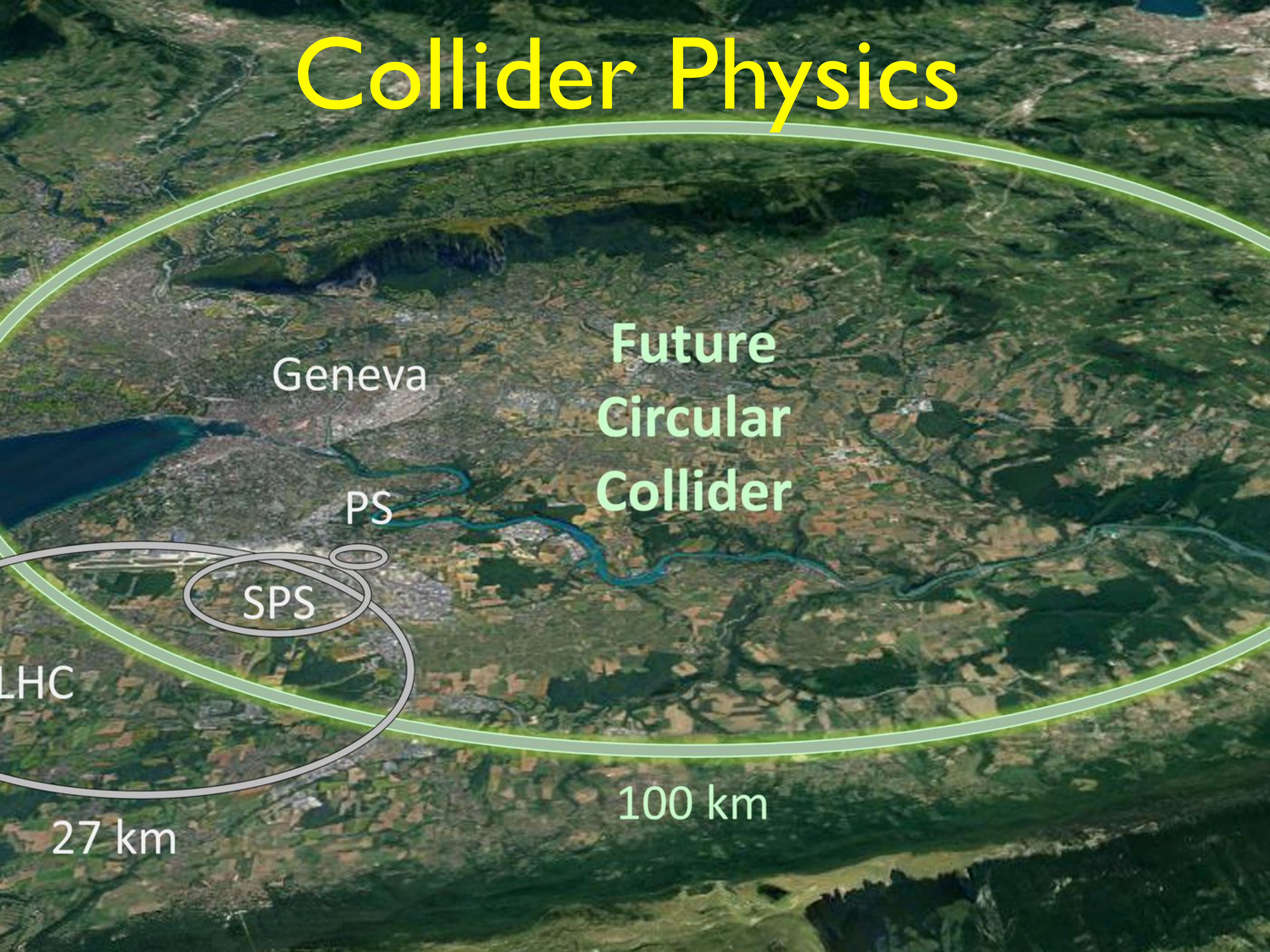
# Collider Physics



# 11.B Phase space

- Evaluation of cross sections
- Two-particle final states
- Three-particle final states
- Multi-particle final states

Reference: Byckling ans Kajantie: Particle Kinematics

# Cross section evaluation

- Consider a process $p_a + p_b \to p_1 + p_2 + \cdots + p_n$

- Given the scattering amplitude M the total cross section is

$$\sigma = \frac{1}{F} \int \prod_{j=1}^{n} \frac{d^3 p_j}{2E_j} \; \delta^4(p_a + p_b - p_1 - \cdots - p_n) \; |M|^2$$

where we added all possible final states and

$$F = 2\lambda(s, m_a^2, m_b^2)(2\pi)^{3n-4}$$

with $$\lambda(x,y,z) = (x-y-z)^2 - 4yz$$

- We use the continuum normalization
- For decays: $p_a + p_b = p$ and $F = 2m(2\pi)^{3n-4}$

- The factor $\dfrac{d^3p}{2E}$ is Lorentz invariant

$$\frac{d^3p}{2E} = \int d^4p\ \delta(p^2 - m^2)\Theta(p_0)$$

[show it ;-)]

usually omitted

- Differential cross sections are given by

$$\frac{d\sigma}{dx} = \frac{1}{F} \int \prod_{j=1}^{n} \frac{d^3p_j}{2E_j}\ \delta^4(p_a + p_b - p_1 - \cdots - p_n)\ \underline{\delta(x - x(p_k))}\ |M|^2$$

- Sometimes it is possible to remove the delta analytically

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- Sometimes it is possible to remove the delta analytically

this is the phase space integral

# The phase space integral

- Taking $p = p_a + p_b$ and $s = p^2$

$$R_n(s) = \int \prod_{j=1}^{n} \frac{d^3 p_j}{2E_j}\, \delta^4\left(p - \sum_k p_k\right)$$

$$= \int \prod_{j=1}^{n} d^4 p_j \delta(p_j^2 - m_j^2)\delta^4\left(p - \sum_k p_k\right)$$

- It is Lorentz invariant
- The number of integrals is (3n- 4) is hard to do analytically for n large
- The choice of variables related to the physics problem!
- It is a bit of an art!

# Two-particle final states

$$R_2(s, m_1^2, m_2^2) = \int d^4p_1 \; d^4p_2 \; \delta(p_1^2 - m_1^2) \; \delta(p_2^2 - m_2^2) \; \delta^4 \left(p - p_1 - p_2\right)$$

- Let's assume that p is timelike and integrate over $p_2$

$$R_2(s, m_1^2, m_2^2) = \int d^4p_1 \; \delta(p_1^2 - m_1^2) \; \delta((p - p_1)^2 - m_2^2)$$
$$= \int \frac{d^3p_1}{2E_1} \; \delta((p - p_1)^2 - m_2^2)$$

going to the CM system $p^\mu = (\sqrt{s}, \vec{0})$

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$$R_2(s, m_1^2, m_2^2) = \frac{1}{2}\int \; d\Omega_1^\star dE_1^\star \; P_1^\star \; \delta(s - 2E_1^\star\sqrt{s} + m_1^2 - m_2^2)$$

$$= \frac{P_1^\star}{4\sqrt{s}}\int d\Omega_1^\star$$

where

$$E_1^\star = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}$$

fixed in the CM

$$P_1^\star = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}}$$

going to the CM system $p^\mu = (\sqrt{s}, \vec{0})$

$R_2(s, m_1^2$ … $n_1^2 - m_2^2)$

**Obtain a phase space**

**when p is spacelike and**

**timelike**

where

in the CM

# The Mandelstam variables for $p_a + p_b \to p_1 + p_2$

$$\begin{aligned}
\hat{s} &= (p_a + p_b)^2 = (p_1 + p_2)^2 = E_{cm}^2, \\
\hat{t} &= (p_a - p_1)^2 = (p_b - p_2)^2 = m_a^2 + m_1^2 - 2(E_a E_1 - p_a p_1 \cos\theta_{a1}), \\
\hat{u} &= (p_a - p_2)^2 = (p_b - p_1)^2 = m_a^2 + m_2^2 - 2(E_a E_2 - p_a p_2 \cos\theta_{a2})
\end{aligned}$$

with $\hat{s} + \hat{t} + \hat{u} = m_a^2 + m_b^2 + m_1^2 + m_2^2$ [this is used to name channels]

## s-channel physical region

$\hat{s} \geq (m_a + m_b)^2$ why?

$$-\frac{\lambda(s, m^2, \mu^2)}{s} \leq \hat{t} \leq 0 \quad \text{for} \quad m_a = m_1 = \mu \; ; \; m_b = m_2 = m$$

$$2m^2 + 2\mu^2 - s \leq \hat{u} \leq \frac{(m^2 - \mu^2)^2}{s}$$

## The Mandelstam variables for $p_a + p_b \to p_1 + p_2$

$\hat{s} \quad = (p$ … 

$\hat{t} \quad = (p$ … $_1 \cos\theta_{a1})$,

$\hat{u} \quad = (p$ … $_2 \cos\theta_{a2})$

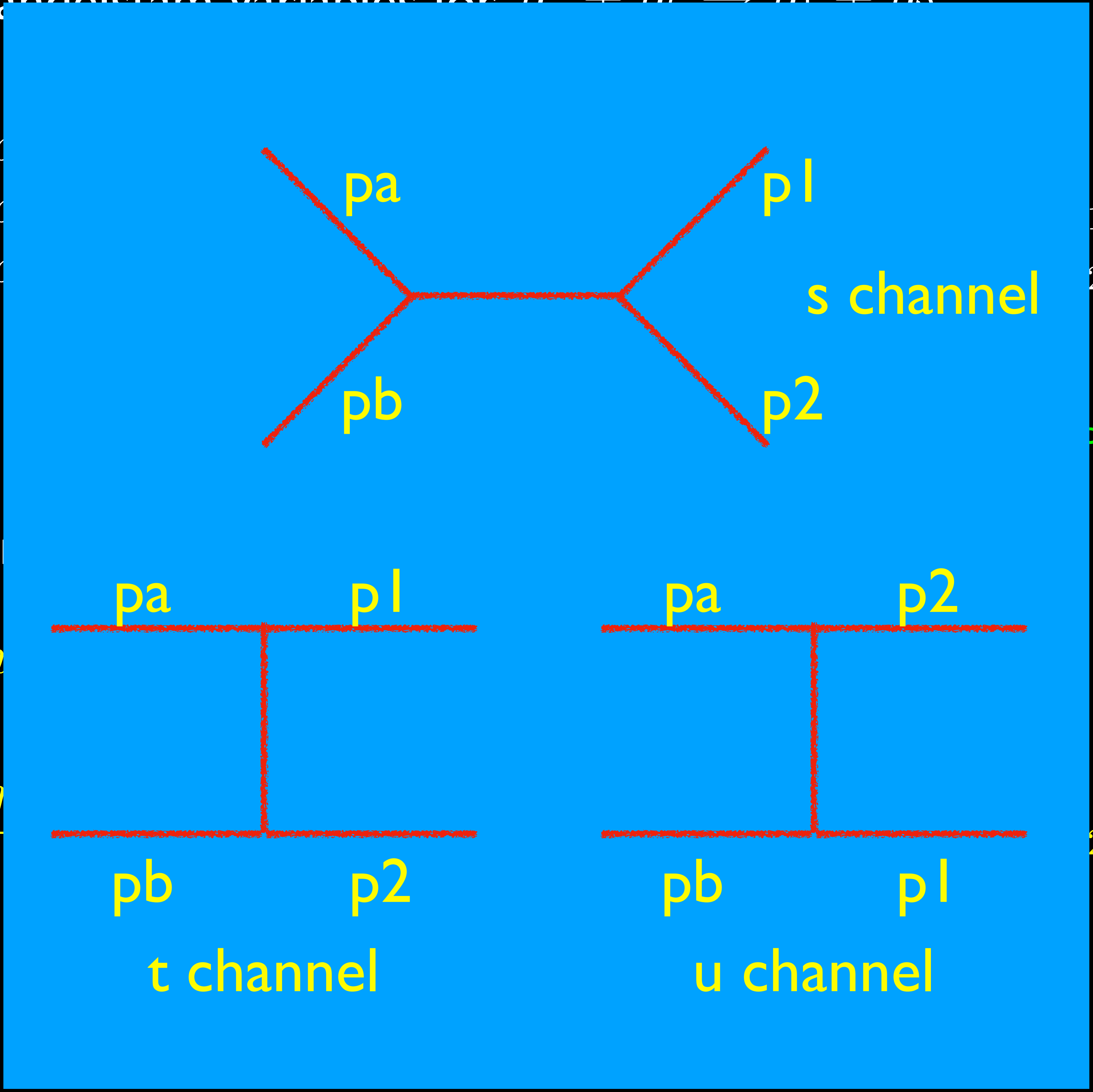
with … o name channels]

- s-chan…

$\hat{s} \geq (m$ …

$-\frac{\lambda(s, m}{}$ … $_2 = m$

$2m^2 +$ …



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$$

with $\hat{s} + \hat{t} + \hat{u} = m_a^2 + m_b^2 + m_1^2 + m_2^2$ [this is used to name channels]

## s-channel physical region

$$\hat{s} \geq (m_a + m_b)^2 \text{ why?}$$

$$-\frac{\lambda(s, m^2, \mu^2)}{s} \leq \hat{t} \leq 0 \quad \text{for} \quad m_a = m_1 = \mu \, ; \; m_b = m_2 = m$$

$$2m^2 + 2\mu^2 - s \leq \hat{u} \leq \frac{(m^2 - \mu^2)^2}{s}$$

We can express the two-particle phase space as

$$dR_2 \equiv \; d\Phi_2 = \frac{1}{4} \; \frac{dt \;\, d\phi_1}{s \;\, \lambda^{1/2}\,(1, m_a^2/s, m_b^2/s)}$$

[Prove this]

[invariant under boosts in the beam direction]

# Three-particle final states

$$R_3(s) = \int d^4p_1\, d^4p_2\, d^4p_3 \delta(p_1^2 - m_1^2)\, \delta(p_2^2 - m_2^2)\, \delta(p_3^2 - m_3^2)\, \delta^4\,(p - p_1 - p_2 - p_3)$$

- There are many ways to evaluate $R_3$
- Let's group $p_2$ and $p_3$

$$q = p_2 + p_3 \text{ and } 1 = \int d^4q\, dM^2\ \delta^4(q - p_2 - p_3)\ \delta(q^2 - M^2)$$

so

$$R_3 = \int dM^2 \left( \int d^4p_1\, d^4q\ \delta(p_1^2 - m_1^2)\ \delta(q^2 - M^2) \times \int d^4p_2\, d^4p_3\ \delta(p_2^2 - m_2^2)\ \delta(p_3^2 - m_3^2)\ \delta^4(q - p_2 - p_3) \right)$$

and

$$R_3(s, p_1, p_2, p_3) = \int dM^2\ R_2(s, p_1, q) \otimes R_2(q^2, p_2, p_3)$$

- So we simplify to the calculation of two-particle states!

- The grouping of particles is not unique:

$$p_1 + p_2 \quad \text{or} \quad p_2 + p_3 \quad \text{or} \quad p_1 + p_3$$

- We could also start by integrating over $p_1$

$$R_3(s) = \int d^4p_1\, d^4p_2\, d^4p_3 \delta(p_1^2 - m_1^2)\, \delta(p_2^2 - m_2^2)\, \delta(p_3^2 - m_3^2)\, \delta^4\left(p - p_1 - p_2 - p_3\right)$$

$$\begin{aligned} R_3(s) &= \int d^4p_1\ \delta(P_1^2 - m_1^2) \int d^4p_2\ d^4p_3\ \delta(p_2^2 - m_2^2)\ \delta(p_3^2 - m_3^2)\ \delta^4\left((p - p_1) - p_2 - p_3\right) \\ &= \int d^4p_1\ \delta(p_1^2 - m_1^2)\ R_2\left((p - p_1)^2, p_2, p_3\right) \end{aligned}$$

with $$\frac{d^3\vec{p}}{E} = p_T dp_T d\phi\ d\eta = E_T dE_T d\phi\ d\eta$$

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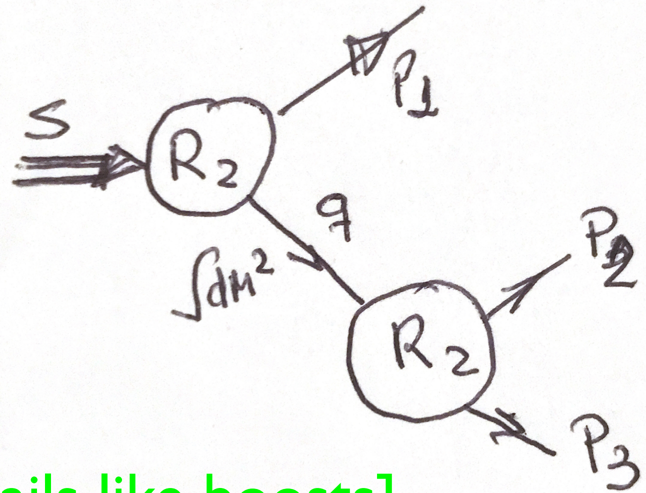
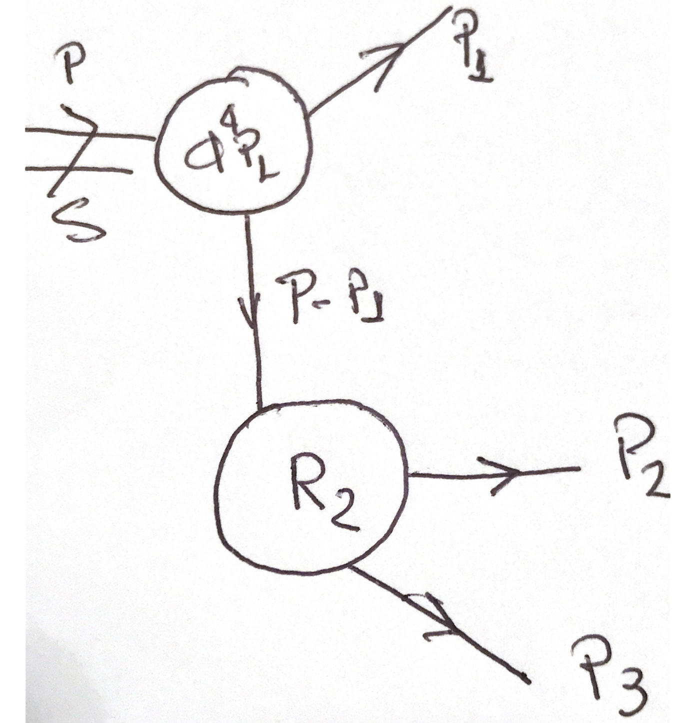
$$R_3(s) = \int d^4p_1\ \delta(P_1^2 - m_1^2) \int d^4p_2\ d^4p_3\ \delta(p_2^2 - m_2^2)\ \delta(p_3^2 - m_3^2)\ \delta^4\left((p - p_1) - p_2 - p_3\right)$$

$$= \int d^4p_1\ \delta(p_1^2 - m_1^2)\ R_2\left((p - p_1)^2, p_2, p_3\right)$$

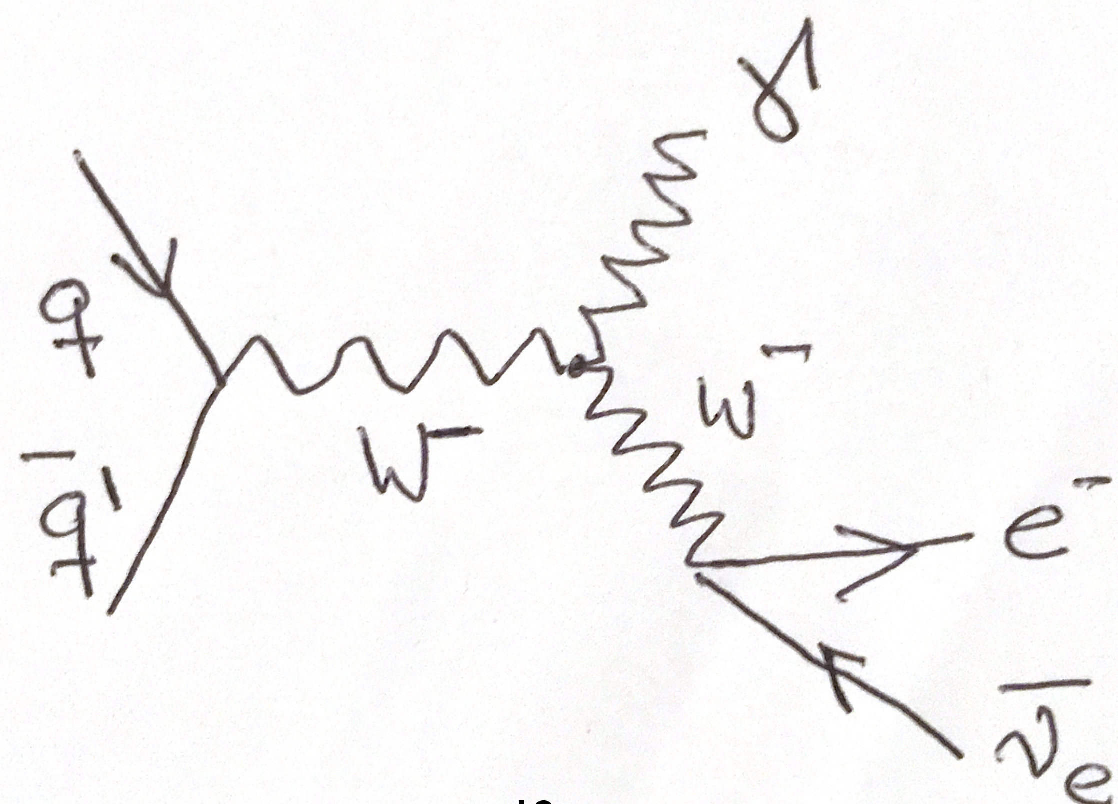
- These choices can be shown graphically as




[mention details like boosts]

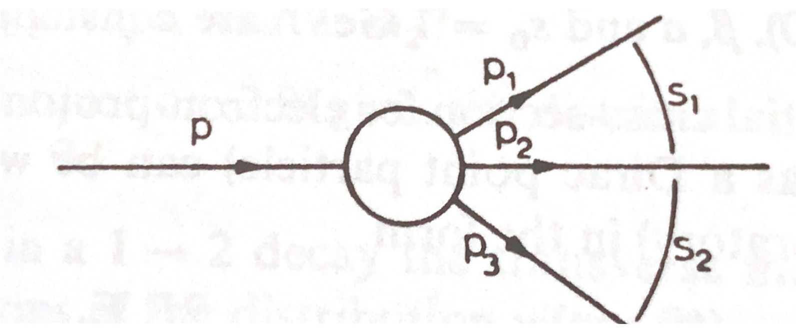
- For instance, we can use the first choice to evaluate the




[there are additional diagrams]

# A bit of history: Dalitz plot $p \to p_1 + p_2 + p_3$

- Invariant variables



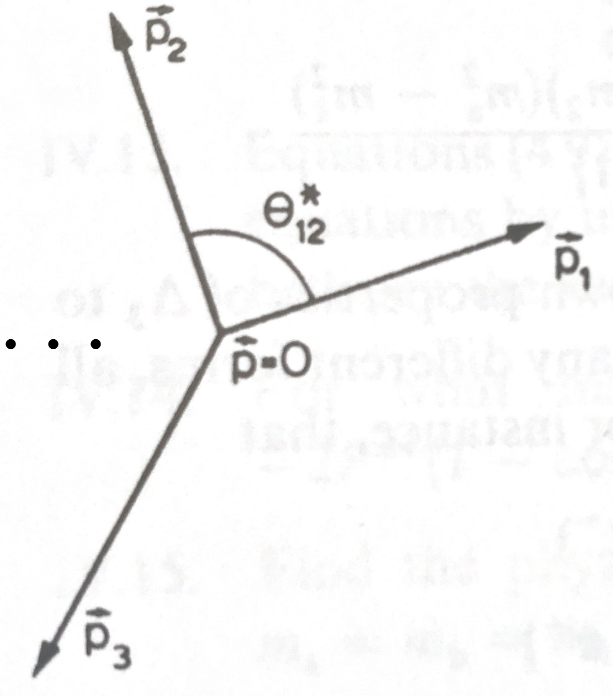

$$s_{12} \equiv s_1 = (p_1 + p_2)^2 = (p - p_3)^2$$
$$s_{23} \equiv s_2 = (p_2 + p_3)^2 = (p - p_1)^2$$
$$s_{31} \equiv s_3 = (p_3 + p_1)^2 = (p - p_2)^2$$

there are more invariants in collisions. Notice that

$$s_1 + s_2 + s_3 = s + m_1^2 + m_2^2 + m_3^2$$

- In the CM (non-invariant variables)

$$E_1 = \frac{s + m_1^2 - s_2}{2\sqrt{s}} \qquad P_1 = \frac{\lambda^{1/2}(s, m_1^2, s_2)}{2\sqrt{s}} \ldots\ldots$$




$$E_2 = \frac{s + m_2^2 - s_3}{2\sqrt{s}}$$

$$E_3 = \frac{s + m_3^2 - s_1}{2\sqrt{s}}$$

- Notice we can go back to invariant variables

$$s_1 = m_1^2 + m_2^2 + 2E_1E_2 - 2P_1P_2\cos\theta_{12} \implies$$

$$\cos\theta_{12} = \frac{(s + m_1^2 - s_2)(s + m_2^2 - s_3) + 2s(m_1^2 + m_2^2 - s_1)}{\lambda^{1/2}(s, m_1^2, s_2)\lambda^{1/2}(s, m_2^2, s_3)}$$

- Physical region (anologous/equal to 2 to 2 case)

$$G(s_1, s_2, s, m_1^2, m_2^2, m_3^2) \leq 0$$

$$G(x,y,z,u,v,w) = \frac{1}{2}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & v & x & z \\ 1 & v & 0 & u & y \\ 1 & x & u & 0 & w \\ 1 & z & y & w & 0 \end{vmatrix}$$

- In the CM frame

$$R_3(s) = \int \prod_{j=1}^{3} \frac{d^3 p_j}{2E_j}\ \delta^3(\vec{0} - \vec{p}_1 - \vec{p}_2 - \vec{p}_3)\delta(\sqrt{s} - E_1 - E_2 - E_3)$$

- Integrating over $\vec{p}_2$

$$R_3(s) = \int \frac{d^3P_1 d^3p_3}{8E_1E_2E_3}\ \delta(\sqrt{s} - E_1 - E_2 - E_3)$$

with $\quad E_2^2 = P_1^2 + P_3^2 + 2P_1P_3\cos\theta_{13} + m_2^2$

- Write the integrals as $\ d^3p_1 d^3p_3 = P_1^2 dP_1 d\Omega_1\ P_3^2 dP_3 d\cos\theta_{13} d\varphi_3$

and use the delta: $\quad \dfrac{dE_2}{d\cos\theta_{13}} = \dfrac{P_1P_3}{E_2}$

$$R_3(s) = \frac{1}{8}\int dE_1 dE_3 d\Omega_1 d\varphi_3\ \Theta(1-\cos^2\theta_{13})$$

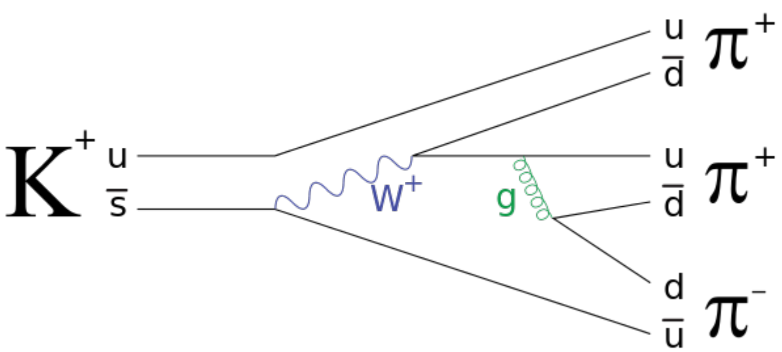
- Now $\quad dE_1 = \dfrac{ds_2}{2\sqrt{s}}\ \text{ and }\ dE_3 = \dfrac{ds_1}{2\sqrt{s}}$

- Finally,

$$R_3(s) = \frac{1}{32s} \int ds_1 ds_2 d\Omega_1 d\varphi_3 \; \Theta(G(s_1, s_2, s, m_1^2, m_2^2, m_3^2))$$

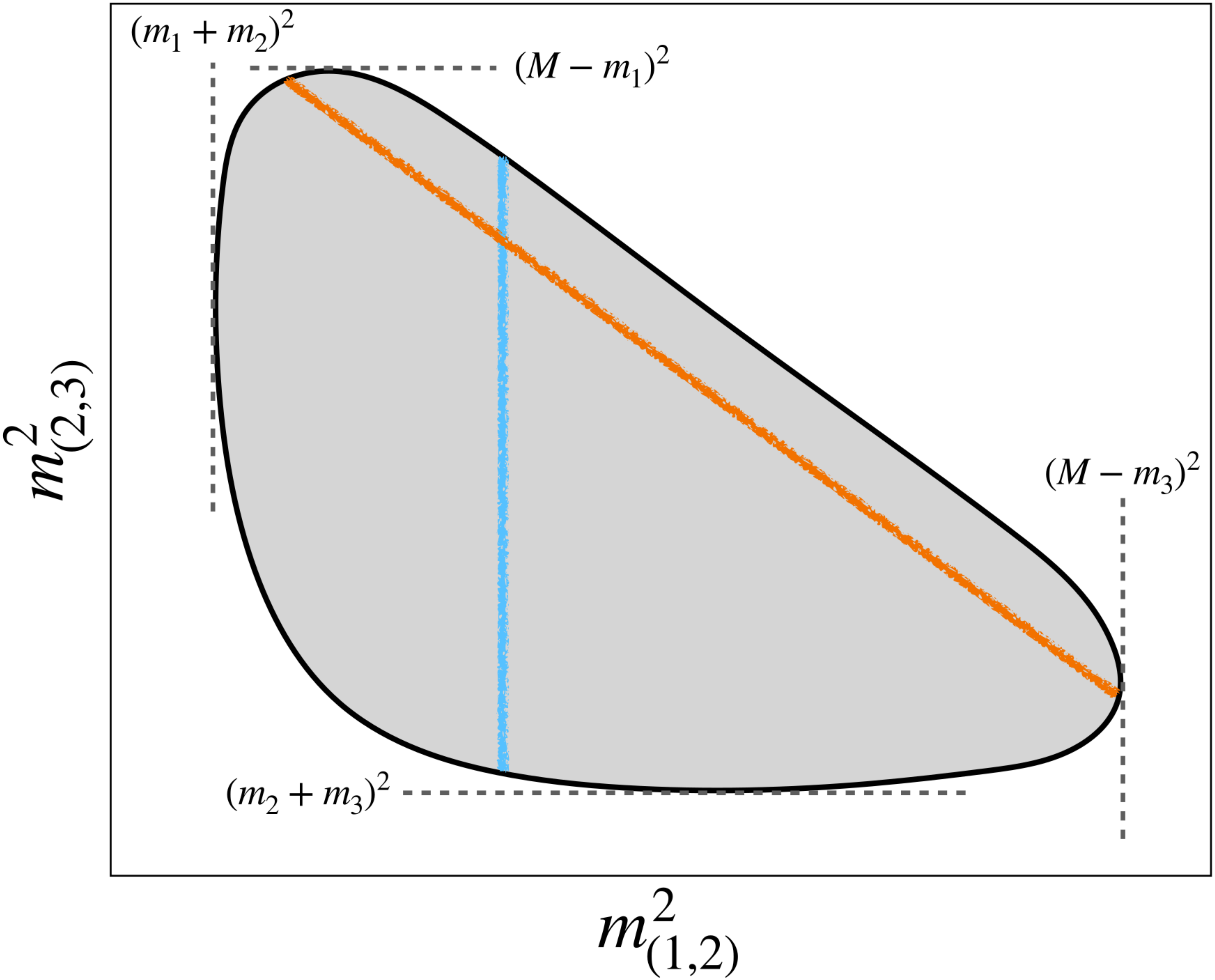
- A bit of Physics: some decays don't have a special direction

$$K^+ \to \pi^+\pi^+\pi^- \quad \Longrightarrow \quad d\Omega_1 = 4\pi \quad \text{and} \quad d\varphi_3 = 2\pi$$



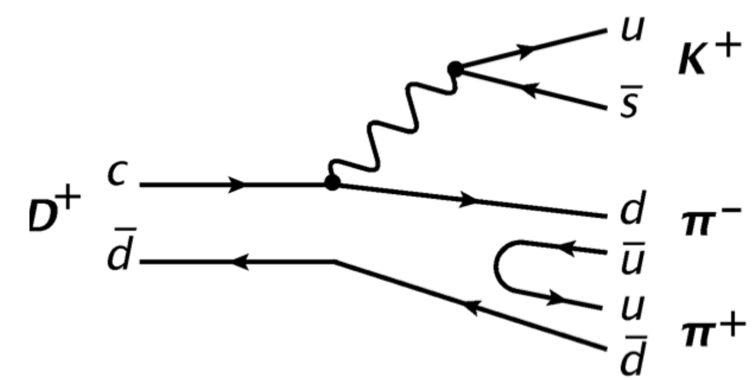

$$\frac{d^2 R_3}{ds_1 ds_2} = \frac{\pi^2}{4s}$$

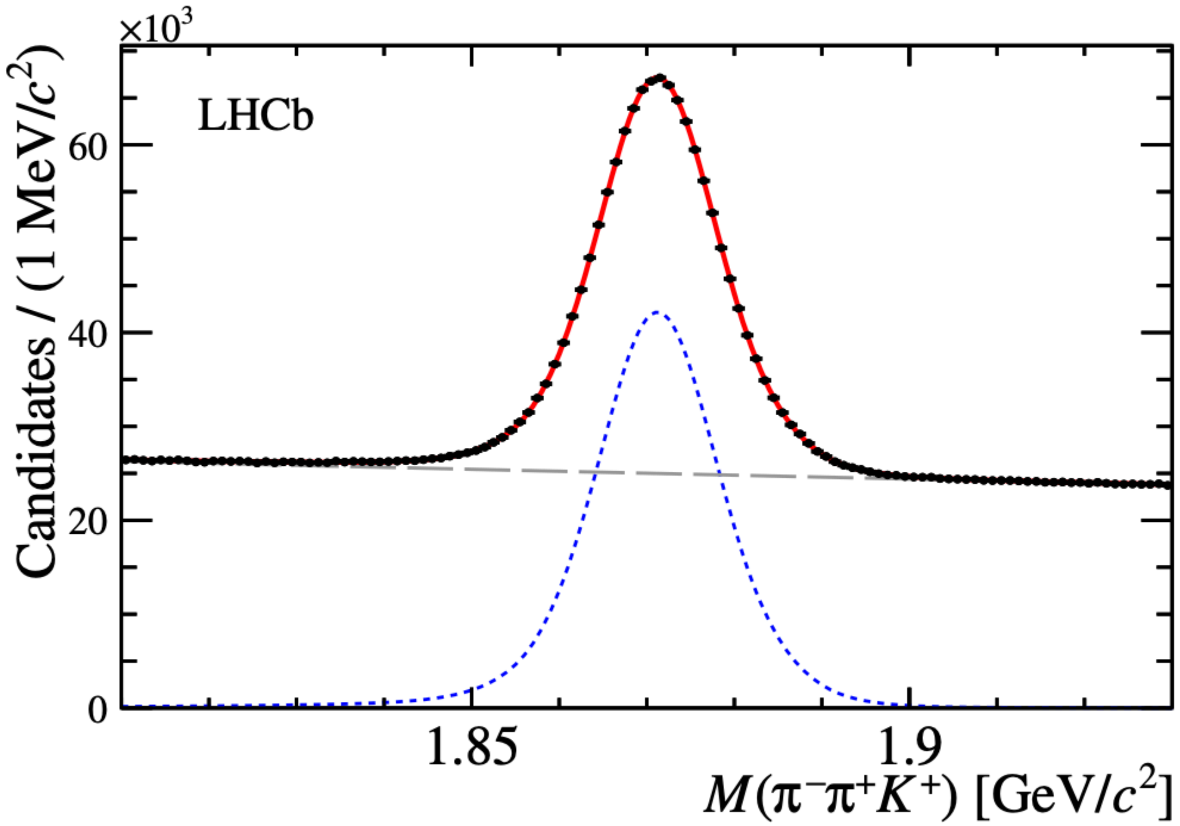
- We can gain information on the scattering amplitude from data

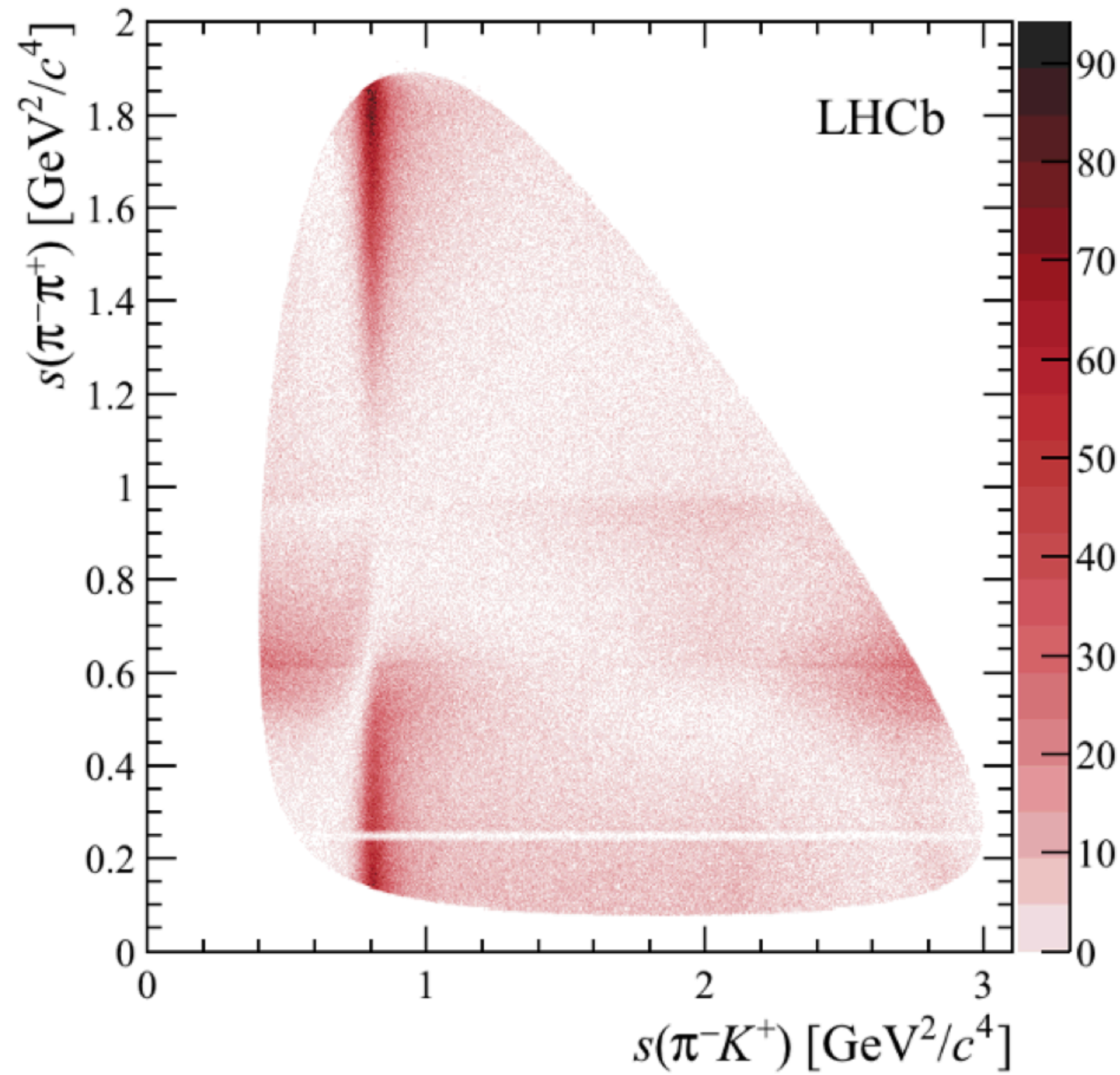


$(m_1 + m_2)^2$
$(M - m_1)^2$
$(M - m_3)^2$
$(m_2 + m_3)^2$
$m^2_{(2,3)}$
$m^2_{(1,2)}$

- Dalitz plot is still used in real life!
- LHCb analysis of $D^+ \to K^+\pi^+\pi^-$

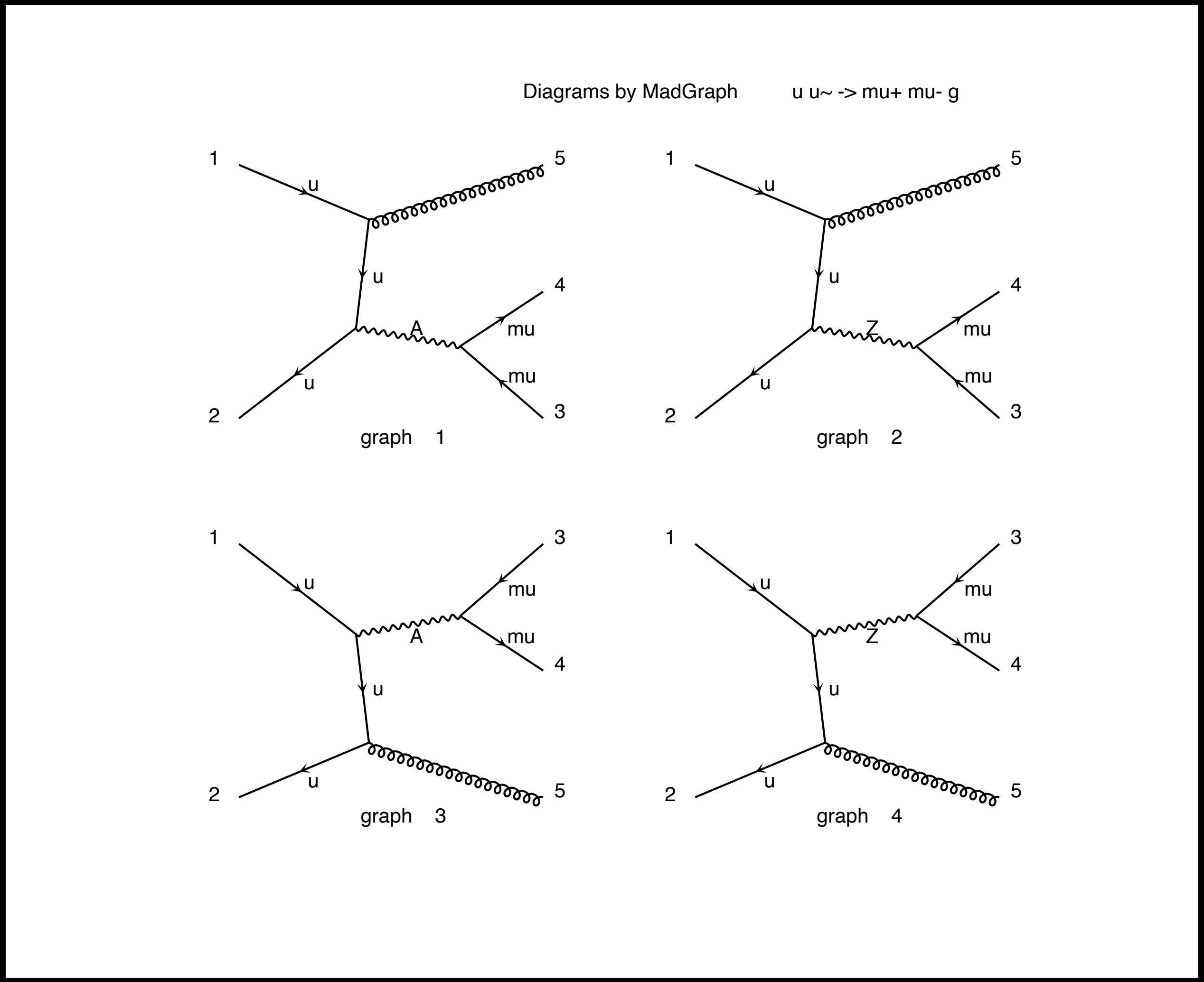


# Multiparticle Phase Space

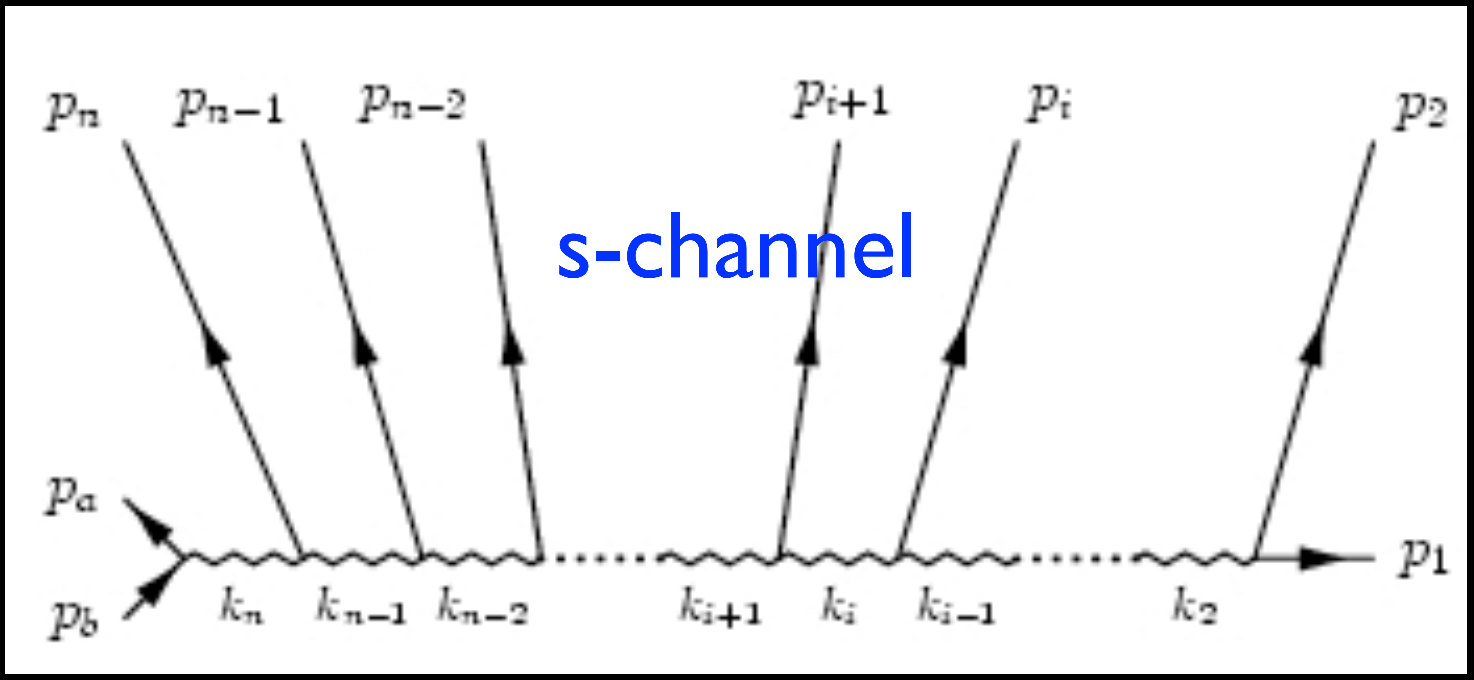
- Choice of variables decided by physics
- e.g., m34 in this example

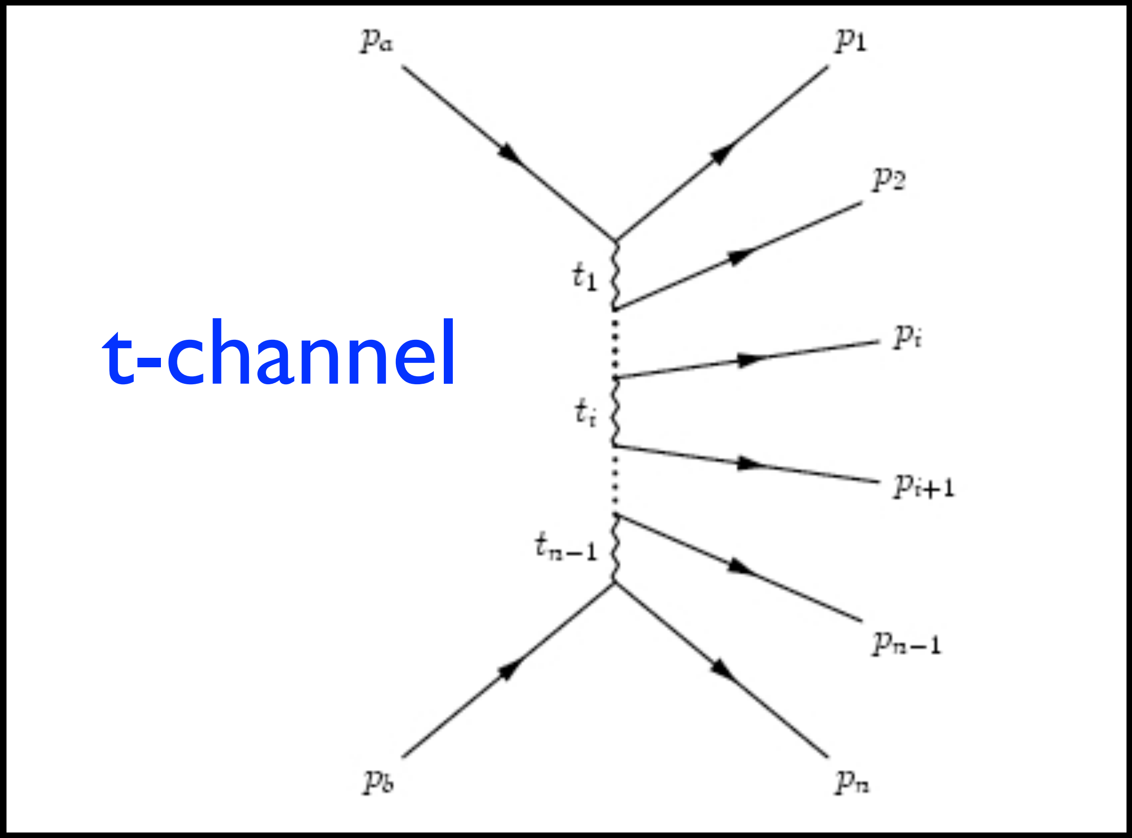



- a possible starting point is the relation: X=1+...j and Y=(j+1)+...+n

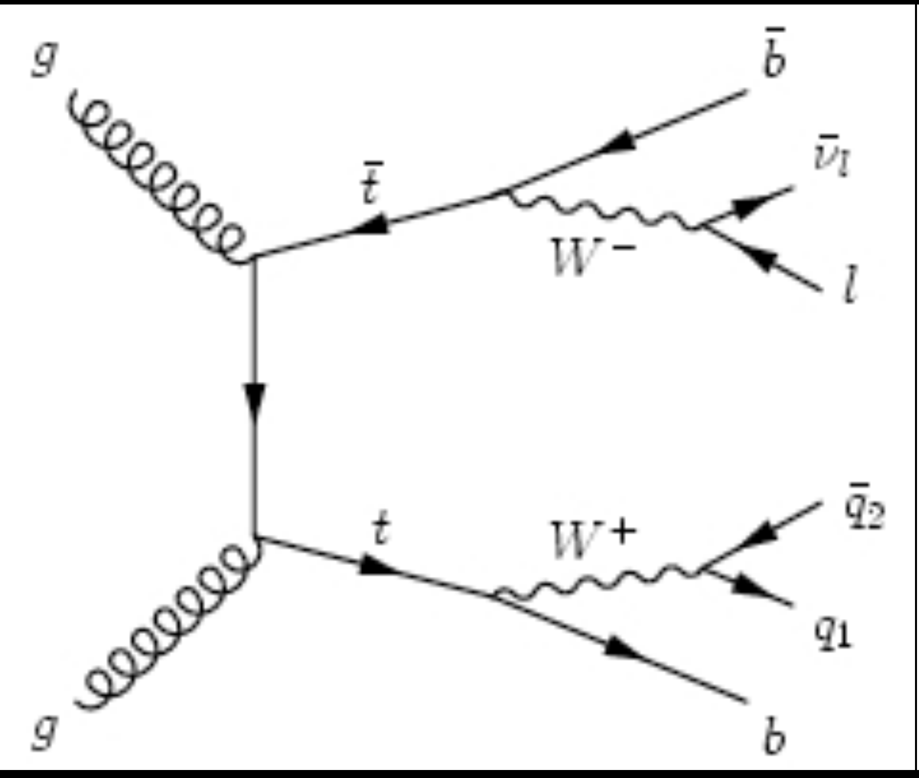
$$d\Phi_n(ab \to 1\dots n) = d\Phi_2(ab \to XY) \times dM_X^2\, dM_Y^2 \times d\Phi_j(X \to 1\dots j) \times d\Phi_{n-j}(Y \to j+1\dots n)$$

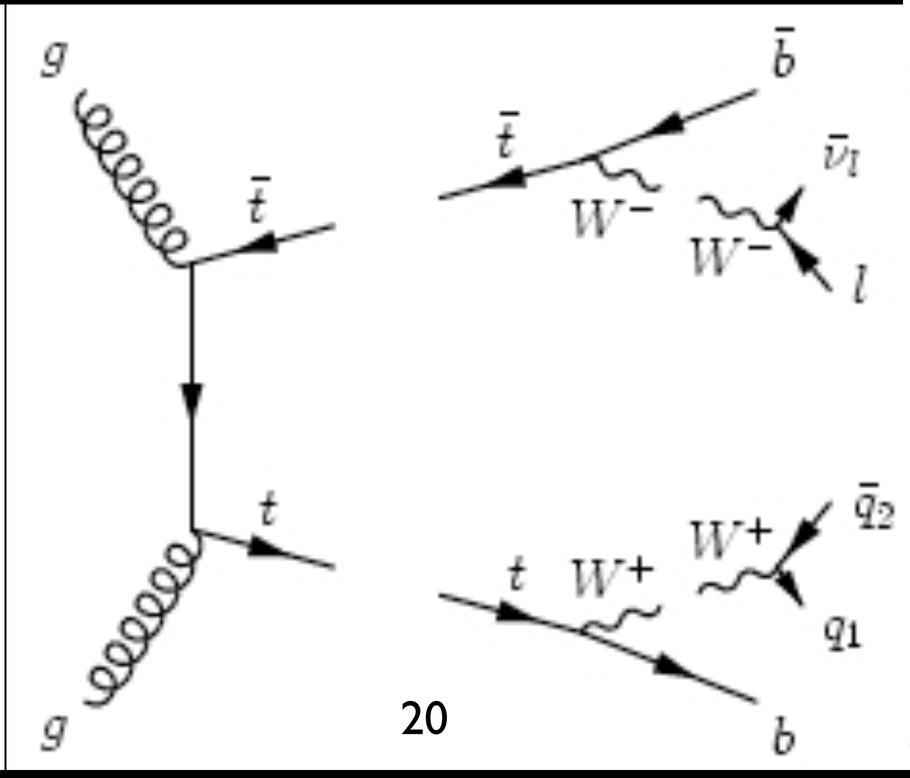
• it is possible to generate "generic" phase spaces

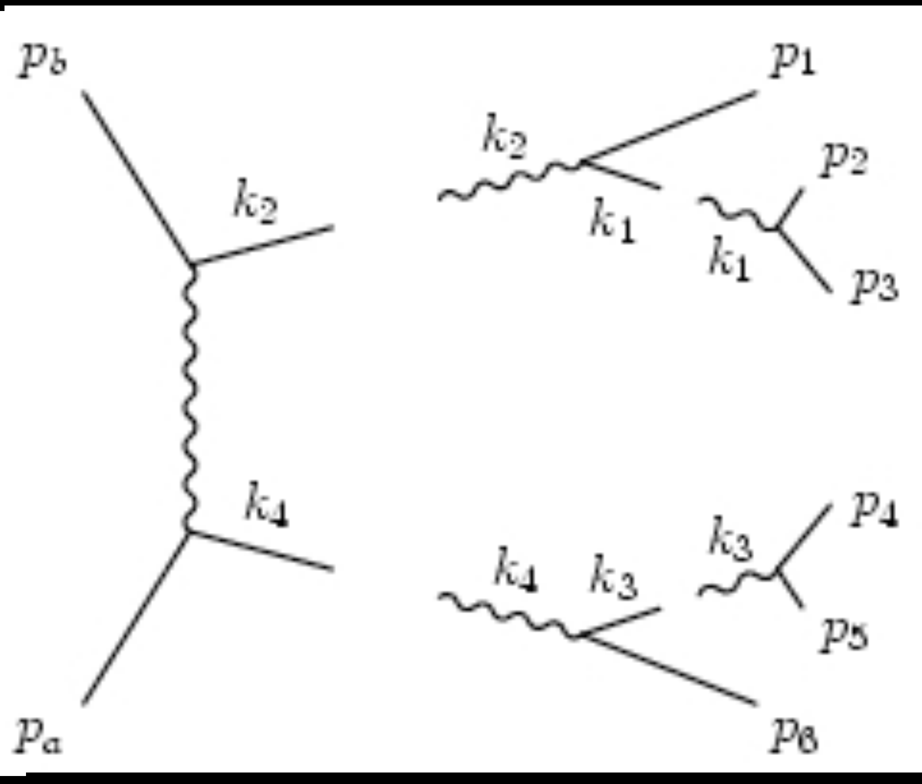



or more customized choices







$$R_3(s) = \int \; d^4p_2 \; d^4p_3 \; \delta((p - p_2 - p_3)^2 - m_1^2) \; \delta(p_2^2 - m_2^2) \; \delta(p_3^2 - m_3^2$$