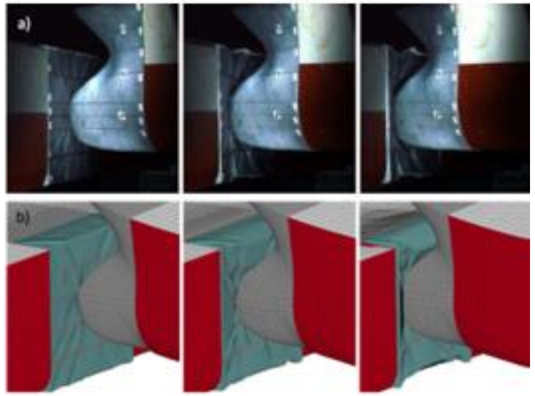
Vibração de Hastes

Sistemas Dinâmicos II

Profs. Ettore de Barros,
Larissa Driemeier e
Marcílio Alves



IMPACT ENGINEERING

Fundamentals

Experiments

Nonlinear Finite Elements

Marcílio Alves

Let us start by analysing the simplest case of a wave propagating in a straight rod, see next figure. A small compressive disturbance applied in the rod will, so to say, pushes the material ahead, which in turn will move the next material front, and so on. We have then a compressive propagating pulse, causing a compressive stress, $-\sigma$, at, say, x. As the pulse travels, the stress level at $x = x + \mathrm{d}x$ will change from $-\sigma$ to $\sigma + (\partial\sigma/\partial x)\mathrm{d}x$, such that force equilibrium gives

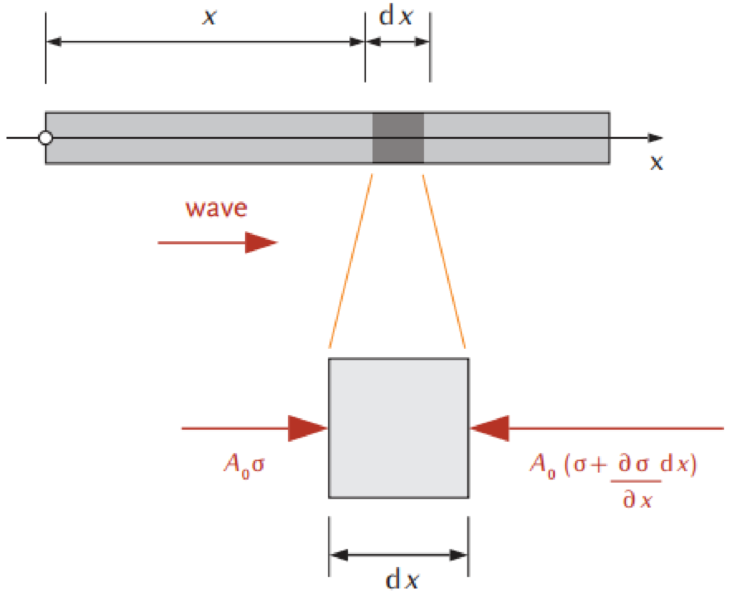
Vector bold notation is not used in this chapter until section 9.

$$-\sigma A + \left(\sigma + \frac{\partial \sigma}{\partial x}\mathrm{d}x\right) A = \rho A \mathrm{d}x \frac{\partial^2 u}{\partial t^2}$$

or

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2},$$

where ρ is the rod density.



Equilibrium of a rod.

Returning now to our previous equilibrium equation and changing σ by $E\varepsilon$, we obtain the so called wave equation

$$\frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

We can find the solution of this equation by inspection. Considering the function

$$u = f\left(x - \sqrt{E/\rho}\, t\right)$$

and differentiating it twice with respect to time and space, we can see that such a solution satisfies the wave equation. Another function that is also the solution of the wave equation is

$$u = g\left(x + \sqrt{E/\rho}\, t\right).$$

Examining close one of these solutions, say $u = f$, at $t = t_1$ and $x = x_1$, the resulting magnitude is u_1. Such a magnitude is also attained at $t = t_2$ when $x = x_2$, as depicted in the next figure, so that

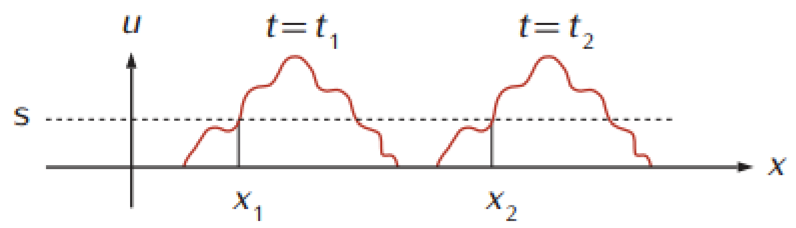
$$u_1 = f\left(x_1 - \sqrt{E/\rho}\, t_1\right) = f\left(x_2 - \sqrt{E/\rho}\, t_2\right),$$

or

$$\sqrt{\frac{E}{\rho}} = \frac{x_2 - x_1}{t_2 - t_1},$$

implying that the ratio

$$c = \sqrt{E/\rho}$$



Material	E GPa	ρ kg/m^3	c m/s	ν	σ_y MPa	σ_u MPa
Acrylic	3	1180	1594	0.35	124	70
Aluminium	70	2700	5092	0.22	95	110
Beryllium	303	1844	12819	0.03	240	370
Brass	97	8670	3345	0.34	112	37
Cast iron	130	7810	4080	0.21	100	1650
Copper	130	8930	3815	0.34	33	210
Diamond	1200	3520	18464	0.20	-	60000
Epoxy resin	2.41	2600	963	0.40	85	97
Glass	62	2400	5083	0.25	-	33
Gold	77	19300	1997	0.42	-	100
Ice	9	897	3168	0.33	-	1
Iron	211	7850	5184	0.29	100	350
Lead	14	11400	1108	0.42	19	32
Magnesium	44	1740	5029	0.35	105	205
Molybdenum	330	1022	17969	0.38	415	515
Nickel	207	8880	4828	0.31	59	317
Platinum	171	21500	2820	0.39	140	180
Plexiglass	3	1190	1588	0.37	80	100
Porcelain	104	2400	6583	0.17	-	130
PVC	3	2500	1095	0.40	52	59
Quartz	76	2650	5355	0.17	-	48
Steel	200	7870	5041	0.29	300-1000	500-2000
Titanium	116	4500	5077	0.34	880	950
Tungsten	400	19250	4558	0.28	750	980
Uranium	208	18950	3313	0.30	200	500
Zinc	85	7100	3460	0.33	300	400

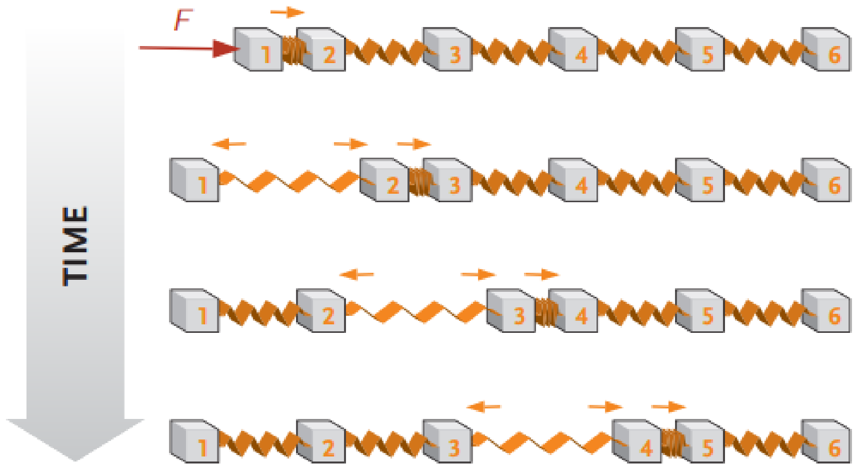
Since the material particles change their rest position, there is some straining in the material and, hence, stress. The stress level caused by a wave propagating in a linear elastic material can be quantified considering that

$$u = f\,(x - ct)\,, \qquad \frac{\partial u}{\partial x} = \frac{\partial f}{\partial x}, \qquad \frac{\partial u}{\partial t} = v = -c\frac{\partial f}{\partial x},$$

from which it follows

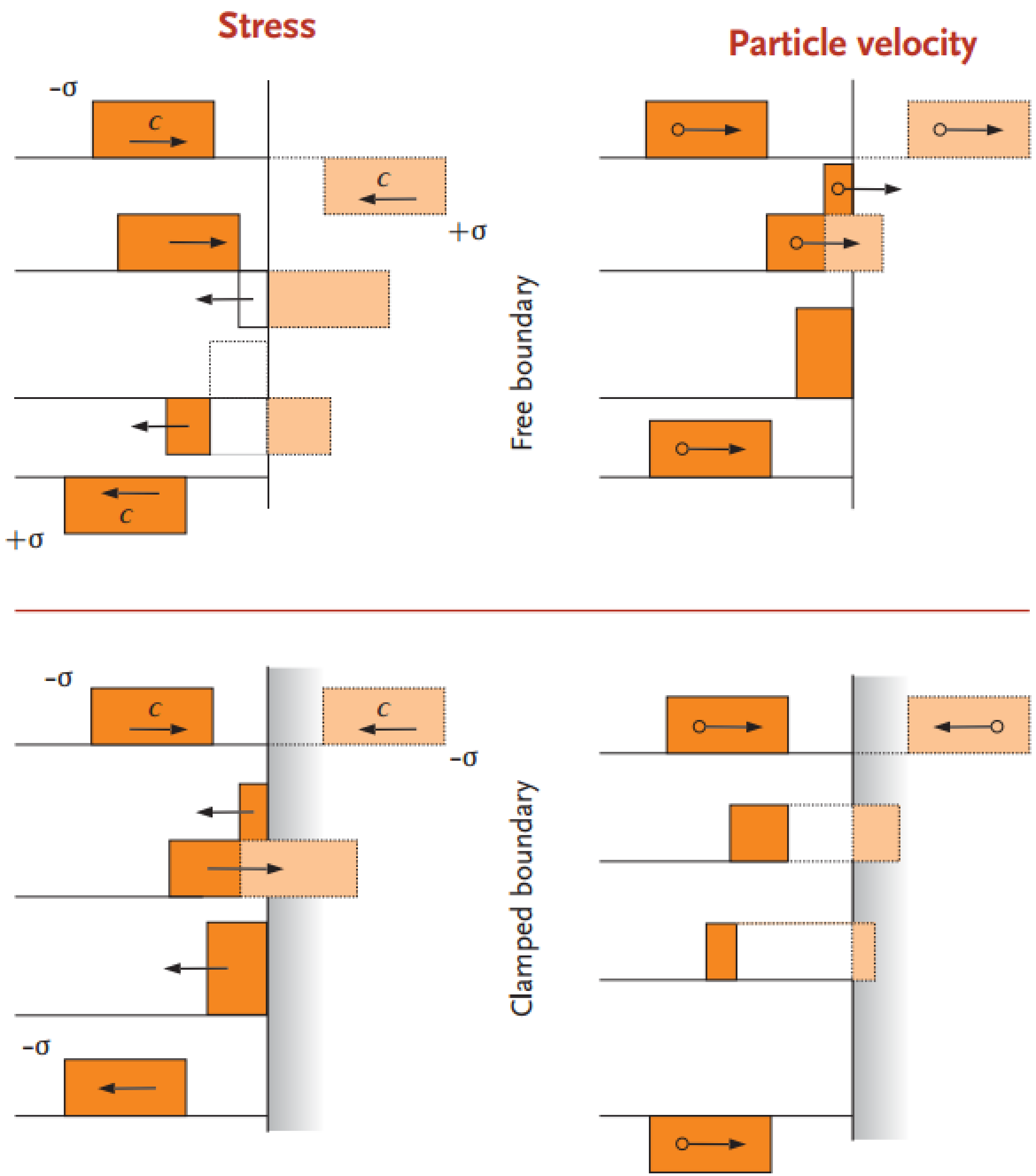
$$\sigma = E\frac{\partial u}{\partial x} = -E\frac{v}{c} = -\rho c v,$$

with ρc being called mechanical impedance.



A row of masses and springs being disturbed by the sweeping of a wave.

also evident from the previous figure that a longitudinal disturbance moving the particles in the same (opposite) direction it travels is a compressive (tensile) wave. Transverse movement of particles causes material shearing and the associate waves are called shear or torsional waves and are not considered here.



Reflection of waves in free (top) and fixed (bottom) boundaries. Adapted from M.A. Meyers, Dynamic Material Behavior, Wiley, 1994.

There are many practical examples where, instead of colliding, a given bar may be loaded by some pulse or other initial condition and its dynamics is of interest to us. We can expand our analysis of waves in bars by exploring the so called vibration solutions to the equilibrium equation

$$\frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} + \frac{1}{\rho A} f(x,t) = \frac{\partial^2 u}{\partial t^2},$$

which has now the forced term $f(x,t)$. Dropping it for the moment we can examine the axial free vibration of the bar, whose displacement depends on time and position and it can be expressed as the combination of two functions $U(x)$ and $T(t)$ such that,

$$u(x,t) = U(x)T(t).$$

Accordingly, the wave equation now reads

$$\frac{c^2}{U(x)}\frac{\mathrm{d}^2 U(x)}{\mathrm{d}t^2} = \frac{1}{T(t)}\frac{\mathrm{d}^2 T(t)}{\mathrm{d}t^2}.$$

Note that its left side depends only on x, while the right one on t. This is only possible if these sides equal to a constant term, that we set as $-\omega^2$ for convenience. This leads to the equations

$$\frac{\mathrm{d}^2 U(x)}{\mathrm{d}t^2} + \frac{\omega^2}{c^2}U(x) = 0$$

and

$$\frac{\mathrm{d}^2T(t)}{\mathrm{d}t^2} + \omega^2 T(t),$$

whose solutions are

$$U(x) = a_1 \cos\frac{\omega x}{c} + a_2 \sin\frac{\omega x}{c} \quad \text{and} \quad T(t) = a_3 \cos\omega t + a_4 \sin\omega t,$$

giving

$$u(x,t) = \left(a_1 \cos\frac{\omega x}{c} + a_2 \sin\frac{\omega x}{c}\right)(a_3 \cos\omega t + a_4 \sin\omega t)$$

with the constants a_1, a_2, a_3, a_4 being obtained from the boundary conditions, *ie* the way the rod is held in space, and from the initial conditions, *ie* the way the rod is set in motion at $t = 0$, respectively.

Consider, as an example of application of the above solution, a horizontal bar of length L supported by strings and lightly hit at one of its end. This will set it to vibrate and the aim is to calculate the natural frequencies and vibration modes of the bar in its free condition.

The boundary conditions for this case is that of a free–free bar, meaning that at the extremes there is no stress. Since $\sigma = E\varepsilon$ and $\varepsilon = \mathrm{d}u/\mathrm{d}x$, we can write

$$\frac{\mathrm{d}U(0,t)}{\mathrm{d}x} = 0 \quad \text{and} \quad \frac{\mathrm{d}U(L,t)}{\mathrm{d}x} = 0.$$

The first equation at $x = 0$ implies that

$$\frac{\mathrm{d}U(0)}{\mathrm{d}x} = \frac{-a_1\omega}{c}\sin\frac{\omega x}{c} + \frac{a_2\omega}{c}\cos\frac{\omega x}{c} = 0 \rightarrow a_2 = 0$$

and the second boundary condition yields

$$\sin\frac{\omega L}{c} = 0,$$

since $a_1 = 0$ implies a trivial solution. The above equation has the solution

$$\frac{\omega_n L}{c} = n\pi, \qquad n = 1, 2, 3, \ldots$$

and the bar can vibrate at various natural frequencies, each one associate to the respective natural mode

$$U_n(x) = a_1 \cos\frac{n\pi x}{L}.$$

The complete motion of the bar is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_1 \cos \frac{n\pi x}{L} T(t),$$

with $T(t)$ being the harmonic function seen before. Observe that this solution depends on a_1, obtained once a known excitation is applied to the bar

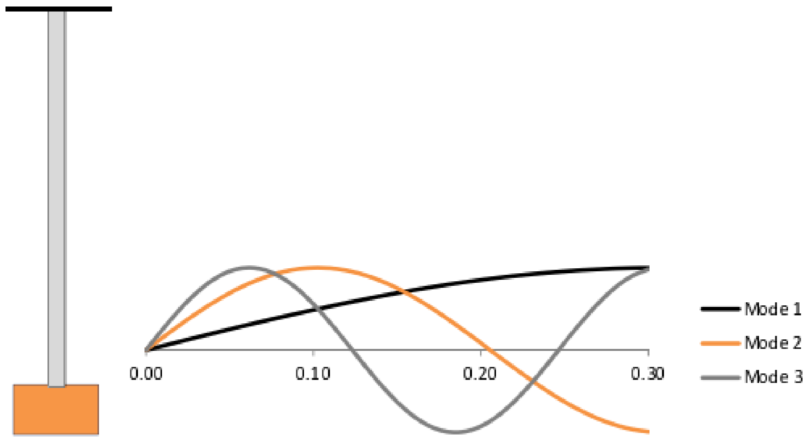
The above example indicates that, for each natural frequency, also called eigenvalues, there is an associate natural mode of vibration.

Note the use of

$$u(x,t) = \sum_{n=1}^{\infty} U_n(x) T_n(t),$$

ie the bar will vibrate in a way that entails a combination of the basic vibration modes. This advanced idea is known as the expansion theorem. A rigorous proof of this theorem is not given here but we indicate that the eigenvectors are orthogonal and form a basis in the n-dimensional space. The solution of a linear dynamic problem is a combination of these eigenvectors, the weight of each vector being an unknown determined by the boundary and initial conditions of a given problem.

A mass G is attached at the free end of a vertical rod fixed on a ceiling. Evaluate the influence of the mass on the natural frequencies of the rod.



The figure indicates that the boundary conditions at the bar support and at the mass extreme are,

$$u(0, t) = 0$$

and

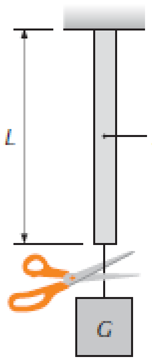
$$F(L, t) = A\sigma(L, t) = AE\frac{\partial u(L, t)}{\partial x} = -M\frac{\partial^2 u(L, t)}{\partial t^2}.$$

By using them in the equilibrium equations we obtain the transcendental frequency equation

$$\beta_n \tan \beta_n = m_r, \quad n = 1, 2, \ldots$$

with $\beta_n = \omega_n L/c$ and $m_r = m/G$, the ratio between the rod mass, m, and the suspended mass, G. As before, each n represents one of the infinity natural frequencies, an eigenvalue.

A bar is vertically fixed on a ceiling and it has elastic modulus E, density ρ and cross section A. At its opposite extreme it is loaded by a mass G via a massless string, as depicted in the next figure. Obtain the dynamic displacement field of the bar when the string is cut at $t = 0$.



A vertical bar loaded by a mass.

A known boundary condition for this problem is that, at the support,

$$u(x = 0, t) = 0,$$

which implies that, in our general solution $u(x, t)$ seen before, $a_1 = 0$.

We also know that, once the string is cut, the bar is set in motion. The bar end at $x = L$ is now free to vibrate and there is no more stress there, so that

$$\frac{\mathrm{d}u(L,t)}{\mathrm{d}x} = \sum_{n=1}^{\infty} \frac{\omega_n}{c} a_2 \cos \frac{\omega_n L}{c} T_n(t) = 0,$$

giving the natural frequencies of the bar as

$$\omega_n = \frac{n\pi c}{2L}, \qquad n = 1, 3, 5, \ldots$$

The bar motion reads now

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{2L} \left(a_3 \cos \omega_n t + a_4 \sin \omega_n t \right),$$

with the constant a_2 being incorporated by a_3 and a_4.

We also have that the strain ε caused by the hanged mass, the bar weight not being considered, affects the axial displacement such that, at $t = 0$,

$$\varepsilon x = \sin \frac{n\pi x}{2L} a_3,$$

which can be multiplied by $\sin \frac{n\pi x}{2L}$ and integrated from 0 to L to give

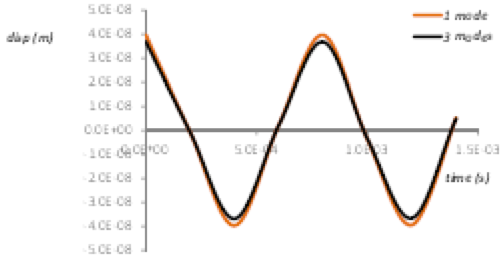
$$a_3 = \frac{8\varepsilon L}{n^2\pi^2}(-1)^{\frac{n-1}{2}}.$$

Finally, considering that $a_4 = 0$ since $\mathrm{d}u(x,0)/\mathrm{d}t = 0$, we have the final motion of the bar as

$$u(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{8L}{n^2\pi^2} \frac{Gg}{EA} (-1)^{\frac{n-1}{2}} \sin \frac{n\pi x}{2L} \cos \frac{n\pi c t}{2L},$$

where use was made of the fact that the strain is $\varepsilon = \sigma/E = Gg/EA$.

The next figure plots for some elected parameters ($L = 1$m, $A = 1\text{mm}^2$, $G = 1$ kg) the motion of a steel bar after the string is cut.



Motion of the extreme of a bar after a mass there connected is cut.

In the examples above, loading was imposed to the bar via an initial condition. It was not necessary to have in the wave equation the force term, $f(x,t)$. For a forced vibration however, $f(x,t)$ comes into play and we need to solve the partial differential equation

$$\frac{E}{\rho}\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{\rho A} f(x,t) = \frac{\partial^2 u(x,t)}{\partial t^2}.$$

To solve it generically, let us assume, as we did before, that the solution is of the type $u(x,t) = \sum_{n=1}^{\infty} U_n(x) T_n(t)$. Substituting it in the above equation, multiplying by $U_m(x)$ and integrating along the bar length, it follows that

$$\sum_{n=1}^{\infty}\left[-\frac{E}{\rho}T_n(t)\int_0^L \frac{\mathrm{d}^2 U_n(x)}{\mathrm{d}x^2}U_m(x) + \frac{\mathrm{d}^2 T_n(t)}{\mathrm{d}t^2}\int_0^L U_n(x)U_m(x)\mathrm{d}x\right]$$

$$= \frac{1}{\rho A}\int_0^L U_n(x) f(x,t)\mathrm{d}x.$$

We will see in Chapter 4 that $\int U_n(x)U_m(x) = 0$ for $n \neq m$ and $\int U_n(x)U_m(x) = 1$ for $n = m$, *ie* the natural modes of vibration form a orthonormal basis, with this orthogonality conditions being also valid for the derivatives of the eigenfunctions. These properties render the above equation as

$$\frac{\mathrm{d}^2 T_n(t)}{\mathrm{d}t^2} + \omega_n^2\, T_n(t) = \frac{1}{\rho A}\int_0^L U_n(x) f(x,t)\mathrm{d}x.$$

whose solution is

This is known as the Duhamel integral.

$$T_n(t) = \frac{1}{\rho A \omega_n}\int_0^L U(x)_n \int_0^t f(x,\tau)\sin\omega_n(t-\tau)\mathrm{d}\tau\mathrm{d}x.$$

The final sought forced solution becomes then

$$u(x,t) = \sum_{n=1}^{\infty}\frac{U_n(x)}{\rho A\omega_n}\int_0^L U_n(x)\int_0^t f(x,\tau)\sin\omega_n(t-\tau)\mathrm{d}\tau\mathrm{d}x.$$

Obtain the ensued motion of the free–free bar in the next figure suddenly pulled in its extreme $x = L$ by a force F.

We note first that the vibration modes for this free–free bar is $U_n(x) = a_1 \cos n\pi x/L$, with the natural frequencies $\omega_n = n\pi c/L$. Normalizing $U_n(x)$ according to

$$\int_0^L U_n(x)\mathrm{d}x = 1,$$

we have that $a_1 = \sqrt{2/L}$ and we obtain

$$u(x,t) = \sum_{n=1}^{\infty} \frac{\sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}}{\frac{n\pi c}{L}} \int_0^L \sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}\left[\int_0^t \frac{F}{\rho A}\sin\frac{n\pi c}{L}(t-\tau)\mathrm{d}\tau\right]\mathrm{d}x$$

The time integral above amounts to $\frac{F}{\rho A}\frac{L}{n\pi c}(1-\cos\frac{n\pi c}{L}t)$ and the integral along the length is simply $U_n(x = L) = \sqrt{\frac{2}{L}}(-1)^n$ since the force is concentrated. Hence, the vibrational motion of this beam is given by

$$\bar{u}(x,t) = \frac{2LF}{\pi^2 c^2 \rho A}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{L}\left(1-\cos\frac{n\pi ct}{L}\right).$$

Observe that any force applied to the beam will also move it as a rigid body and this can be quantified by solving the equilibrium equation for a rigid rod

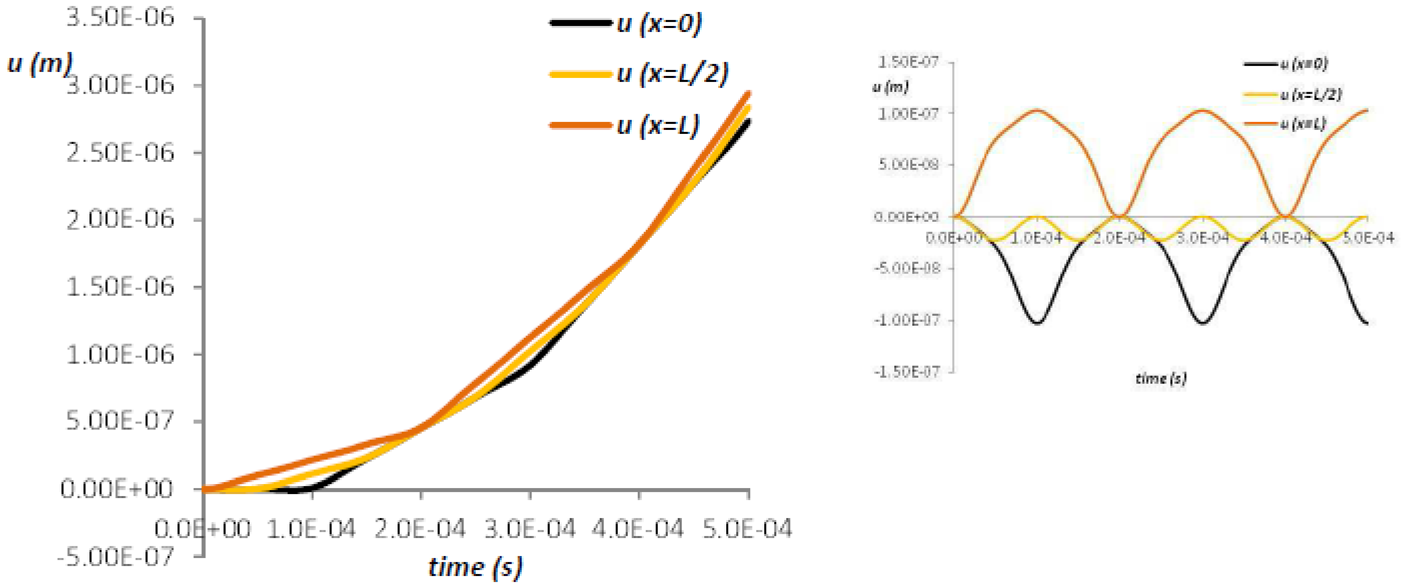
$$\rho AL\frac{\mathrm{d}^2\bar{\bar{u}}(t)}{\mathrm{d}t^2} = F,$$

whose solution

$$\bar{\bar{u}}(t) = \frac{F}{2\rho AL}t^2$$

needs to be added to $\bar{u}(x,t)$ to give the total motion of the bar as

$$u(x,t) = \frac{F}{2\rho AL}t^2 + \frac{2LF}{\pi^2 c^2 \rho A}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{L}\left(1-\cos\frac{n\pi ct}{L}\right).$$



The next figure plots this solution for some specific parameters, a 10 mm diameter, 500 mm long aluminium bar under a force of 10 N. An inset is provided that shows only the vibrational motion, *ie* without the rigid body one.

Demonstração

Abrir garrafa de vinho

https://www.youtube.com/watch?v=hbQQg53VGGc open a wine bottle