4300303 1 Eletromagnetismo I
Segundo Semestre 2014

Primeira Prova

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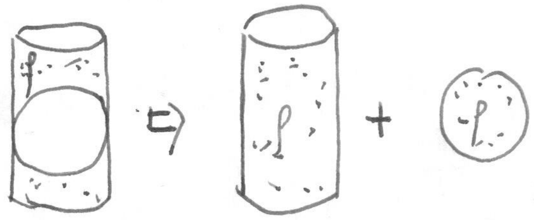
1.



a) Cilindro: $\int_S \vec{E}\cdot\hat{n}\,ds = \frac{q_t}{\epsilon_0} \Rightarrow 2\pi r L E_+ = \frac{\rho}{\epsilon_0}\times(\pi a^2 L) \therefore E_+ = \frac{\rho a^2}{2\epsilon_0 r}$

Esfera: $4\pi r^2 E_- = -\frac{\rho}{\epsilon_0}\left(\frac{4}{3}\pi a^3\right) \therefore E_- = \frac{-\rho a^3}{3\epsilon_0 r^2}$

Campo total $(r = l_A)$: $\vec{E} = \frac{\rho a^2}{\epsilon_0}\left[\frac{1}{2l_A} - \frac{a^3}{3l_A^2}\right]\hat{e}_r \therefore \boxed{\vec{E} = \frac{\rho a^2}{2\epsilon_0 l_A}\left[1 - \frac{2a}{3l_A}\right]\hat{e}_r}$

b) Cilindro: $2\pi r L E_+ = \frac{\rho}{\epsilon_0}(\pi r^2 L) \therefore E_+ = \frac{\rho r}{2\epsilon_0}$

Esfera: $4\pi r^2 E_- = -\frac{\rho}{\epsilon_0}\left(\frac{4\pi r^3}{3}\right) \therefore E_- = -\frac{\rho r}{3\epsilon_0}$

$\therefore \underset{(r=l_B)}{\vec{E}} = \frac{\rho l_B}{\epsilon_0}\left[\frac{1}{2} - \frac{1}{3}\right]\hat{e}_r \therefore \boxed{\vec{E} = \frac{\rho l_B}{6\epsilon_0}\hat{e}_r}$

2.

a) $W = \frac{1}{2}\int \rho(\vec{r}')\phi(\vec{r}')d\tau' = \frac{1}{2}\sum_i q_i \int \delta(\vec{r}'-\vec{r}_i)\phi(\vec{r}')d\tau' \therefore \boxed{W = \frac{1}{2}\sum_i q_i\phi(\vec{r}_i)}$

b) $W = \frac{1}{2}\int \rho(\vec{r})\phi(\vec{r}')d\tau' = \frac{\epsilon_0}{2}\int \phi(\nabla\cdot\vec{E})d\tau' = \frac{\epsilon_0}{2}\left[\int \nabla\cdot(\phi\vec{E})d\tau' - \int \vec{E}\cdot\nabla\phi\, d\tau'\right]$

$\therefore W = \frac{\epsilon_0}{2}\int_S (\phi\vec{E})\cdot\hat{n}\,ds + \frac{\epsilon_0}{2}\int \vec{E}\cdot\vec{E}\,d\tau'$

$R\to\infty \Rightarrow \int_S (\phi\cdot\vec{E})\,\hat{n}\,ds \sim \frac{1}{R}\frac{1}{R^2}R^2 \to 0 \therefore \boxed{W = \int \frac{\epsilon_0|E|^2}{2}d\tau}$

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a) $\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$; $\vec{E} = -\nabla\phi$ $\therefore$ $\nabla^2\phi = -\frac{\rho}{\epsilon_0}$; $\rho = \rho_{livre} + \rho_{pol}$

$\rho_{livre} = 0$; $\rho_{pol} = -\nabla\cdot\vec{P}$; $\vec{P} = cte \Rightarrow \rho_{pol} = 0$ $\therefore$ $\boxed{\nabla^2\phi = 0}$

b) $\nabla^2\phi = 0 \Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ $\therefore$ $\phi(x,y) = X(x)Y(y)$ $\therefore$ $\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$

$\therefore$ $\frac{d^2X}{dx^2} = \alpha^2 X$; $\frac{d^2Y}{dy^2} = -\alpha^2 Y$ $\therefore$ $X(x) = A e^{\alpha x} + B e^{-\alpha x}$; $Y(y) = C\cos(\alpha y) + D\sin(\alpha y)$

$y = 0 \Leftrightarrow \phi(x,y) = 0$ $\therefore$ $\boxed{C = 0;\ y = b \Rightarrow \phi(x,y) = 0 \Rightarrow \alpha = \frac{m\pi}{b},\ m = 1, 3, \dots}$

$\therefore$ $\phi(x,y) = \sum_{m=1}^{\infty}\left[A_m e^{\frac{m\pi x}{b}} + B_m e^{-\frac{m\pi x}{b}}\right]\sin\left(\frac{m\pi y}{b}\right)$

c) $x \to \infty \Rightarrow \phi_{II}(x,y) \sim A_m^{II} e^{\frac{m\pi x}{b}} \Rightarrow \boxed{A_m^{II} = 0}$

d) $\phi_I(0,y) = V \Rightarrow V = \sum_{m=1}\left[A_m^I + B_m^I\right]\sin\left(\frac{m\pi y}{b}\right)$

$\therefore$ $V\int_0^b \sin\left(\frac{m'\pi y}{b}\right)dy = \sum_{m=1}\left[A_m^I + B_m^I\right]\int_0^b \sin\left(\frac{m'\pi y}{b}\right)\sin\left(\frac{m\pi y}{b}\right)dy$ (Integrais dadas)

$\therefore$ $\frac{2b}{m\pi}V = \left[A_m^I + B_m^I\right]\frac{b}{2}$ (m impar) $\therefore$ $\boxed{A_m^I + B_m^I = \frac{4V}{m\pi}}$; m impar

e) $x = a$ $\quad D_{nI} = D_{nII}$ $\therefore$ $D_{xI} = D_{xII}$ $\therefore$ $\epsilon_0 E_{xI} + P_0 = \epsilon_0 E_{xII}$

$E_{tI} = E_{tII}$ $\therefore$ $E_{yI} = E_{yII}$

$\vec{E} = -\nabla\phi$ $\therefore$ $\vec{E}_I = \sum_{m=1}\left\{\frac{m\pi}{b}\left[-A_m^I e^{+\frac{m\pi x}{b}} + B_m^I e^{-\frac{m\pi x}{b}}\right]\sin\left(\frac{m\pi y}{b}\right)\hat{e}_x -\right.$

$\left. - \frac{m\pi}{b}\left[A_m^I e^{\frac{m\pi x}{b}} + B_m e^{-\frac{m\pi x}{b}}\right]\cos\left(\frac{m\pi y}{b}\right)\hat{e}_y\right\}$

$\vec{E}_{II} = \sum_{m=1}\frac{m\pi}{b}B_m^{II} e^{-\frac{m\pi x}{b}}\left[\sin\left(\frac{m\pi y}{b}\right)\hat{e}_x - \cos\left(\frac{m\pi y}{b}\right)\hat{e}_y\right]$

$\therefore$ $E_{yI} = E_{yII} \Rightarrow \boxed{A_m^I e^{\frac{m\pi a}{b}} + B_m^I e^{-\frac{m\pi a}{b}} = B_m^{II} e^{-\frac{m\pi a}{b}}}$

$\epsilon_0 E_{xI} + P_0 = \epsilon_0 E_{xII} \Rightarrow$

$\epsilon_0\sum_{m=1}\frac{m\pi}{b}\left[-A_m^I e^{\frac{m\pi a}{b}} + B_m^I e^{-\frac{m\pi a}{b}}\right]\sin\left(\frac{m\pi y}{b}\right) + P_0 = \epsilon_0\sum_{m=1}\frac{m\pi}{b}B_m^{II} e^{-\frac{m\pi a}{b}}\sin\left(\frac{m\pi y}{b}\right)$

$$\therefore\ P_0 = \epsilon_0 \sum_{m=1} \frac{m\pi}{b}\left[-A_m^{I} e^{\frac{m\pi a}{b}} + \left(B_m^{I} - B_m^{II}\right) e^{-\frac{m\pi a}{b}}\right] \sin\left(\frac{m\pi y}{b}\right)$$

Aplicando novamente a integral de Fourier

$$\therefore\ P_0 \underbrace{\int_0^b \sin\left(\frac{m'\pi y}{b}\right) dy}_{2b/m'\pi} = \epsilon_0 \sum_{m=1} \frac{m\pi}{b}\left[-A_m^{I} e^{\frac{m\pi a}{b}} + \left(B_m^{I} - B_m^{II}\right) e^{-\frac{m\pi a}{b}}\right] \times \underbrace{\int_0^b \sin\left(\frac{m'\pi y}{b}\right) \sin\left(\frac{m\pi y}{b}\right) dy}_{b/2\ \delta_{m'm}}$$

$$\therefore\ \frac{2b}{m\pi} P_0 = \epsilon_0 \left[-A_m^{I} e^{\frac{m\pi a}{b}} + \left(B_m^{I} - B_m^{II}\right) e^{-\frac{m\pi a}{b}}\right] \frac{b}{2}$$

$$\therefore\ \boxed{-A_m^{I} e^{\frac{m\pi a}{b}} + \left(B_m^{I} - B_m^{II}\right) e^{-\frac{m\pi a}{b}} = \frac{4P_0}{m\pi\epsilon_0}}$$

f) $\vec{D}_I = \epsilon_0 \vec{E}_I + P_0 \hat{e}_x$

$$\therefore\ \boxed{\begin{aligned}\vec{D}_I = &\left[\epsilon_0 \sum_{m=1} \frac{m\pi}{b}\left(-A_m^{I} e^{+\frac{m\pi x}{b}} + B_m^{I} e^{-\frac{m\pi x}{b}}\right) \sin\left(\frac{m\pi y}{b}\right) + P_0\right] \hat{e}_x \\ &- \left[\epsilon_0 \sum_{m=1} \frac{m\pi}{b}\left(A_m^{I} e^{\frac{m\pi x}{b}} + B_m^{I} e^{-\frac{m\pi x}{b}}\right) \cos\left(\frac{m\pi y}{b}\right)\right] \hat{e}_y\end{aligned}}$$