may object that all we have done is invent a new symbol, and that this does not constitute A PROOF THAT THERE EXISTS a number *i* with the very unusual property $i^2 = -1$.
>
> [*Ibidem*, p. 2]

It is not clear whether he himself is raising the question or whether he expects someone else will raise it. Anyway, he takes this criticism into account in his discourse, for afterwards he will give Hamilton's definition as an alternative.

Nehari affirms that Hamilton's definition solves the problems of existence and legitimacy of use:

> **The question as to the existence of these numbers and as to the legitimacy of their use in algebraic operations** is, however, easily settled once it has been realized that operating with a number of the form (2) is the same as operating with the two real numbers *a* and *b* according to certain simple rules. It is entirely possible to discuss

these numbers without ever mentioning the "imaginary" number i.

[*Ibidem*, p. 2]

The problems pointed out by Nehari are of different natures: the problem of the existence of complex numbers is a philosophical problem, while the problem of legitimacy of their use is sociological. Here Nehari is being rhetorical since legitimacy is established by consensus and not by logic.

After all, Nehari is pragmatic: he resolves the dilemma by accepting both definitions. The first, for convenience of use; the second, because of rigour.

COMPARING NEHARI'S AND AHLFORS' DISCOURSES

Ahlfors also acknowledges that his definition of complex numbers as symbols was defective:

> So far our approach to complex numbers has been completely uncritical. We have not questioned the existence of a number system in which the equation $x^2 + 1 = 0$ has a solution while all the rules of arithmetic remain in force.
>
> [Ahlfors, p. 4]

However Ahlfors shows that there is no need to adhere to the concept of complex number as a pair of real numbers in order to solve this sophisticated problem of existence. He constructs a field that satisfies the required condition in the following way:

> Consider all expressions of the form $\alpha + i\beta$ where α,β are real numbers while the sign + and i are pure symbols (+ does *not* indicate addition, and i is *not* an element of a field). These expressions are elements of a field F in which addition and multiplication is defined by
>
> (1) $(\alpha + i\beta) + (\gamma + i\delta) = (\alpha + \gamma) + i(\beta + \delta)$,
>
> (2) $(\alpha + i\beta)(\gamma + i\delta) = (\alpha\gamma - \beta\delta) + i(\alpha\delta + \beta\gamma)$

(observe the two different meanings of the sign +).
[Ahlfors, p. 6]

Here Ahlfors calls attention to the problem of homonymy of the symbol (+). This problem is similar to the case of vector space which I have discussed in chapter 7.

Ahlfors' and Nehari's approaches are to some extent similar. Both consider the definition of complex numbers as expressions as valid, and both criticize this definition saying that there is a problem of existence in this approach. However they diverge about the way of solving this problem.

Nehari changes the notation and the conception in order to put his approach right. On the other hand, Ahlfors proves that his approach can be fixed without any change of notation or conception. Ahlfors has then showed that Hamilton's definition can be avoided, because it is not necessary even for introducing 'rigour' in the discourse. However it is not clear from Ahlfors' text why he avoids the use of Hamilton's definition.

MILLER'S DISCOURSE

K.S. Miller expects that his reader is already familiar with complex numbers:

> For example, the reader has an INTUITIVE notion of a complex number a + bi where $i = \sqrt{(-1)}$ and a and b are real.
> [Miller, p. 1]

In my view, Miller is being rhetorical by using the term 'intuitive' since there is no intuition involved with complex number, unless one is provided with a geometric interpretation.

He suggests then that a complex variable $z = a + bi$ can be seen as a pair of real variables, however he warns that this approach should not be carried to extremes:

> We do not wish to mislead the reader. While the technique of considering a complex number as two real numbers enables us to prove many theorems, this method of approach should not be carried to extremes. The essential beauty and utility of the theory of analytic functions rests on the treatment of the complex variable as an *entity* and not as a pair of real variables.
>
> [Miller 60, p. 1]

Miller is affirming then that a function of a complex variable is not exactly the same as a function of two real variables. For Miller, the problem of considering complex number as a pair of real numbers is really definitional, and not only notational. Miller's last sentence is completely rhetorical since he appeals to 'beauty' and 'utility', and says that a pair of real numbers is not an entity. For any mathematician, a pair of real numbers is an entity, if entity means a mathematical object.

Nevertheless, Miller will define complex numbers as ordered pairs of real numbers!

• TENSION BETWEEN GAUSS' AND HAMILTON'S DEFINITIONS

Other authors acknowledge the existence of this conflict between Gauss' and Hamilton's definitions, or at least between the notations $a + bi$ and (a,b).

PENNISI'S DISCOURSE

For example, L.L. Pennisi starts his introduction to complex numbers by saying that he will adopt Hamilton's definition instead of Gauss' definition, without any justification:

> An approach to complex numbers is to consider the properties of expressions of the form $a + bi$, where a and b are real numbers and i is one of the imaginary roots of $x^2 + 1 = 0$. We shall approach the subject of complex numbers, however,

> by considering the set of all ordered pairs of real numbers.
> [Pennisi, p. 1]

Pennisi only says that there are different approaches to complex numbers; as he does not give more details, it seems that such discourse is directed to the instructor, and not to the student.

This decision becomes clear on page 3, where Pennisi deduces that the pair [a,b] can be written as a + ib:

$$[a,0] + [0,1]\,[b,0] = [a,0] + [0,b] = [a,b]$$

> hence

$$[a,b] = a + ib$$

> Thus [a,b] may be denoted by a + ib with the understanding that a is [a,0], b is [b,0], and a + ib is a plus i times b.
> [*Ibidem*, p.3]

Pennisi does not accept Gauss' definition because of formal rigour: complex numbers should be set in terms of real numbers only and not in terms of expressions. Pennisi accepts a + bi as a notation, but not as a definition of complex numbers.

MORETTI'S DISCOURSE

Contrary to Pennisi, another author G. Moretti criticises the notation of complex number as an ordered pair of real numbers:

> Much of the practicality and simplicity of the theory would be obscured if another notation, for example $z = (x,y)$ were used instead of $z = x + iy$.
> [Moretti, p. 27]

MARSDEN'S DISCOURSE

J.E. Marsden utilizes Hamilton's notation in his definition, however he soon adopts Gauss' notation:

> Rather than using (x,y) to represent a complex number, we will find it more convenient to return to more standard notation as follows ... A single symbol such as $z = a + bi$ is generally used to indicate a complex number ... The reason for using the expression $a + bi$ is twofold. First it is conventional. Second, the rule $i^2 = -1$ is easier to use than the rule $(a,b)(c,d) = (ac - bd, bc + ad)$ although both rules produce the same result.
>
> [Marsden, p. 3]

Therefore, there exists conflict between mathematicians about Gauss' definition and Hamilton's definition.

9.1.2 GEOMETRIC DEFINITIONS

Gauss' and Hamilton's definitions are algebraic. There are two geometric definitions as well: complex numbers as points in the plane (Argand), and complex numbers as vectors in the plane (Wessel).

MARSDEN'S DEFINITION

J.E. Marsden defines complex numbers geometrically, as Argand did:

> The basic idea of complex number is credited to Jean Robert Argand, who suggested using points in the plane to represent complex numbers. The student will recall that the plane, denoted by $\mathbb{R}^2$, consists of all ordered pairs (x,y) of real numbers.
>
> [Marsden, p. 3]

Marsden's definition is the following:

> The system of complex numbers, denoted $\mathbb{C}$, is the set $\mathbb{R}^2$ together with the usual rules of vector addition and scalar multiplication by a real number *a*:

$$(x_1,y_1) + (x_2,y_2) = (x_1 + x_2, y_1 + y_2)$$

$$a(x,y) = (ax,ay)$$

and the operation of complex multiplication defined by

$$(x_1,y_1)(x_2,y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$$

[Marsden, p. 3]

It is noteworthy that Marsden suggests the definition of complex numbers as points in the plane, but in fact he defines $\mathbb{C}$ simultaneously as a real vector space and as a field. It is then implicitly assumed that vectors in the plane correspond to points in the plane.

In this way, he will easily give geometric meaning to the addition of complex numbers and to the multiplication of complex numbers by real numbers; however, in order to give some meaning to the multiplication, he will revert to the notation a + bi.

Marsden's definition in terms of pairs of real numbers formally coincides with Stewart & Tall's definition: however it has a different character. Marsden's definition is essentially geometric, whereas Stewart & Tall's definition is essentially algebraic.

DEFINITION, INTERPRETATION, OR REPRESENTATION?

Many authors use the words *interpretation* and *representation* instead of *definition*, when they refer to geometric objects.

Pennisi, for example, says that 'according to Gauss, complex numbers may be interpreted as points in a two-dimensional plane'. [Pennisi, p. 9]

Ahlfors explains why he adopts the term 'geometric interpretation' of complex number, instead of definition:

> The geometric representation derives its usefulness from the vivid mental pictures associated with geometric language. We take the point of view, however, that ALL CONCLUSIONS IN ANALYSIS SHOULD BE DERIVED FROM THE PROPERTIES OF REAL NUMBERS, AND NOT FROM THE AXIOMS OF GEOMETRY. For this reason we shall use geometry only for descriptive purposes, and not for valid proof, unless the language is so thinly veiled that the analytic interpretation is self-evident. This attitude relieves us from the exigencies of rigor in connection with geometric considerations.
>
> [Ahlfors, p. 12]

Ahlfors's analytic rigour is not followed by some authors who use geometric reasoning to prove theorems. For example, Nehari proves the triangle inequality using geometric reasoning:

> It is also known from elementary geometry that the sum of two sides of a triangle is larger than the third side, provided the three vertices of the triangle do not lie on a straight line. Applying this to the triangle $(0,\alpha,\beta)$ and noting that its sides are $|\alpha|$, $|\beta|$, $|\alpha - \beta|$, we find that $|\alpha - \beta| \leq |\alpha| + |\beta|$. If we replace β by $-\beta$ and observe that $|\beta| = |-\beta|$, we obtain the important inequality
>
> $$|\alpha + \beta| \leq |\alpha| + |\beta|.$$
>
> [Nehari, p. 8]

Therefore, authors who adopt algebraic definitions refer to geometric definitions as geometric interpretation or representation of complex numbers; on the other hand, authors who adopt geometric definitions refer to algebraic definitions as notations for complex numbers.

OTHER DEFINITIONS

• COMPLEX NUMBER AS A MATRIX

E.T. Copson does not accept the definition of complex number as a symbol:

> It is very desirable that we should provide a definition of such numbers which depends only on real numbers, instead of the formal introduction of the symbol $\sqrt{(-1)}$ of elementary algebra.
>
> [Copson, p. 2]

Copson regards complex numbers as geometric transformations. He defines complex numbers as matrices of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

where a,b are real numbers. In this definition, the imaginary unit is the matrix

$$i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

[Copson, p. 5]

Alternative definition in terms of matrices is given by B. Artmann: he adopts as the imaginary unit the transpose of the matrix used by Copson [Artmann, p. 77].

Copson's and Artmann's definitions are based on the same idea of rotation and enlargement, the difference being just notational. Copson represents a transformation by the following notation

$$(x', y') = (x, y) \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

whereas Artmann uses the notation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

• COMPLEX NUMBER AS A POLYNOMIAL

Ahlfors asks readers to prove that the complex-number system can be thought as the field of all polynomials with real coefficients modulo the irreducible polynomial $x^2 + 1$. [Ahlfors, p. 6]

This definition, due to Cauchy (1847), can be adequate for algebra, but not for analysis or geometry.

RETROSPECTIVE

There are at least six different definitions of complex numbers:

1. complex numbers as symbols of the form a + bi;
2. complex numbers as pairs of real numbers (a,b);
3. complex numbers as points in the plane;
4. complex numbers as vectors in the plane;
5. complex numbers as matrices;
6. complex numbers as classes of polynomials.

The algebraic definitions 1 and 2 are the most used in Complex Analysis textbooks.

Definition 1 is proper for algebraic manipulation; however, it lacks 'reality' and meaning.

Definition 2 has more 'reality' than definition 1; it facilitates the connection between complex numbers and points in the plane; however, multiplication is artificial, has no plausibility.

Definition 3 is adequate for the introduction of topological notions; however, there is no meaning in adding and multiplying points.

Definition 4 explains addition very well, however multiplication of vectors seems artificial.

Definition 5 is adequate to explain addition and multiplication of complex numbers; however it seems sophisticated to look at complex numbers as matrices.

Definition 6 is more adequate for algebraic treatment, rather than for analysis.

Geometric definitions (3,4,5) give geometric intuition and meaning to complex numbers, whereas algebraic definitions (1,2,6) give us ease of manipulation.

In summary, the adoption of a definition of complex number is not only a matter of taste; it involves the defence of a perspective, a point of view on complex numbers. There are advantages and disadvantages in adopting a particular one. In my view, teachers should show to the students the existence of this variety of perspectives, instead of choosing a particular one from the outset. Equally important, as has been stressed, is a realisation of the nature of the heuristic and rhetorical devices used to advance a particular perspective.

9.1.3 Preliminary Discourse on Complex Numbers

I am now interested in showing the variety of introductory discourse to complex numbers. Churchill & Brown and Stewart & Tall represent opposite points of views.

On the one hand, Churchill & Brown give no preliminary discourse. They go straight to the definition of complex numbers as an ordered pair of real numbers.

Stewart & Tall, on the contrary, are well aware of the rhetorical aspects of mathematical discourse. They have devoted a whole chapter to introduce the reader to the subject; to do what in rhetoric is called the *exordium* or the *proemium*.

Stewart & Tall evoke historical events to negotiate their definition of complex number. In the first chapter of their textbook, they give an account of the history of complex numbers and complex analysis, pointing to some definite steps in this history.

The first historical step was formal manipulation of complex numbers. In sixteenth century Cardan and Bombelli manipulated symbols such as $5 + \sqrt{(-15)}$ and $5 - \sqrt{(-15)}$ as if they were ordinary numbers.

The second step was formal manipulation of complex functions. In the eighteenth century John Bernoulli, Leibniz and Euler integrated and differentiated complex functions.

The third step was geometric interpretation of complex numbers; this was done by Wessel, Argand and Gauss at the beginning of the nineteenth century.

The fourth step was the formal definition of complex numbers, which according to Stewart & Tall, 'satisfies our modern standards of rigour' [Stewart & Tall, p. 1]. Hamilton's definition of complex numbers as ordered pairs of real numbers subject to certain explicit rules of manipulation 'has placed the complex numbers on a firm *algebraic* basis'. [*Ibidem*, p. 5]

An interesting point in Stewart & Tall's account is the divergence between John Bernoulli who affirmed that the logarithm of a negative number is real and Leibniz who affirmed that it is complex.

What is log(-1)? This could be a good starting point for an interactive teaching of Complex Analysis.

APPLICATIONS

J. Marsden's approach is tripartite: (a) he presents a historical sketch before defining complex number, (b) he mentions applications of complex numbers and complex analysis to other branches of mathematics, and to non-mathematical fields as well, and (c) he talks about the importance of the subject.

THE DISCOURSE ON EXTENSION

The majority of authors of Complex Analysis textbooks do not use the historical approach in the introduction of complex numbers. History is replaced by hindsight, and this causes some 'strangeness' in their preliminary discourse. They try to justify the introduction of complex numbers based on logical arguments.

- DERRICK'S PRELIMINARY DISCOURSE

Derrick, for example, says that complex numbers are defined in order to remedy a defect of real numbers:

> The real numbers have one basic FLAW: They do not provide all possible solutions to polynomial equations. For example, the equation $x^2 + 1 = 0$ cannot be solved using real numbers, since the square of any real number is nonnegative. To remedy this defect, we define the set of *complex numbers* $\mathbb{C}$ consisting of all ordered pairs $z = (x,y)$ of real numbers ... [Derrick, p. 2]

Derrick says that the non-existence of roots to the equation $x^2 + 1 = 0$ is a defect of real numbers. Following this reasoning we could say that real numbers have also many other 'defects': the equation $1/x = 0$ has no real solutions; the equation $1^x = 0$ has no solution, etc. However, no one will try to construct an extension of real numbers that includes a solution, for example, to the last equation.

Therefore the notion of 'defect' is a purely rhetorical device. What is its function? Does the author merely feel that he has to provide some motivation, or does the use of the word 'defect' have a definite purpose? It may be impossible to decide from the text, but the discourse analysis provides the questions which a teacher has to be aware of.

● MORETTI'S PRELIMINARY DISCOURSE

Moretti is also rhetorical:

> So far, the operation $\sqrt{(-1)}$ is meaningless because no real number can be found whose square is a negative number. IF SOME SENSE HAS TO BE GIVEN TO THE OPERATION, it is necessary to step off the real axis, to find more points to associate with the square roots of negative real numbers. However, the points in a plane are defined by two coordinates, thus we are NATURALLY led to the *complex numbers*, the word "complex" meaning that one of these numbers is actually a set of two real numbers and can be associated with a point in a plane. [Moretti, p. 26]

Moretti asserts that it is 'natural' what took 200 years to be realized: to think of associating complex numbers with points of a plane.

FOUNDATIONS

W. J. Thron prefers to construct complex numbers axiomatically; he starts talking about axioms of set theory, and develops the theory of real numbers and cardinal numbers before introducing Hamilton's definition.

I have shown five distinct discursive procedures:

1. no preliminary discourse;
2. historical perspective;
3. reference to applications;
4. mathematical reasons (extension);
5. construction from the axioms.

The first and the last procedures are authoritarian; the learner has to accept the complex numbers as they are.

The third procedure, reference to applications, is just rhetoric. To say that complex numbers are applied in electrical engineering or in fluid dynamics without providing details of the interaction between mathematics and other disciplines is not enough.

The fourth procedure, to give intrinsic mathematical reasons for the study of complex numbers, is also rhetoric. Hindsight in this case seems artificial; it cannot substitute for history properly.

Among these five procedures, the only one that is educationally interesting is the second procedure, historical perspective, as was done by Stewart & Tall.

DUREGE'S APPROACH

In my view, better than Stewart & Tall's historical account is the introduction of H. Durege's textbook **Elements of the Theory of Functions of a Complex Variable**, published in 1896.

Durege gives a historico-critical account of the development of complex numbers. He starts his introduction in the following way:

> To follow the gradual development of the theory of imaginary quantities is especially interesting, for the reason that we can clearly perceive with what difficulties is attended the introduction of ideas, either not at all known before, or at least not sufficiently current. [Durege, p. 1]

Durege talks about Hankel's principle of the permanence of the formal law, which he asserts 'is of the greatest importance for mathematics, notwithstanding the fact that its assumption is by no means necessary but arbitrary'.

Durege as well as Stewart & Tall does not hide the fact that complex numbers were controversial. He cites for example that Euler taught that, if a and b denote two positive quantities,

$$\sqrt{(-a)}.\ \sqrt{(-b)} = \sqrt{(ab)}.$$

However this view was not generally accepted because it would be absurd to assume that the product of two impossible quantities should not also be impossible. According to Durege, Hutton's **Mathematical Dictionary** (1796) says that in his time the views of mathematicians were about equally divided on this point.

HISTORICAL DEVELOPMENT AND EDUCATION

Why do the authors of Complex Analysis textbooks hide the fact that concepts such as complex numbers are the final results of the struggle of mathematicians to understand them, and not the starting points, as they present them?

In my view, this is done because, for the majority of teachers, lecturers and textbooks authors, mathematical education means the teaching of truths and certainty, and not of controversies and doubts.

Authors have then to resort to rhetorical devices in order to justify the inversion of history, replacing the description of historical events by its rationalization; replacing conjectures and refutations by analytic rigour. History shows that the way to truth is sometimes tortuous and difficult; hindsight 'corrects' history giving it order and 'naturality'.

The ideology of many textbook authors is precisely to present this 'corrected' view of the subject.

In the past polemics between scientists were more overt; in present day texts authors take positions on polemical issues in a covert, implicit way. One of the tasks of discourse analysis is to detect them, uncovering their implicit messages.

9.2 DISCOURSE ANALYSIS OF A CONCEPT: RIEMANN SURFACES

In section 9.1 I have analysed the discourse of the first concept given in Complex Analysis textbooks: complex numbers. In this section I will analyse the discourse of the last concept given in such textbooks: Riemann surfaces.

Formal definition of Riemann surface is considered difficult by the majority of textbook authors and in general is not presented in introductory texts on Complex Analysis. Authors usually negotiate the intuition of Riemann surfaces, by means of analogy, examples and figures.

In the following I will show that different authors conduct this negotiation in different ways, with different perspectives.

THRON'S PRESENTATION

Thron's project is the axiomatization of Complex Analysis, as I discussed in section 8.3. Thron gives the following definition of Riemann surface:

> DEFINITION 31.8. The *Riemann surface of an analytic function F(z)* consists of its Riemann configuration and those singular points which are poles or algebraic branch points.
>
> [Thron, p. 225]

The difficulty of this definition is on the definition of Riemann configuration:

> DEFINITION 31.3. Let F(z) be an analytic function. Consider the set S consisting of all pairs $<z, P_z>$, where z is any complex number in the domain of existence of F(z) and P_z is

a functional element of $F(z)$ at z. Let $\zeta = \langle z, P_z \rangle$; then we call z the *place coordinate of* ζ and P_z the *element coordinate of* ζ. For every point ζ of S we define neighborhoods $N_{\zeta,\rho}$, $\rho > 0$ but less than the radius of convergence of P_z , to consist of all points $\xi = \langle x, P_x \rangle \in S$ for which $|x - z| < \rho$ and P_x is a direct analytic continuation of P_z . If ∞ belongs to the domain of existence of $F(z)$, all possible $\langle \infty, P_\infty \rangle$ are also considered as points of S. Let $\eta = \langle \infty, P_\infty \rangle$; then we define $N_{\infty,d}$ to be the set of all ξ such that $|x| > d$; d is to be taken large enough so that P_∞ converges for all $|x| > d$, and such that P_x can be obtained by direct analytic continuation from P_∞ . The space S with the topological structure defined above is called the *Riemann configuration of the function F(z).*

[Thron, p. 221]

To complete the understanding of the definition of Riemann surface, it is also necessary to see the definitions of algebraic branch point (definition 31.7) and pole (definition 31.6), which involve more and more definitions.

In order to facilitate readers' acceptance of the formal definition, Thron attempts to make appeal to intuition, before the formal definition:

An intuitive picture of what a Riemann surface is can be obtained by the following consideration. Since every analytic function can be generated by its power series expansions, **we provide ourselves with circular pieces of paper**, one for each element at each point of the domain of existence of the function. The radii of our pieces we make equal to the radii of convergence of the corresponding functional element. Starting with some element **we glue to its piece of paper (with the right amount of overlapping) all pieces that belong to elements that are direct continuations of the original element and proceed with this process until all pieces have been glued together.** The resulting surface (**the process can in general not be carried out in three dimensions without having pieces interpenetrate each other, which should not touch each other**) is, apart from certain boundary points to be added later, the Riemann surface of the function.

[Thron, p. 222]

His 'intuitive' description of Riemann surface is inadequate for several reasons.

(a) Thron attempts to give intuition to the readers without using examples. The readers have then to apply the same intuition to $\sqrt{z}$ and log z for example, which are qualitatively different multifunctions.

(b) In the first emphasized segment (we provide ourselves with circular pieces of paper) Thron gives the impression that readers are supposed to provide themselves with real pieces of paper; readers can therefore confound mental picture with reality. The mental picture proposed by Thron does not work in reality for example in the case of log z, since infinite pieces of paper are necessary.

(c) Since Thron does not use figures in his text, it is difficult to understand how one piece is glued to another: his comment 'with the right amount of overlapping' is simply rhetorical and not descriptive.

(d) The last emphasized segment 'the process in general can not be carried out in three dimensions without having pieces interpenetrate each other, which should not touch each other' is abstruse. Is Thron describing a process that in general is impossible? Does this affirmation mean that the process he is describing is mental and not real?

Thron's appeal to intuition is not in tune with the rest of book where he is extremely formal. This is a sign that the formal definition of Riemann surface does not work, it is not clear. Thron's attempt to transmit his intuition verbally to the reader fails because he does not use the appropriate tools (examples and figures).

This failure proves that in the case of Riemann surfaces adequate intuitive explanation is essential for its understanding: formal definition is not clear at all.

PRIESTLEY'S PRESENTATION

Priestley presents a modernized intuition of Riemann surface:

> A multi-storey car park provides a good mental picture. The floors of the car park represent copies of the plane, and the ramps one up and down between levels indicate how these copies are pasted together. The Riemann surface for the logarithm is modelled by a car park with infinitely many floor each of infinite extent, with a ramp joining each floor to the next; see the figure ... For more complicated multifunctions the car park designer might be said to have a warped sense of humour.
>
> [Priestley, p. 101]

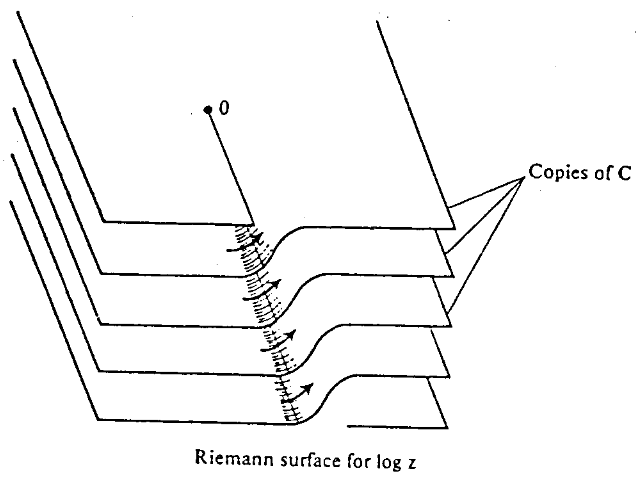


Figure 9.1 Priestley's illustration

Priestley does what Thron has not done. He tells readers that the picture is 'mental' and not 'real', he uses an example, he indicates how the branches are pasted together, and he uses a figure. Above all, Priestley use a heuristic device: the analogy of Riemann surface with a car park.

This is, in my view, an example of good combination of intuition and heuristic that convey understanding of a concept, without logical formalism: a necessary doorstep to the entrance into the rather abstract theory of Riemann surfaces.

STEWART & TALL' S PRESENTATION

They stress that they will give an intuitive picture, and the reader should not confound it with a rigorous definition:

> Riemann invented a geometrical way to envisage multiform functions, much more intuitively appealing than an equivalence class of function elements, which involves replacing $\mathbb{C}$ by a more complicated 'Riemann surface'. In the case of the logarithm we can describe it informally in the following terms, which should not be subjected to too deep scrutiny of a logic-chopping kind. We are not attempting a rigorous definition at this point: the informal description, though it may sound far-fetched, is in fact capable of a precise and rigorous rendering.
>
> [Stewart & Tall, p. 268]

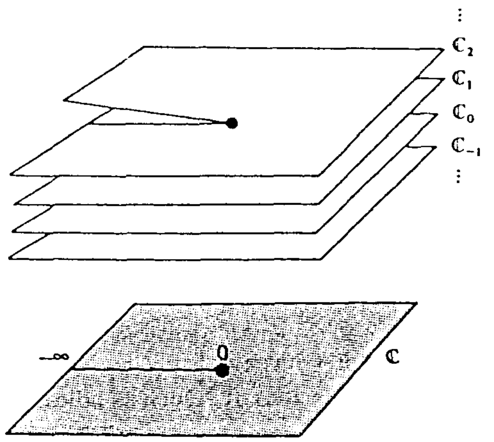


Figure 9.2 Stewart & Tall's illustration

Stewart & Tall make explicit that they are negotiating intuition. They make clear that decisions are being taken with respect to the conflicts between logic and intuition, analytical approach and geometric approach, and between formal and informal.

Im my view, Stewart & Tall use the right amount of rhetorical devices in their negotiation of Riemann surfaces.

DEPREE & OEHRING'S PRESENTATION

Depree & Oehring make analogy of Riemann surface of the function z^2 with a Japanese fan:

> At the risk of repetition and of further **degeneration toward the intuitive** in describing **R** [the Riemann surface of z^2], we remark that if we interpret the map $w = z^2$ "dynamically", we can think of it as fanning out a sector of angle 2α, much like the opening of a Japanese fan. (Of course, **the analogy is imperfect** since there is also radial motion.) If we think of the fan as initially constituting a complete circle, then opening it so as to double the angle would wind it above itself to produce a model of **R** (without the final crossing).
> [Depree & Oehring, p. 113]

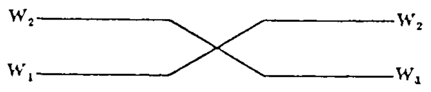


Figure 9.3 Depree & Oehring's illustration

In the first paragraph we can 'feel' that Depree & Oehring somewhat despise intuition: 'degeneration toward the intuitive' is a really strong utterance against intuition.

This discourse against intuition is so frequent in their text that they warn the reader in the preface:

> Finally, a word to the student. There are many allusions to "rigor" and "intuition" throughout the book, and perhaps you will feel their purpose is for lauding the former and deprecating the latter. On the contrary, each has an important place in mathematics.
>
> [Depree & Oehring, preface]

Depree & Oehring use the rhetoric of ambiguity; they warn about the danger of 'degeneration toward the intuitive', and at the same time, call attention on the subjectivity of a proof:

> ... proofs which are considered rigorous, that is, acceptable, by one generation often lose that status with the passage of time and, occasionally, vice versa. In fact, proof techniques which are accepted by certain mathematicians may be rejected by their contemporaries. Thus the decision whether or not a proof is rigorous seems to be largely determined by intuition tempered by training: in any event it is clearly a subjective decision.
>
> [Depree & Oehring, preface]

Depree & Oehring are an example of authors oscillating between intuition and logic, which proves that mathematical discourse is a dialectical process.

MORETTI'S PRESENTATION

He uses an ad hoc figure, that is, a figure that serves only for the specific example $w = \sqrt{z}$.

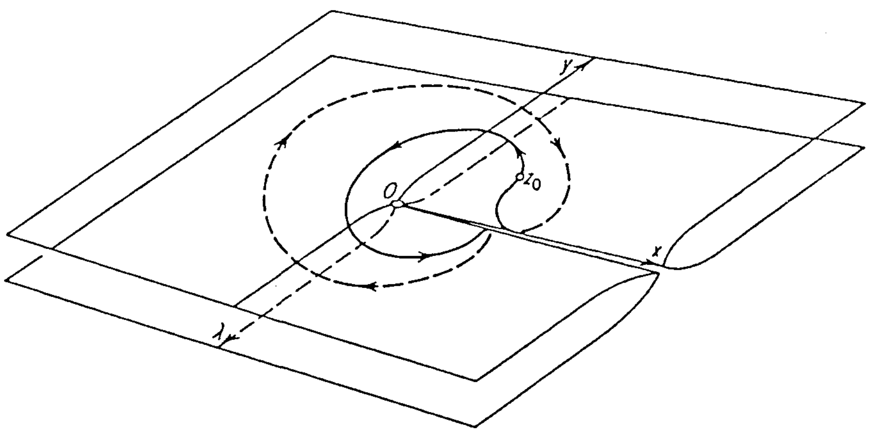


Figure 9.4 Moretti's illustration

In my view, Moretti's 'original' way of illustrating Riemann surface is not adequate, since it is not generalizable. Moretti opts for the particular at the expense of the general. Other authors do a better job than Moretti's because they give a scheme that can be utilized in other examples, whereas Moretti's construction only works for the particular example.

FUCHS & SHABAT'S APPROACH

They illustrate the Riemann surface for the function $w = z^2$ by making an interesting analogy with the real function $y = x^2$.

The real analogue of $w = z^2$, the function $y = x^2$, maps the x-axis onto the non-negative y-axis. This mapping is one-valued, but not one-to-one; each of the pair of points A and B, corresponding to abscissae differing only in sign, is transformed onto the same point A" = B". However, by taking two separate representations of the positive y-axis, joined together at the points $y = 0$ and $y = +\infty$ we may consider that A" lies on the one and B" on the other; the function $y = x^2$ can then be considered as giving a biuniform mapping of the x-axis onto the "doubled" y-axis $(y \geq 0)$.

[Fuchs & Shabat, p. 82]

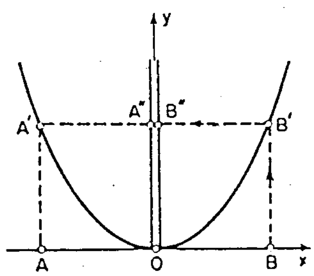


Figure 9.5 Fuchs & Shabat's illustration

Fuchs & Shabat have devoted a whole chapter to the construction of 'mental pictures' of Riemann surfaces of elementary functions. They show clearly that it is possible to develop deep mathematical reasoning by using heuristic and intuition, without appealing to a logico-formal apparatus.

This discussion on Riemann surfaces suggests that Geometry is an area where heuristic, intuition and rhetoric will be more pronounced than in some other topics, and this would provide a productive area of further research.

Complex Analysis was chosen as one where it might have been thought that there was less room for non-logical components of discourse. In my view, the fact that this has proved not to be the case is significant.

9.3 DISCOURSE ANALYSIS OF A TEXTBOOK: STEWART & TALL

My aim in this section is to apply the principles and techniques I have propounded in previous sections to the discourse analysis of a particular textbook on Complex Analysis.

Stewart & Tall's textbook, **Complex Analysis** (1988) is a relatively new textbook; it was first published in 1983. Both authors are British mathematicians. Ian Stewart is author of several books. David Tall is also recognized as a mathematics educationalist; he regularly publishes papers on educational themes. Stewart & Tall's textbook can be seen as a modern textbook, since it introduces some novelties in terms of organisation of mathematical discourse.

I will use another textbook, Churchill & Brown's textbook, by way of counterpoint. I will compare some aspects of Stewart & Tall's book with Churchill & Brown's, seeking to show more differences than similarities.

I will use the fifth edition of Churchill & Brown's textbook, **Complex Variables and Applications** (1990). This is one of the most used textbooks on Complex Analysis, whose two first editions (1948, 1960) were written by the late Ruel V. Churchill alone. The authors are American mathematicians.

I will divide Stewart & Tall's discourse into two parts: discourse on non-mathematical themes, and negotiation (how they negotiate proofs, definitions, etc.).

For ease of reference, I will sometimes use the abbreviations C & B and S & T to refer respectively to Churchill & Brown and Stewart & Tall.

9.3.1 Discourse

In this section I will analyse some aspects of Stewart & Tall's discourse on non-mathematical themes, which is an important component of their overall discourse.

DISCOURSE ON THE SIMPLICITY OF COMPLEX ANALYSIS

One of the philosophical theses of Stewart & Tall is that complex analysis is simpler or easier than real analysis. They start to defend this thesis from the first paragraph of the preface:

> Students faced with a course on 'Complex Analysis' often find it to be just that - complex. In the sense of 'complicated'. It is true, of course, that the proofs of some of the major theorems in the subject can demand a certain technical versatility. But in many ways, on a conceptual level, complex analysis is actually *easier* than real analysis; it just isn't always taught that way.
>
> [Stewart & Tall, preface]

In section 4 of chapter 0 Stewart & Tall say that complex numbers are simpler than real numbers, and complex analysis is simpler than real analysis:

> To the modern ear the very *name* 'complex analysis' carries misleading overtones: it suggests complexity in the sense of complication ... And in fact complex numbers are *not* more complicated than reals: in some ways they are simpler. For instance, polynomials always have roots. Likewise complex analysis is often simpler than real analysis: for example, every differentiable function is differentiable as often as we please and has a power series expansion.
>
> [S & T, p. 8]

According to Stewart & Tall's argument, complex numbers are simpler than real numbers, because all polynomials with real coefficients have complex roots, whereas there exist polynomials with real coefficients without real roots.

By extending this reasoning, real numbers would be simpler than rational numbers, because more polynomials have roots in the set of real numbers than in the set of rational numbers. By the same token, rational numbers would be simpler than integer numbers, and these in turn would be simpler than natural numbers. Consequently, the natural numbers would be the most complicated of number sets!

In section 6 of chapter 4, Stewart & Tall reinforce the argument that complex analysis is simpler than real analysis because the behaviour of complex functions is better than the behaviour of real functions. They construct a sequence of real functions (b_n) such that b_n is differentiable everywhere precisely n times, but differentiable nowhere $n + 1$ times. Such 'pathological' functions do not exist in complex analysis, which implies that complex analysis is simpler than real analysis:

> Real analysis is a very hairy subject indeed. But what is the relevance of such bizarre functions in complex analysis? The answer is: **none whatsoever**. They have been mentioned once only to be dismissed. We shall find that no such animals live in the complex world; as we have said in Chapter 0, complex analysis is *simple* (relative to the real case).
>
> [S & T, p. 78]

In section 2 of chapter 10, after the proof of Taylor's theorem, Stewart & Tall come back to their thesis:

> At a stroke he [Cauchy] showed that *complex analysis is simpler than real analysis* by reducing the general study of differentiable complex functions to particular computations with power series.
>
> [S & T, p. 183]

Stewart & Tall's discourse on the simplicity of complex analysis seems to be an exorcism; they attempt to exorcize the mystery and mystification which has accompanied the development of complex numbers and complex analysis, since Cardan's use of such numbers in his **Ars Magna** (1545).

The use of the term 'simple' as a negotiating tool has some interesting features which we should take care to look for in analysing this kind of discourse. It seems to be using a multiple 'register' device (see [Pimm 87]).

The use of the word 'simple' in the text is clearly applied to some structural aspects internal to the mathematics, whereas its rhetorical use indicates that the learner is supposed to find the text *simple to comprehend.*

It is possible to admit this as a legitimate device for cultivating a receptive attitude by the learner. The important thing here is to recognise its presence. It falls within the logic vs. rhetoric conflict discussed in section 2.3, and in fact exemplifies the way in which such conflicts relate to different linguistic 'registers'.

• NEGOTIATION OF IMPORTANCE

This discourse on the simplicity of complex analysis can also be regarded as part of the negotiation of the importance of Taylor's theorem or The Fundamental Theorem of Algebra.

Compared to Stewart & Tall, Churchill & Brown are very laconic. With regard to Taylor's theorem, C & B just say that it is one of the most important results of the chapter on series [Churchill & Brown, p. 138]. With regard to The Fundamental Theorem of Algebra, they just say that in elementary algebra courses, it is often stated without proof [Churchill & Brown, p. 131].

METADISCOURSE: DISCOURSE ON ORGANISING PRINCIPLES

In the preface Stewart & Tall promise to decomplicate the discourse of Complex Analysis by using two rhetorical or heuristic principles:

> To exhibit this inherent simplicity of complex analysis we have organized the material around two basic principles: (1)

> **generalize from the real case**; (2) when that reveals new phenomena, **use the rich geometry of the plane** to understand them. [Stewart & Tall, preface]

In section 4 of chapter 0 Stewart & Tall explain the meaning of the first principle:

> The first [principle] is **the direct generalization to the complex case of real analysis.** Definitions of limits, continuity, differentiation and integration are the NATURAL extension of the real notions, provided that we view them in suitable terms.
>
> [S & T, p. 8]

This means that Stewart & Tall will use ANALOGY to introduce concepts of complex analysis. In section 3.2 I have already shown how varied are the uses of analogy by Stewart & Tall in their textbook.

The second principle becomes: '*geometric insight is valuable and should be cultivated*'.

Thus, Stewart & Tall will emphasize ANALOGY and GEOMETRIC INSIGHT (I prefer to say INTUITION). However, this is just rhetorical, since in fact their discourse is governed by the permanent tension between rigour and intuition: sometimes they emphasize analytic rigour, sometimes they emphasize geometric intuition.

In my view, it is this tension that is educationally significant. In this sense, Stewart & Tall's practice is fortunately better than their rhetoric.

DISCOURSE ON THE CONFLICT INTUITION VS. RIGOUR

Stewart & Tall are aware of the tension between intuition and rigour. In the preface, S & T declare that their aim is 'to encourage geometric thinking, with the proviso that it must be adequately backed up by analytic rigour'. Here I understand that 'geometric thinking' means 'geometric insight' or 'geometric intuition', since they repeat the same thing in section 4 of chapter 0 in the following way:

> ... and this brings us to our second major organizing principle: geometric insight is valuable and should be cultivated. Of course, this insight must be translated into sound formal arguments; this can be done using modern topological notions.
>
> [S & T, p. 8]

Sometimes S & T give priority to logical formalism. For example, they do not accept the geometric proof of the triangle inequality as a proof but only as an interpretation:

> The triangle inequality is a little harder to prove directly, although its geometric interpretation is the obvious fact that one side of a triangle is no longer than the sum of the lengths of the other two sides.
>
> [S & T, p. 15]

After this utterance, they give an algebraic proof of the triangle inequality, which, in my view, is rather non-insightful.

Sometimes they opt for geometric reasoning, at the expense of analytic reasoning. For example, after giving two methods of computing winding number, the first analytic, and the second geometric, they affirm:

> Comparison of the first 'bad' method shown with the final 'good' one gives a striking illustration of the dangers of blind 'formula-crunching' analysis. **Complex analysis is a highly geometric subject, and the geometry should not be despised.**
>
> [S & T, p. 136]

At the end of their textbook Stewart & Tall discourse on the value of mathematical intuition:

> Towards the end of nineteenth century Felix Klein offered a 'proof' of a theorem along the following lines: *think of the Riemann surface as being made of thin metal, and an electric current flowing through it* ... It would not, today, be considered a logically convincing argument; but **the physical intuition certainly revealed some important mathematical ideas**. Today we are seeing the converse process, with **mathematical intuition providing important concepts for physics**. It is a two-way trade. And, whatever the attractions of beauty for its own sake, it is vital to the health of both mathematics and science that this trade be maintained.
>
> [S & T, p. 282]

DISCOURSE ON APPLICATIONS

In the preface of their textbook, Churchill & Brown fix two objectives:

> As was the case with the earlier editions, the *first objective* of this edition is to develop in a rigorous and self-contained manner those parts of the theory which are prominent in the applications of the subject. The *second objective* is to furnish an introduction to applications of residues and conformal mapping.
>
> [Churchill & Brown, preface]

Churchill & Brown's textbook is then clearly orientated towards applications.

Stewart & Tall, contrary to Churchill & Brown, neglect applications. In their textbook, there is only a short section on potential theory that deal with applications. Despite this, Stewart & Tall discourse several times on applications.

They start talking about applications in the first chapter:

> The abstruse invention of complex numbers, once described by our mathematical forbears as 'impossible' and 'useless', has become part of an aesthetically satisfying theory with **eminently practical applications in aerodynamics, fluid mechanics and many other areas.**
>
> [S & T, p. 6]

In the introduction to the chapter on conformal transformations, Stewart & Tall talk again about applications:

> By studying differentiable functions one may prove theorems about curves; by studying curves one may prove theorems about differentiable functions. The latter technique is of great importance in the advanced 'geometrical' theory of differentiable functions, but only the former falls within our present scope. The method has **interesting applications to potential theory and fluid dynamics**, and we shall outline the beginnings of these.
>
> [S & T, p. 238]

It should be said that S & T use only four pages to develop potential theory. S & T give only one 'applied' exercise:

> Verify that the Joukowski transformation does, as claimed above, give rise to an aerofoil shape. Look up pp 131-4 of A. Kyrala, *Applied Functions of a Complex Variable*, Wiley-Interscience, New York 1972, and see how to compute flow-lines round it.
>
> [S & T, p. 256]

This exercise appears to be suggesting that the readers should consult other textbooks if they wish to learn applications of complex analysis.

S & T talk again on application at the end of the textbook. Their aim is to affirm the potential applicability of abstract mathematical concepts such as Riemann surfaces, functions of several complex variables and complex manifolds:

> As an example, very recently complex manifolds and automorphic functions have turned out to be important in Quantum Field Theory, in the study of 'Gauge Fields'.
>
> [S & T, p. 282]

Unfortunately this rhetoric is not followed by action. We only can regard it as a component of their negotiation with the reader on the importance of Complex Analysis.

DISCOURSE ON METHODS IN MATHEMATICS

S & T affirm the advantage of general principles over manipulative ingenuity:

> The tasks to which complex analysis may be set include the explicit computation of definite integrals and the summation of series. Although such problems are not as important a part of pure mathematics as they once were, they are still very useful in practical applications. Further, the power of the method [calculus of residues] and its wide applicability demonstrate **the advantage of general principles and deep theorems over any amount of manipulative ingenuity.**
>
> [S & T, p. 212]

This principle seems to be contradicted by the following principle:

> If one formalizes these ideas it becomes clear that the restriction to power series and discs is inessential. **this is often the way in mathematics: the solution to a special problem turns out to apply in a much more general setting.**
>
> [S & T, p. 261]

In the first quotation S & T affirm that it is important to investigate general principles; in the second quotation, they affirm that it is important to deal with special problems.

This demonstrate that in fact S & T oscillate between general and particular, which is a characteristic of mathematical thinking (see section 3.2). Sometimes it is better to attack a particular problem, sometimes it is better to invoke general principles: mathematical reasoning develops through these oscillations from particular to general and vice-versa.

CONCLUSION

I have shown that Stewart & Tall's textbook is permeated by meta-mathematical discourses. With these discourses the authors attempt to communicate philosophical messages such as the simplicity of complex analysis, or the applicability of abstract mathematical theories, and methodological messages such as the value of analogy and geometric thinking.

9.3.2 Negotiation

Undergraduate mathematics textbooks are in general written in terms of theorems, proofs, definitions, examples, etc. Discourse analysis should take into account this discourse format.

I have propounded in the first two parts of this work that discourse is a negotiation between author and reader; hence, discourse analysis should analyse features of this negotiation such as
negotiation of truth (or analysis of theorems and proofs),
negotiation of concepts (or analysis of definitions),
negotiation of activities (analysis of exercises).

NEGOTIATION OF TRUTH

Stewart & Tall negotiate the truth of theorems in different ways: giving formal proofs, not giving proofs, giving two proofs, indicating how to prove, etc. This was discussed in section 5.1.

Here I am interested to show some examples of alternative proofs of theorems, and how they justify its presentation.

ALTERNATIVE PROOFS

EXAMPLE 1. S & T first prove the formula

$$\exp(z + w) = \exp(z) \exp(w)$$

using the formula for the product of two series [S & T, p. 60]. The second proof uses the formula for the derivative of the exponential function. There S & T argue that the first proof is CUMBERSOME [S & T, p. 83]. Here they are negotiating cognition (see section 3.3).

EXAMPLE 2. In chapter 6 on integration S & T offer two alternative definitions of integral of a complex function. Because of this, they are obliged to furnish two different proofs for some theorems. For example, the additivity of the integral has two proofs [S & T, p. 107]. Another theorem with two proofs is the estimation lemma [S & T, p. 111].

EXAMPLE 3. S & T give two proofs of the continuity of the argument in the cut plane. On the first proof they say:

> The proof that follows is an **inelegant** 'bare hands' reduction to properties of real functions: for a more elegant approach see § 8.4.
>
> [S & T, p. 123]

So, elegance justifies the presentation of a new proof.

EXAMPLE 4. S & T prove Cauchy's Residue Theorem in two ways; the first proof uses the generalized Cauchy theorem, while the second proof uses Laurent series. They say the second is 'instructive but less elegant' [S & T, p. 214]. This means that elegance in mathematics is not all.

In section 5.1 I have shown that Stewart & Tall use almost all possible strategies in the presentation of theorems. This means that their presentation has more rhetoric and intuition than is supposed because of their defence of analytic rigour.

NEGOTIATION OF THE CONCEPT OF COMPLEX NUMBER

Stewart & Tall are well aware of the rhetorical aspects of mathematical discourse. They have reserved, not only the preface, but a whole chapter to do what in rhetoric is called the *exordium* or the *proemium*: introduce the reader to the subject.

• HISTORICAL NOTICES

Stewart & Tall evoke historical events to negotiate their definition of complex number. In the first chapter of their textbook, they give an account of the history of complex numbers and complex analysis, pointing out to some definite steps in this history, as I have expounded in section 9.1.2.

STEWART & TALL'S DEFINITION OF COMPLEX NUMBER

In chapter 1 Stewart & Tall adopt Hamilton's definition of complex numbers as ordered pairs of real numbers, as pointed out in section 9.1.1. After this definition they comment:

> This definition is the culmination of several centuries of struggle to understand complex numbers, and it shows how elusive a simple idea can be'.
>
> [S & T, p. 11]

ARTIFICIAL VS. NATURAL

Hamilton's definition of complex number has the disadvantage of introducing the product of complex numbers in an artificial way. Stewart & Tall however declare that complex numbers are not more artificial than real numbers:

> The logical status of complex numbers, which caused so much distress during the eighteenth century, is now seen to be very much on a par with that of the 'real' numbers. What puzzled the ancients was the obvious ARTIFICIALITY and ABSTRACTION of the complex number system, in contrast to the apparently NATURAL and CONCRETE *real* number system: but the mathematician of today sees even the real numbers as

possessing a similar artificiality and abstraction.

[S & T, p. 10]

In my view, the important educational question is not whether complex numbers are more artificial than real numbers or not, but which definition among the many existing definitions of complex number is the most natural. I have investigated the discourse on complex numbers with more details in section 9.1.

Contrary to Stewart & Tall, Churchill & Brown neglect completely the historical aspects of Complex Analysis. C & T go straight to the definition of complex numbers as ordered pairs of real numbers; they are interested in giving mathematical information and nothing else, while Stewart & Tall are concerned in negotiating the whole subject of Complex Analysis, and particularly the concept of complex number.

NEGOTIATION OF READERS' ACTIVITIES

Authors negotiate activities with the readers by means of exercises given at the end of sections or chapters, and gaps left in the text to be filled by the readers.

The importance of exercises in the learning of mathematics is obvious: they serve to help learners to construct their own mathematical knowledge, to acquire practice in techniques, to gain manipulative ability, to increase their comprehension of definitions and theorems, to conjecture, to prove, etc.

In my view, it is important to know what kind of activity or reasoning exercises demand from the learners. It is convenient then to establish criteria for the classification of exercises.

George Polya has propounded to divide problems into 'problems to find' and 'problems to prove'. In the context of exercises in textbooks, I will say practical or routine exercises are what Polya calls 'problems to find', and theoretical exercises are what corresponds to 'problems to prove'.

Practical exercises are exercises demanding calculation, manipulation of formulae, classification, etc.; they are orientated towards particular objects rather than general concepts. The aim of practical exercises is to give practice in manipulative techniques.

Theoretical exercises will be classified as closed or open exercises; closed exercises are those which explicitly say what to prove, while in open exercises the reader has to make a conjecture, and prove that this conjecture is true. Theoretical exercises are orientated towards general results instead of particular results. The aim of theoretical exercises is to give practice in logic and heuristic: proving, and conjecturing.

Discourse analysis of exercises should detect the balance or imbalance between general and particular, practical and theoretical, closed and open, etc. More detailed analysis of exercises was carried out in chapter 6.

EXAMPLE OF ANALYSIS OF EXERCISES

Let us see, for example, the exercises at the end of chapter 11, Laurent series, of Stewart & Tall's textbook.

Exercises 1 to 9, 11, 14, 16, 17 and 18 can be considered routine practical exercises. For example,

Ex. 1: Find Laurent expansions for the following around $z = 0$ ...

Ex. 2: Find Laurent expansions for the given functions on the stated annuli ...

Ex. 9: Describe the type of singularity at 0 of each of the following functions ...

Exercise 12 is a non-routine practical exercise, because the reader is asked to construct an example of function satisfying given conditions. It is more demanding than the use of formulae, it demands imagination.

Exercises 10, 19 to 22 are exercises to prove. They are closed in the sense that the reader has to prove ready made theorems; the reader knows in advance what to prove.

Exercises 13 and 15 are open. Exercise 15, for example, asks whether circles and straight lines in the complex plane have circles as images in the Riemann sphere. Readers have to make a conjecture and prove the conjecture; the result is not apparent.

In conclusion, the exercises are well distributed between practical exercises (15 out of 22), theoretical closed exercises (5), and theoretical open exercises (2).

Note: S & T are not very friendly with the readers, since they do not furnish any solutions to the exercises. This fact makes learners more dependent on instructors, the owners of the answers.

9.3.3 Organisation of Discourse

Stewart & Tall's textbook, with regard to discourse organisation, is exemplary. It is a model that should be followed by other authors.

The first chapter is a historical introduction to the subject, with some methodological considerations, and a preview of the subject. It works as an enlarged preface.

The last chapter is an informal treatment of some advanced topics, analytic continuation and Riemann surfaces, without precise definitions, theorems or rigorous proofs. It looks like a conversation as well as the first chapter. It ends with a section called *the road goes ever on ...*, suggesting that mathematics is a never ending story.

All intermediate chapters have an introduction and some of them contain concluding remarks or concluding sections. For example, chapter 4 on differentiation contains, besides introduction and text, a concluding section called *a glimpse into the future*, where the authors make a philosophical comparison between complex analysis and real analysis.

It is interesting to observe how they end the chapter 6 on integration:

With minor exceptions, this chapter completes the natural analogies between the real and complex theories of differentiation and integration. From now on, new possibilities will unfold [S & T, p. 117].

CHAPTER 10 CONCLUSIONS AND EDUCATIONAL IMPLICATIONS

10.1 CONCLUSIONS OF THE PRESENT INVESTIGATION

The theme of investigation of this thesis has been mathematical discourse, more precisely, the discourse underlying undergraduate mathematics textbooks.

This thesis was divided into three parts. In the first part, my aim was to establish theoretical frameworks for the analysis of mathematical discourse.

In chapter 1, I have characterized discourse as a social interaction through messages, that is, as a negotiation between addresser and addressee mediated by a text. Therefore, analysis of the discourse underlying mathematics textbooks means analysis of the negotiations that the author establishes with the readers. I have argued that this discourse is a complex amalgamation of the discourses of scientific and pedagogic processes.

In chapter 2, I have characterized mathematical discourse as a dialectical process. Contrary to common expectation, mathematical discourse is not 'standard', objective, uniform, dominated by precision and logic. It is driven by opposing forces analogous to those regulating the development of mathematics.

Mathematics textbooks are regarded as the battlefields of the struggle between logic, heuristic, intuition and rhetoric. More precisely, mathematical discourse is controlled by the tensions logic vs. heuristic (information vs. know-how), logic vs. intuition (rigour vs. comprehension), and logic vs. rhetoric (proof vs. rhetorical argument).

In chapter 3, I have described logical, heuristic and rhetorical schemes used by the authors in their discourse. A great educational problem in the mathematical discourse is that it is often hard for the learners and perhaps also for the teachers to know whether the author is reasoning logically, heuristically or rhetorically.

In the second part, I have specified some items that are constantly negotiated between author and reader.

In chapter 4, I have shown that the concept of mathematical object is at the root of philosophical divergences. I have argued that mathematical objects are not Platonic objects but cultural objects (social constructs). This viewpoint facilitates the acceptance of the multiplicity of definitions, proofs, and perspectives. Multiplicity is not the exception, but the rule.

In chapter 5, I have discussed another crucial point, the negotiation of truth and comprehension. Theorems presented in undergraduate mathematics textbooks in general are not completely proved, and the proofs sometimes are not rigorous at all. The truth of the theorems is then negotiated through a mixture of formal proof, intuitive explanation and rhetorical argumentation.

Proving is not everything; logical proofs are not necessarily convincing. In fact authors need to complement their logical reasoning in the proof with other resources in order to obtain learners' comprehension and acceptance: negotiation of truth is intimately linked with negotiation of comprehension and acceptance. To improve comprehension authors use examples, figures, and appeals to intuition. To improve acceptance authors use rhetorical devices. This means that rhetorical, heuristic and logical devices are all mixed together.

In chapter 6, I have discussed the negotiation of activities, arguing that exercises, besides being auxiliary for the learners' construction of mathematics, are also at the service of the authors, since they use them to complete the text, and to convey new information as well. I have suggested a hierarchical classification of exercises, (manipulative, ontological (definitional), and relational (inferential)) as a device for the discourse analysis of exercises. In the same chapter I have discussed the negotiation of applications, prerequisites and historical contextualization.

In chapter 7, I have discussed the ways authors deal with the problem of ambiguity in terminology and notation. Multiplicity of meanings is a discursive device used by mathematicians in order to take advantage of analogies. Authors oscillate between tolerance of ambiguity, which can become confusing for the learners, and intolerance, which can turn into pedantry.

In chapter 8, I have discussed organisational features of mathematical discourse. I have called *style* the way the author organises his or her discourse, i.e., how he or she copes with the conflicts between the three purposes of mathematical discourse: transmission of information, construction of knowledge and negotiation of meanings.

In the third part, I have applied the theoretical tools developed in previous chapters to three case studies. I have investigated the discourse on complex numbers as an introductory discourse to Complex Analysis. The second case study is another limit case, the discourse on Riemann surfaces as a final discourse of Complex Analysis. The third case study is the discourse analysis of a whole textbook on Complex Analysis.

SUMMARY OF CONCLUSIONS

1. Mathematical discourse is not dominated by logical reasoning; it is a complex combination of logical, heuristic, intuitive and rhetorical reasonings.

2. Mathematical discourse is not neutral; on the contrary, it is a dialectical discourse, governed by the tensions between logic and heuristic, logic and intuition, logic and rhetoric.

3. Within mathematical discourse there exists sometimes another explicit discourse, by which authors express their points of view about metamathematical themes such as elegance, geometric intuition, rigour, pedantry, axiomatic method, etc. However, ordinarily such discourse is implicit. It is the attempt to uncover this implicit discourse which is the aim of the tools of analysis developed here.

4. When we look at mathematical discourse as author-reader negotiation, we should pay attention to the authors' rhetoric: how they motivate, persuade, express feelings, judge values.

10.2 EDUCATIONAL IMPLICATIONS

It is likely that the discoveries made through the application of discourse analysis to just one area, Complex Analysis, would have analogues in other fields of mathematics, and even a cursory examination of other textbooks suggests that this is the case.

The analysis has led to the realisation that mathematical text is permeated by the use of heuristic, intuition and rhetoric which all interact, globally or locally, with the logical framework of mathematics.

This challenges the belief systems of many teachers in higher education, who believe that they give logical exposition in their lectures. Textbook authors are usually such teachers, and so these beliefs transferred through the construction of the text imply that the rhetoric and the devices used are likely to be unconscious, and therefore their impact upon readers unconsidered.

The chief aim of the analysis of mathematical discourse is then to give teachers awareness of the use of rhetorical devices in mathematical discourse.

Any mathematical message is transmitted through a discourse; this is inevitable. Such message can be sometimes misunderstood only because of the ignorance of the rhetorical strategies used by the transmitter.

Learning involves acceptance, and acceptance means persuasion; teachers and textbook authors are involved in persuasion rather than logical reasoning. In short, mathematics education is essentially a rhetorical process.

The techniques of discourse analysis I have devised can help teachers and mathematics educationalists to understand teaching and learning mathematics as a negotiation process, as a negotiation between teachers and pupils, as a negotiation between authors and readers.

Consequently, discourse analysis can contribute to the demystification of mathematics teaching and learning, which usually concentrate only on the logical aspects of mathematics.

10.3 FURTHER RESEARCH

I believe that, besides the discourse of undergraduate mathematics textbooks, it is also necessary to develop the discourse analysis of school mathematics textbooks. Some of the theoretical frameworks developed in this thesis will be useful, but it could perhaps prove necessary to change emphasis or develop some new perspectives.

Contrary to undergraduate mathematics textbooks, school mathematics textbooks differ greatly from country to country, in content and in form.

In Brazil teachers of secondary schools depend heavily on textbooks. Textbooks are their Bible: they give them all that they need to teach mathematics: contents and exercises. Textbooks intended for pupils of 11-14 are accompanied by the teachers' manual, which gives the solution to all the exercises propounded in the pupils' textbooks. This manual is extremely important; without it many teachers are unable to teach.

Because of this slavish attitude of teachers with respect to textbooks, it is necessary to demystify textbooks, showing that textbooks reflect perspectives; that many perspectives are possible.

Other types of research should be developed as well. Partially due to my personal inclination for philosophical questions, I have emphasized in this work theoretical aspects of mathematical discourse at the expense of 'fieldwork'. My intention after my return to Brazil is to continue developing research on mathematical discourse.

I hope to conduct in collaboration with my colleagues and students of Universidade de São Paulo investigations on the relationship between the discourse of textbook authors and classroom discourse. The following questions can probably function as starting points of research:

(i) How influential is the discourse of textbook authors on TEACHERS' pedagogic discourse?

(ii) How do STUDENTS/PUPILS perceive rhetorical and heuristic schemes used by authors and teachers?

(iii) How do AUTHORS explain the choices they make about the dialectics between logic, heuristic and rhetoric?

Parallel to this research activities I intend to develop programmes of work with Brazilian teachers to enable them to use textual resources more profitably for themselves and their students.

APPENDIX

LIST OF TEXTBOOKS UTILIZED IN THIS RESEARCH

We list below the sample of textbooks whose discursive aspects I have partially analysed in this work.

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