Stability

Lecture notes:

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Plates

Introduction

Objective:

Compute the buckling load for plates

- **thin plates:** when the thickness/length is small and/or the transverse shear modulus is high the transverse shear is negligible – pure bending - (Kirchhoff theory); higher buckling load
- **thick plates:** when the transverse shear is significant the Reissner-Mindlin theory should be used; lower buckling load

Introduction

Plates:

- the geometry is a flat surface in the x-y plane
- have bending in two planes.
- the moments and displacements are functions of two coordinates (x, y)
- the equilibrium equations are partial differential equations
- the buckling load may not be the failure load
- post-buckling analysis may be necessary

Equilibrium equations (Kirchhoff)

Hypotheses:

- the transverse shear strain (γ_{xz}, γ_{yz}) are negligible; the normal lines to the neutral surface remain normal after the deformation
- the normal stress σ_z and the correspondent strain ε_z are negligible; the transverse deflection w is independent of z ($w = w(x,y)$)
- the material is homogeneous, isotropic and obeys Hooke's law

Plate stress resultants

Definitions:

$$N_x(x,y) = \int_{-t/2}^{t/2} \sigma_x(x,y,z)\ dz$$

$$N_y(x,y) = \int_{-t/2}^{t/2} \sigma_y(x,y,z)\ dz$$

$$N_{xy}(x,y) = N_s(x,y) = \int_{-t/2}^{t/2} \tau_s(x,y,z)\ dz$$

$$M_x(x,y) = \int_{-t/2}^{t/2} z\sigma_x(x,y,z)\ dz$$

$$M_y(x,y) = \int_{-t/2}^{t/2} z\sigma_y(x,y,z)\ dz$$

$$M_{xy}(x,y) = M_s(x,y) = \int_{-t/2}^{t/2} z\tau_s(x,y,z)\ dz$$

Plate stress resultants

Positive orientation:

M_x **positive direction corresponds to the direction of the resultant moment of a positive force** $\sigma_x dz$ **for positive** z

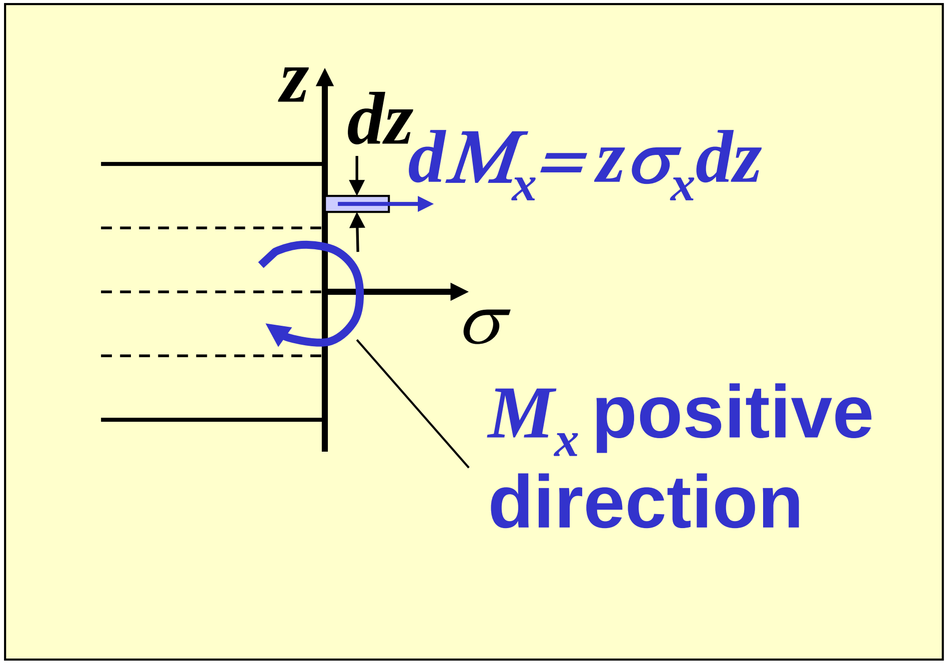


Plate stress resultants

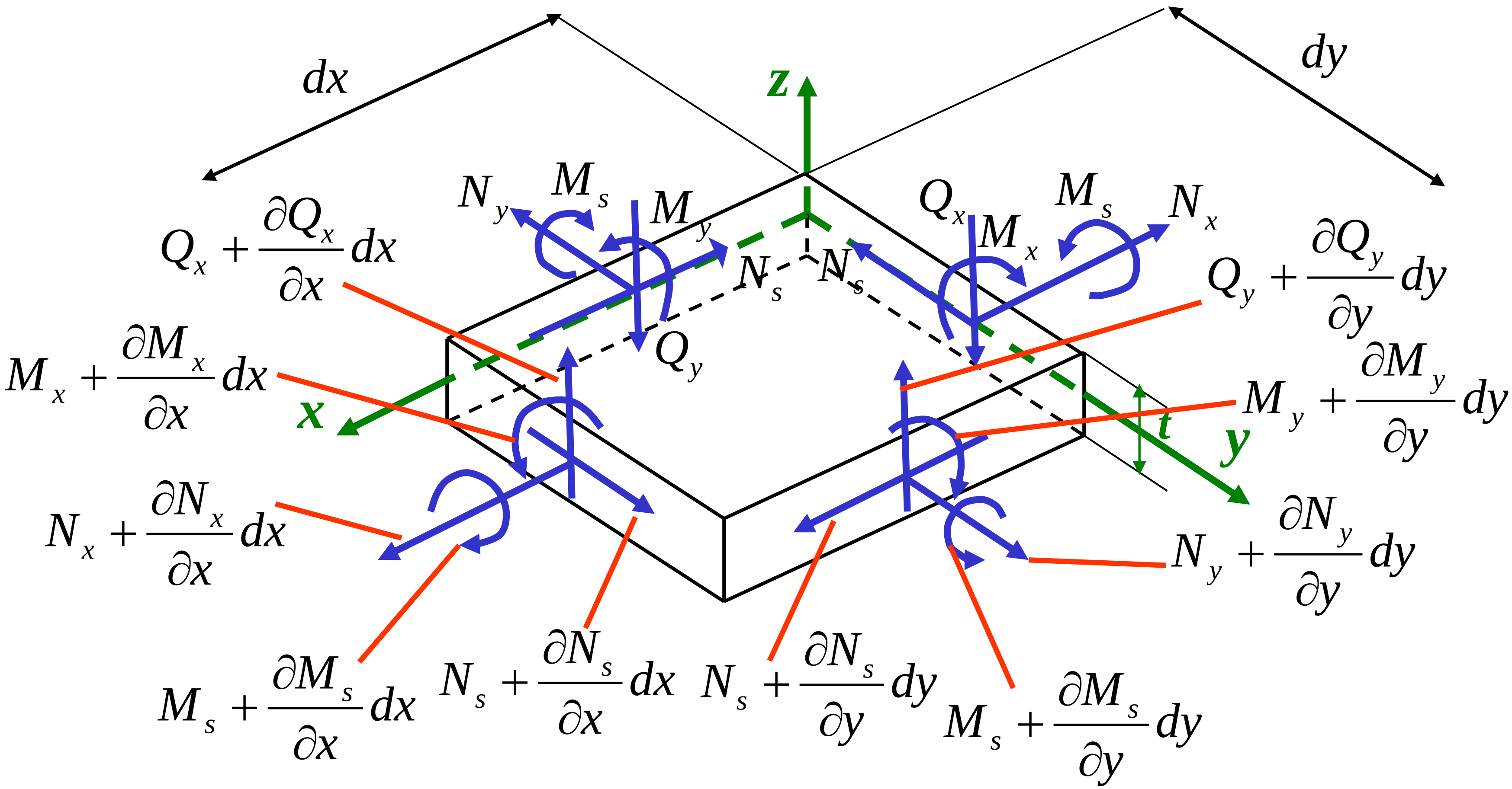
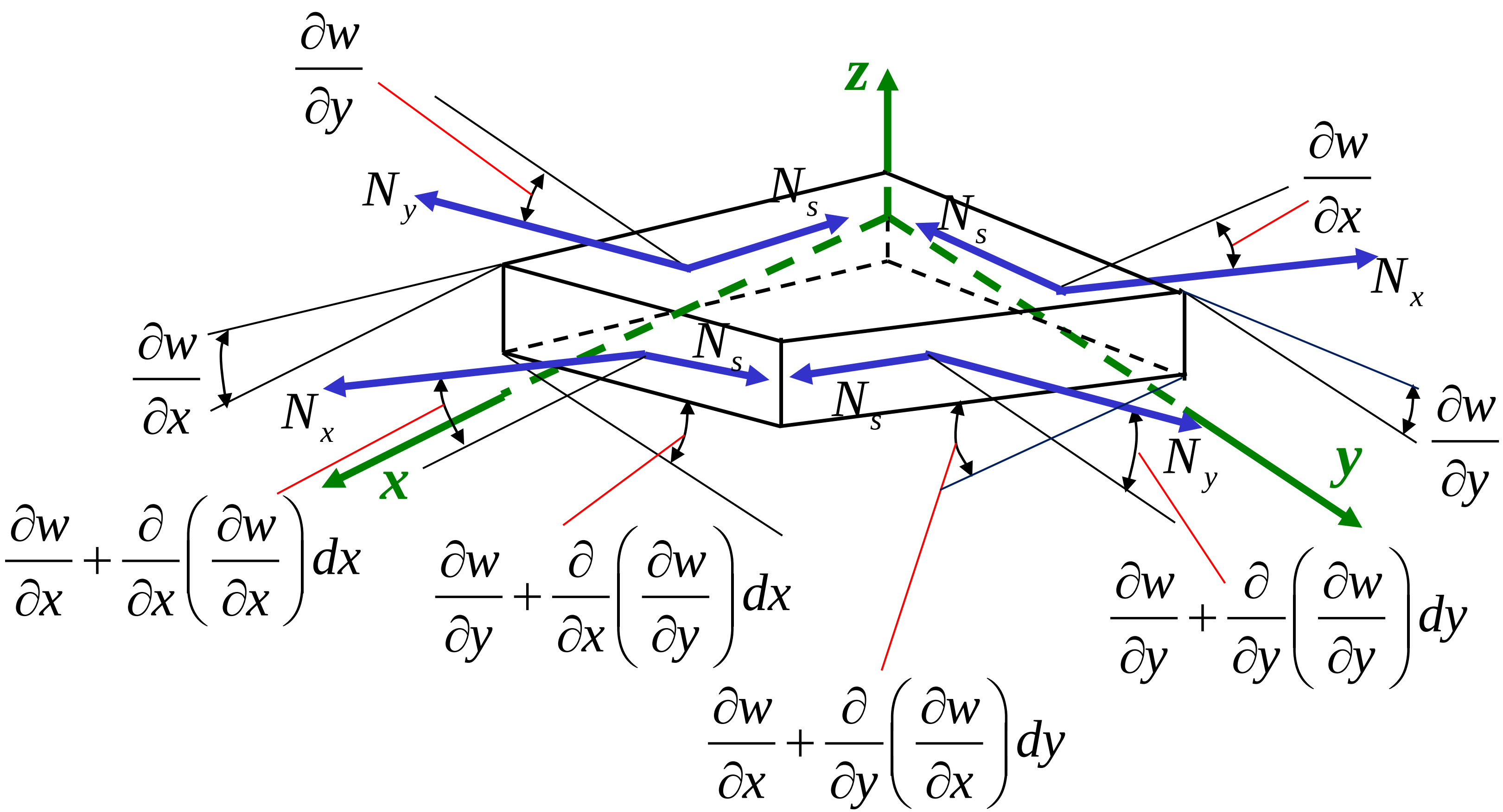


Plate stress resultants



Equilibrium equations

Equilibrium in the z-direction due to the in-plane stress resultants:

$$F_N = N_x\left(\frac{\partial w}{\partial x}+\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)dx-\frac{\partial w}{\partial x}\right)dy+N_y\left(\frac{\partial w}{\partial y}+\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)dy-\frac{\partial w}{\partial y}\right)dx+$$

$$+N_s\left(\frac{\partial w}{\partial y}+\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)dx-\frac{\partial w}{\partial y}\right)dx+N_s\left(\frac{\partial w}{\partial x}+\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)dy-\frac{\partial w}{\partial x}\right)dx$$

$$F_N = N_x\frac{\partial^2 w}{\partial x^2}dxdy+N_y\frac{\partial^2 w}{\partial y^2}dx+N_s\frac{\partial^2 w}{\partial x\partial y}dxdy+N_s\frac{\partial^2 w}{\partial x\partial y}dxdy$$

$$F_N = \left(N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_s\frac{\partial^2 w}{\partial x\partial y}\right)dxdy$$

Equilibrium equations

Equilibrium in the in the z-direction due Q_x and Q_y:

$$F_Q = \left(Q_x + \frac{\partial Q_x}{\partial x} dx - Q_x \right) dy + \left(Q_y + \frac{\partial Q_y}{\partial y} dy - Q_y \right) dx = \left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right) dxdy$$

Equilibrium in the z-direction:

$$F_Q + F_N = 0$$

$$\left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right) dxdy + \left(N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_s \frac{\partial^2 w}{\partial x \partial y} \right) dxdy = 0$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_s \frac{\partial^2 w}{\partial x \partial y} = 0$$

Equilibrium equations

Equilibrium of moments around *y*-axis:

$$\left(M_x + \frac{\partial M_s}{\partial x} dx - M_x \right) dy + \left(M_s + \frac{\partial M_s}{\partial y} dy - Ms \right) dx - Q_x dydx = 0$$

$$\frac{\partial M_x}{\partial x} dxdy + \frac{\partial M_s}{\partial y} dydx - Q_x dydx = 0$$

$$\Rightarrow \quad \frac{\partial M_x}{\partial x} + \frac{\partial M_s}{\partial y} - Q_x = 0$$

Equilibrium equations

Equilibrium of moments around *x*-axis:

$$\left(-M_y - \frac{\partial M_s}{\partial y} dy + M_y\right) dx + \left(-M_s - \frac{\partial M_s}{\partial x} dx + Ms\right) dy + Q_y dxdy = 0$$

$$-\frac{\partial M_x}{\partial x} dydx - \frac{\partial M_s}{\partial y} dxdy + Q_x dxdy = 0$$

$$\Rightarrow \quad \frac{\partial M_y}{\partial y} + \frac{\partial M_s}{\partial x} - Q_y = 0$$

Equilibrium equations

Equilibrium in the z-direction:

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_s\frac{\partial^2 w}{\partial x\partial y}=0$$

$$Q_x=\frac{\partial M_x}{\partial x}+\frac{\partial M_s}{\partial y}\quad\Rightarrow\quad\frac{\partial Q_x}{\partial x}=\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial M_s}{\partial x\partial y}$$

$$Q_y=\frac{\partial M_y}{\partial y}+\frac{\partial M_s}{\partial x}\quad\Rightarrow\quad\frac{\partial Q_y}{\partial y}=\frac{\partial^2 M_y}{\partial y^2}+\frac{\partial M_s}{\partial x\partial y}$$

$$\Rightarrow\quad\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}+2\frac{\partial M_s}{\partial x\partial y}+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_s\frac{\partial^2 w}{\partial x\partial y}=0$$

Stress resultants as functions of displacements

$$M_x = \int_{-t/2}^{t/2} \sigma_x z dz$$

$$M_y = \int_{-t/2}^{t/2} \sigma_y z dz$$

$$M_s = \int_{-t/2}^{t/2} \tau_s z dz$$

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right)$$

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right)$$

$$\tau_s = \frac{E}{2(1+\nu)}\gamma_s$$

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = -z\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_s = -2z\frac{\partial^2 w}{\partial x \partial y}$$

$$\int_{-t/2}^{t/2} z^2 dz = \left.\frac{z^3}{3}\right|_{-t/2}^{t/2} = 2\frac{1}{3}\left(\frac{t^3}{8}\right) = \frac{t^3}{12}$$

$$\int_{-t/2}^{t/2} \frac{E}{1-\nu^2} z^2 dz = \frac{Et^3}{12\left(1-\nu^2\right)} = D$$

Stress resultants as functions of displacements

$$M_x = \int_{-t/2}^{t/2} \frac{E}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) z^2 dz$$

$$M_y = \int_{-t/2}^{t/2} \frac{E}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) z^2 dz$$

$$M_s = \int_{-t/2}^{t/2} \frac{E}{2(1+\nu)}\left(2\frac{\partial^2 w}{\partial x \partial y}\right) z^2 dz$$

$\Rightarrow$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)$$

$$M_s = -2D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

Equilibrium equations

$$\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}+2\frac{\partial M_s}{\partial x\partial y}+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_s\frac{\partial^2 w}{\partial x\partial y}=0$$

$$\frac{\partial^2 M_x}{\partial x^2}=-D\left(\frac{\partial^4 w}{\partial x^4}+\nu\frac{\partial^4 w}{\partial x^2\partial y^2}\right)$$

$$\frac{\partial^2 M_y}{\partial y^2}=-D\left(\frac{\partial^4 w}{\partial y^4}+\nu\frac{\partial^4 w}{\partial x^2\partial y^2}\right)$$

$$\frac{\partial^2 M_y}{\partial x\partial y}=-D(1-\nu)\frac{\partial^4 w}{\partial x^2\partial y^2}$$

Equilibrium equations

$$-D\left(\frac{\partial^4 w}{\partial x^4}+\nu\frac{\partial^4 w}{\partial x^2\partial y^2}\right)-D\left(\frac{\partial^4 w}{\partial y^4}+\nu\frac{\partial^4 w}{\partial x^2\partial y^2}\right)-2D(1-\nu)\frac{\partial^4 w}{\partial x^2\partial y^2}$$

$$+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_s\frac{\partial^2 w}{\partial x\partial y}=0$$

$$-D\left(\frac{\partial^4 w}{\partial x^4}+\nu\frac{\partial^4 w}{\partial x^2\partial y^2}\right)-D\left(\frac{\partial^4 w}{\partial y^4}+\nu\frac{\partial^4 w}{\partial x^2\partial y^2}\right)-2D(1-\nu)\frac{\partial^4 w}{\partial x^2\partial y^2}$$

$$+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_s\frac{\partial^2 w}{\partial x\partial y}=0$$

$$-D\left(\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w}{\partial y^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}\right)-D\nu\left(\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial x^2\partial y^2}-2\frac{\partial^4 w}{\partial x^2\partial y^2}\right)+$$

$$+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_s\frac{\partial^2 w}{\partial x\partial y}=0$$

Equilibrium equations

$$-D\left(\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w}{\partial y^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}\right)+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_s\frac{\partial^2 w}{\partial x \partial y}=0$$

$$\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w}{\partial y^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}-\frac{1}{D}\left(N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_s\frac{\partial^2 w}{\partial x \partial y}\right)=0$$

Boundary conditions at $x = x_0$

Simply supported

$$w(x_0, y) = 0 \; ; \qquad \frac{\partial^2 w(x_0, y)}{\partial x^2} + \nu \frac{\partial^2 w(x_0, y)}{\partial y^2} = 0$$

Clamped

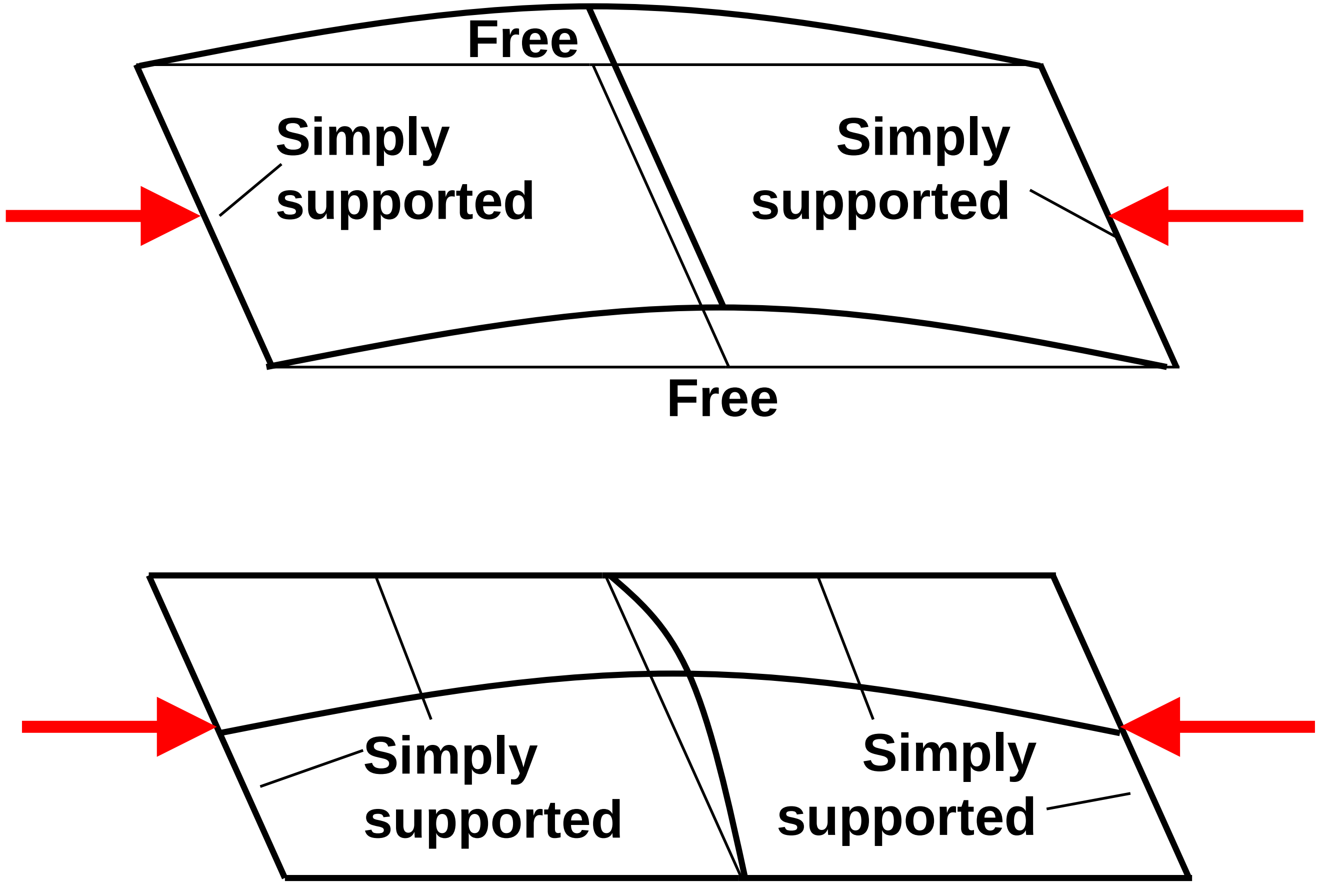
$$w(x_0, y) = 0 \; ; \qquad \frac{\partial w(x_0, y)}{\partial x} = 0$$

Free

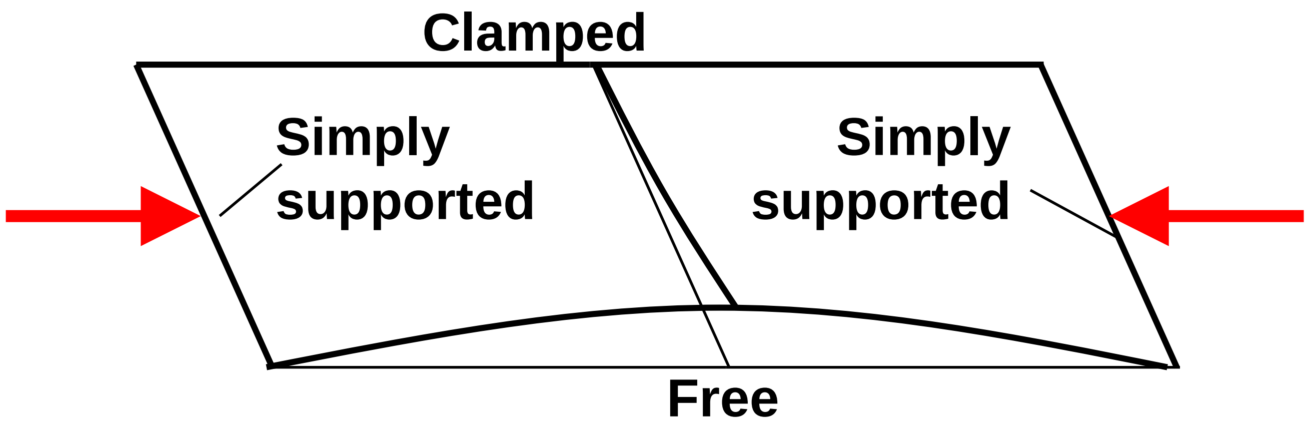
$$\frac{\partial^2 w(x_0, y)}{\partial x^2} + \nu \frac{\partial^2 w(x_0, y)}{\partial y^2} = 0 \; ;$$

$$\frac{\partial^3 w(x_0, y)}{\partial x^3} + 2(1-\nu)\frac{\partial^3 w(x_0, y)}{\partial x \partial y^2} + \frac{N_y}{D}\frac{\partial w(x_0, y)}{\partial y} + \frac{N_s}{D}\frac{\partial w(x_0, y)}{\partial x} = 0$$

Boundary conditions

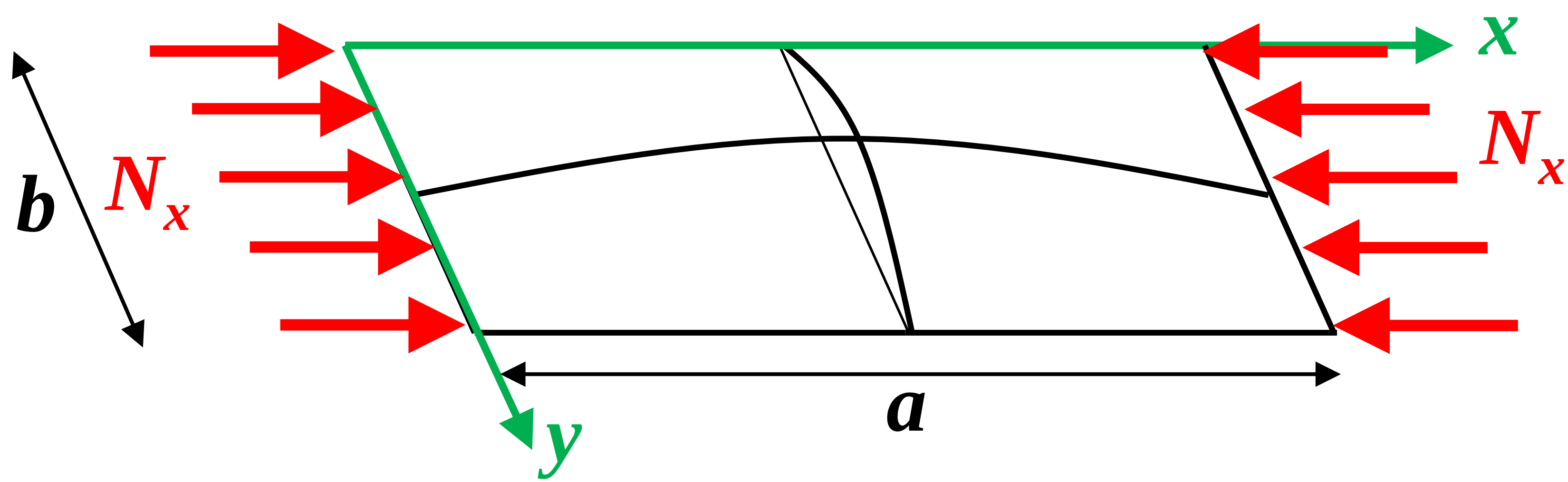


Boundary conditions



Simply supported plate under uniaxial load

Simply supported plate under uniaxial load



Boundary conditions

$$w = 0$$

$$\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} = 0$$

at $y = 0$ and $y = b$

$$w = 0$$

$$\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} = 0$$

at $x = 0$ and $x = a$

Simply supported plate under uniaxial load

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2} = 0$$

$N_x < 0$!

Solution:

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \operatorname{sen}\left(\frac{m\pi x}{a}\right)\operatorname{sen}\left(\frac{n\pi y}{b}\right) \text{ for } m,\ n = 1,\ 2,\ 3\ldots$$

$$\frac{\partial^4 w}{\partial x^4} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^4\pi^4}{a^4} A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\frac{\partial^4 w}{\partial y^4} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{n^4\pi^4}{b^4} A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\frac{\partial^4 w}{\partial x^2 \partial y^2} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2\pi^2}{a^2}\frac{n^2\pi^2}{b^2} A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

Simply supported plate under uniaxial load

$$\frac{\partial^2 w}{\partial x^2} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2\pi^2}{a^2}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

Therefore:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4 - \frac{N_x}{D}\left(\frac{m\pi}{a}\right)^2\right]$$
$$\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) = 0$$

$$\Rightarrow \quad A_{mn}\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 - \frac{N_x}{D}\frac{m^2\pi^2}{a^2}\right] = 0$$

Simply supported plate under uniaxial load

$$A_{mn} \neq 0 \quad \Rightarrow \quad \frac{N_x}{D}\frac{m^2\pi^2}{a^2} = \pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2$$

$$N_x = \frac{\pi^2 D}{b^2}\frac{a^2 b^2}{m^2}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2$$

$$N_x = \frac{\pi^2 D}{b^2}\left(\frac{mb}{a}+\frac{n^2 a}{mb}\right)^2$$

$N_{x,crit}$ is the minimum value of N_x $\Rightarrow$ $n = 1$

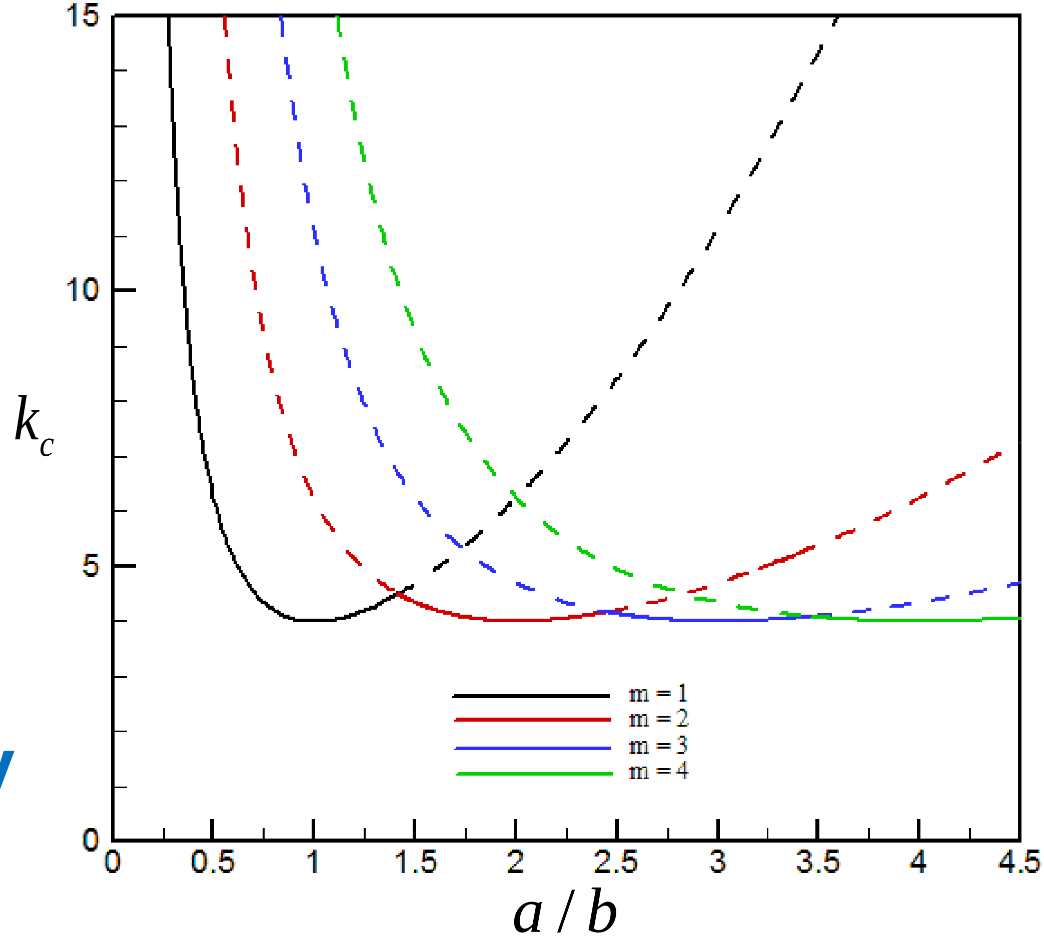
Simply supported plate under uniaxial load

$$N_{x,crit} = \frac{k_c \pi^2 D}{b^2}$$

where n = 1 and:

$$k_c = \left(\frac{mb}{a} + \frac{n^2 a}{mb} \right)^2$$

Similar plots are available for different boundary conditions and loading

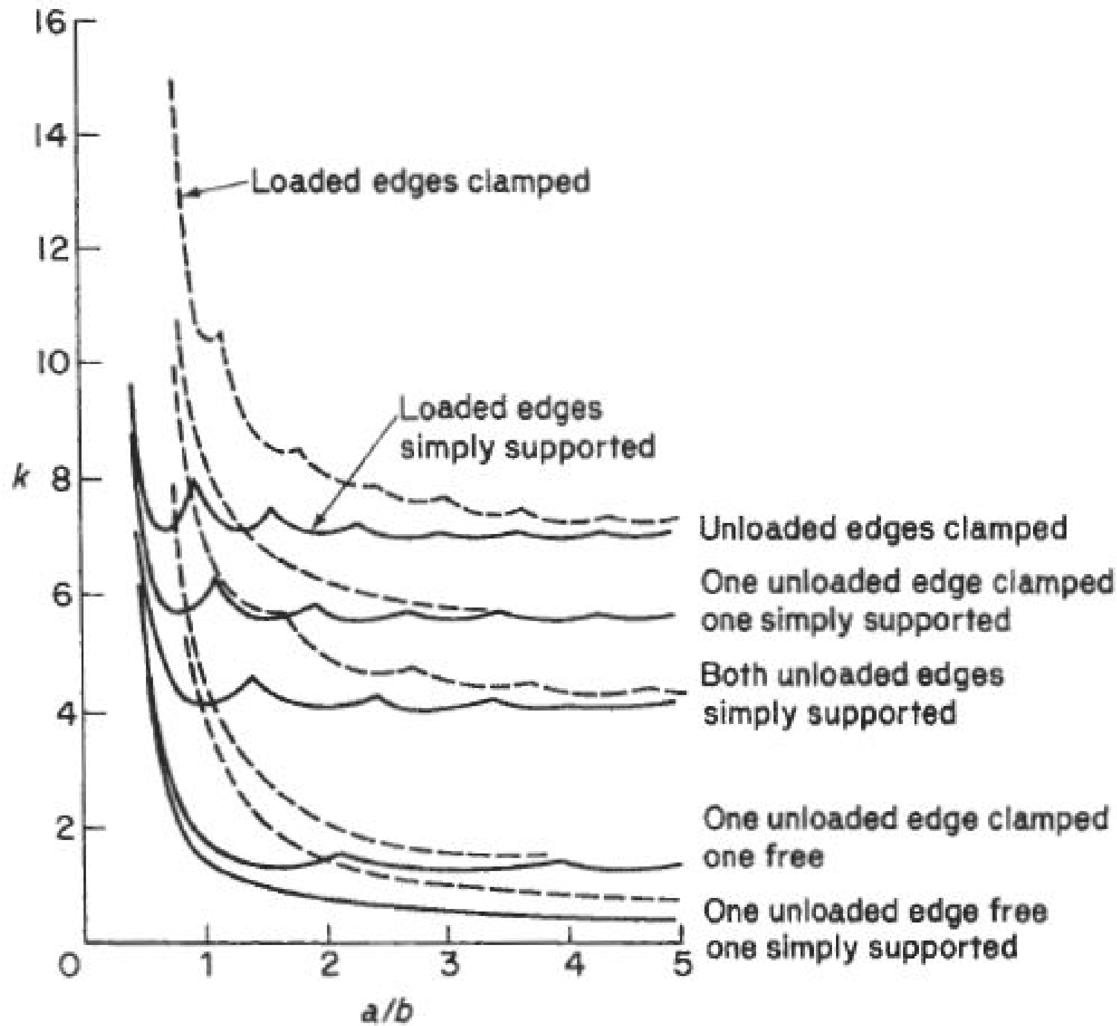


Simply supported plate under uniaxial load

- For very long plates the m approaches ∞
- The tendency is that the plate buckles in a mode where the buckling length is equal to the plate width
- Therefore, the plate would buckle in a series of square plates
- Notice from the previous plot that for a square plate ($a / b = 1$) the buckling load is a local minimum.

Effect of boundary conditions

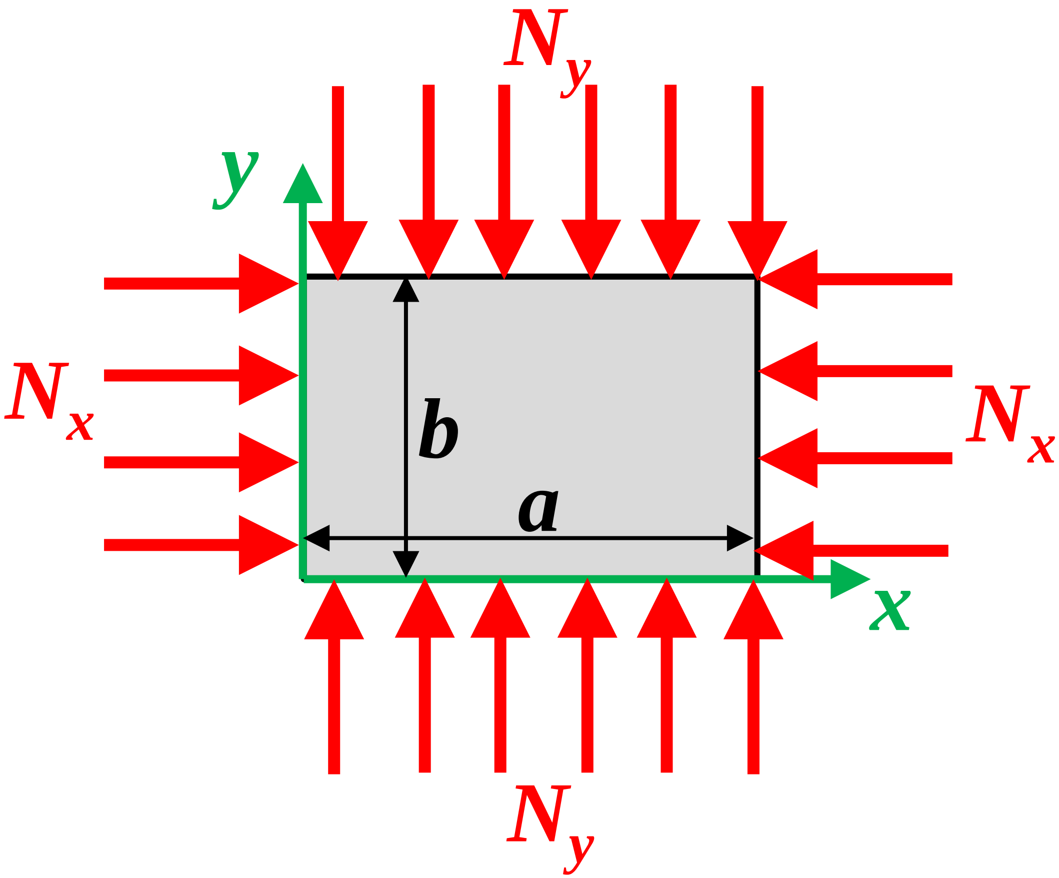
Simply supported plate under uniaxial load



boundary conditions strongly affect the buckling load but they are hard to model

Simply supported plate under biaxial load

Simply supported plate under biaxial load



$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2} + \frac{N_y}{D}\frac{\partial^2 w}{\partial y^2} = 0$$

Simply supported plate under biaxial load

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2} + \frac{N_y}{D}\frac{\partial^2 w}{\partial y^2} = 0$$

Solution:

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \text{ for } m,\ n = 1,\ 2,\ 3\ldots$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4 - \frac{N_x}{D}\left(\frac{m\pi}{a}\right)^2 - \frac{N_y}{D}\left(\frac{n\pi}{b}\right)^2\right] \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) = 0$$

Simply supported plate under biaxial load

$$A_{mn} \neq 0 \quad \Rightarrow \quad \frac{\pi^2 N_x}{D}\left(\frac{m^2}{a^2}+\frac{N_y n^2}{N_x b^2}\right)=\left(\frac{m\pi}{a}\right)^4+2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2+\left(\frac{n\pi}{b}\right)^4$$

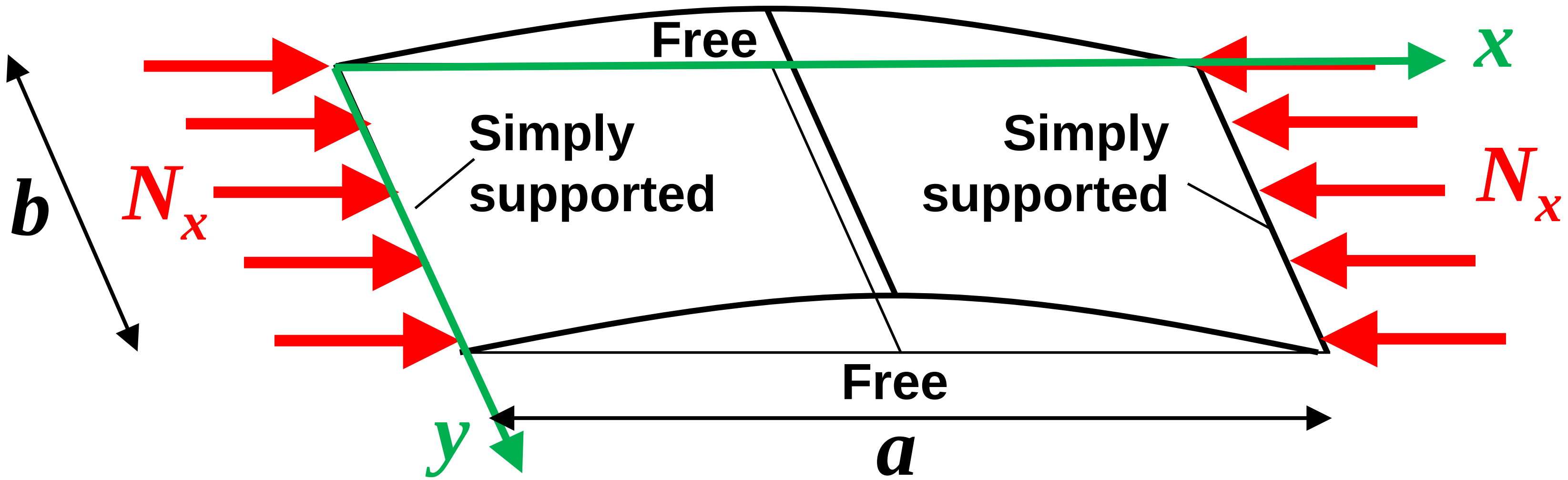
For a given relation N_y / N_x, the minimum value of N_x that satisfies the above equation is $N_{x,crit}$

From this value, $N_{y,crit}$ can be computed

The values of m and n that minimizes N_x define the buckling mode

Simply supported loaded edges

Simply supported loaded edges plate



Equilibrium equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2} = 0$$

Boundary conditions

$$w = \frac{\partial^2 w}{\partial x^2} = 0$$

for $x = 0\,,\,a$

Simply supported loaded edges plate

Solution:

$$w(x,y) = f(y)\,\mathrm{sen}\left(\frac{m\pi x}{a}\right)$$

Substituting in equilibrium equation:

$$\left[\left(\frac{m\pi}{a}\right)^4 f(y) - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2 f(y)}{dy^2} + \frac{d^4 f(y)}{dy^4} - \frac{N_x}{D}\left(\frac{m\pi}{a}\right)^2 f(y)\right]\sin\left(\frac{m\pi x}{a}\right) = 0$$

$$\frac{d^4 f(y)}{dy^4} - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2 f(y)}{dy^2} + \left[\left(\frac{m\pi}{a}\right)^4 - \frac{N_x}{D}\left(\frac{m\pi}{a}\right)^2\right] f(y) = 0$$

Simply supported loaded edges plate

Solution for $f\,(y)$:

$$f\,(y) = C_1 \cosh\left(\frac{\alpha y}{b}\right) + C_2 \sinh\left(\frac{\alpha y}{b}\right) + C_3 \cos\left(\frac{\beta y}{b}\right) + C_4 \sin\left(\frac{\beta y}{b}\right)$$

where:

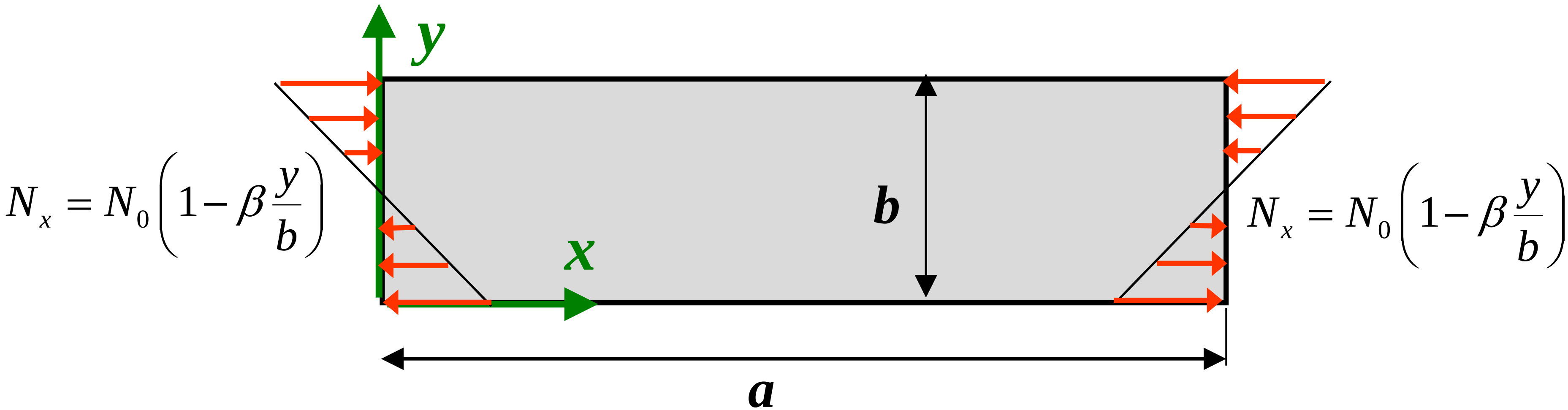
$$\alpha = \pi\left(\frac{mb}{a}\right)^{1/2}\left[\frac{mb}{a} + k_c^{1/2}\right]^{1/2}$$

$$\beta = \pi\left(\frac{mb}{a}\right)^{1/2}\left[-\frac{mb}{a} + k_c^{1/2}\right]^{1/2}$$

$$k_c = \frac{12\left(1-\nu_e^{\,2}\right)\sigma_{cr}}{\pi^2 E}\left(\frac{b}{t}\right)^2$$

Plate under in-plane bending

Plate under in-plane bending



Solution:

$$w(x,y) = \sin\left(\frac{m\pi x}{a}\right)\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{n\pi y}{b}\right)$$

Bending strain energy:

$$U = \frac{1}{2}\int_V \left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_s\gamma_s\right)$$

Plate under in-plane bending

Stress-strain relations:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_s \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{Bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_s \end{Bmatrix}$$

$$\varepsilon_x = \frac{\sigma_x - \nu\sigma_y}{E}$$

$$\varepsilon_y = \frac{\sigma_y - \nu\sigma_x}{E}$$

$$\gamma_s = \frac{\tau_s}{G}$$

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right)$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(\nu\varepsilon_x + \varepsilon_y\right)$$

$$\tau_s = G\gamma_s$$

Plate under in-plane bending

$$\varepsilon_x = u_{,x} - zw_{,xx}$$
$$\varepsilon_y = v_{,y} - zw_{,yy}$$
$$\gamma_s = u_{,y} + v_{,x} - 2zw_{,xy}$$

For pure bending:

$$\varepsilon_x = -zw_{,xx}$$
$$\varepsilon_y = -zw_{,yy}$$
$$\gamma_s = -2zw_{,xy}$$

Therefore:

$$\sigma_x = -\frac{Ez}{1-\nu^2}\left(w_{,xx} + \nu w_{,yy}\right)$$
$$\sigma_y = -\frac{E}{1-\nu^2}\left(w_{,yy} + \nu w_{,xx}\right)$$
$$\tau_s = -\frac{Ez}{2(1+\nu)}2w_{,xy} = -\frac{Ez}{(1+\nu)}w_{,xy}$$

Bending stresses

Plate under in-plane bending

$$U = \frac{1}{2}\int_V \left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_s\gamma_s\right)\, dV$$

$$U = \frac{1}{2}\int_V \left[\sigma_x\left(\frac{\sigma_x - \nu\sigma_y}{E}\right) + \sigma_y\left(\frac{\sigma_x - \nu\sigma_y}{E}\right) + \tau_s\frac{\tau_s}{G}\right] dV$$

$$U = \frac{1}{2}\int_V \left(\frac{\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y}{E} + 2(1+\nu)\frac{\tau_s^2}{E}\right) dV$$

Therefore:

$$U = \frac{1}{2E}\int_V \left(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y + 2(1+\nu)\tau_s^2\right)\, dV$$

Plate under in-plane bending

Substituting the bending stresses:

$$U=\frac{1}{2E}\frac{E^2}{\left(1-\nu^2\right)^2}\int_V\Big[z^2\left(w_{,xx}+\nu w_{,yy}\right)^2+z^2\left(w_{,yy}+\nu w_{,xx}\right)^2-$$

$$\left.-2z^2\nu\left(w_{,xx}+\nu w_{,yy}\right)\left(w_{,yy}+\nu w_{,xx}\right)+z^2 2\left(1+\nu\right)\left(1-\nu^2\right)^2\frac{1}{\left(1+\nu\right)^2}w_{,xy}^2\right]dzdA$$

$$\frac{1}{2E}\frac{E^2}{\left(1-\nu^2\right)^2}\int_{-t/2}^{t/2}z^2dz=\frac{1}{2}\frac{E}{\left(1-\nu^2\right)^2}\left.\frac{z^3}{3}\right|_{-t/2}^{t/2}=\frac{D}{2\left(1-\nu^2\right)}$$

where: $$D=\frac{Et^3}{12\left(1-\nu^2\right)}$$

Plate under in-plane bending

Therefore:

$$U=\frac{D}{2\left(1-\nu^2\right)}\int_A\left[\left(w_{,xx}+\nu w_{,yy}\right)^2+\left(w_{,yy}+\nu w_{,xx}\right)^2-\right.$$
$$\left.-2\nu\left(w_{,xx}+\nu w_{,yy}\right)\left(w_{,yy}+\nu w_{,xx}\right)+2\frac{\left(1-\nu^2\right)^2}{\left(1+\nu\right)}w_{,xy}^2\right]dA$$

$$U=\frac{D}{2\left(1-\nu^2\right)}\int_A\left[\left(1+\nu^2\right)\left(w_{,xx}^2+w_{,yy}^2\right)+4\nu w_{,xx}w_{,yy}-2\nu\left(1+\nu^2\right)w_{,xx}w_{,yy}-\right.$$
$$\left.-2\nu^2\left(w_{,xx}^2+w_{,yy}^2\right)+2\frac{\left(1-\nu^2\right)^2}{\left(1+\nu\right)}w_{,xy}^2\right]dA$$

$$U=\frac{D}{2\left(1-\nu^2\right)}\int_A\left[\left(1-\nu^2\right)\left(w_{,xx}^2+w_{,yy}^2\right)+2\nu w_{,xx}w_{,yy}-2\nu^3 w_{,xx}w_{,yy}+2\frac{\left(1-\nu^2\right)^2}{\left(1+\nu\right)}w_{,xy}^2\ dA\right.$$

Plate under in-plane bending

Simplifying:

$$U=\frac{D}{2\left(1-\nu^2\right)}\int_A\left[\left(1-\nu^2\right)\left(w_{,xx}^2+w_{,yy}^2\right)+2\nu\left(1-\nu^2\right)w_{,xx}w_{,yy}+2\frac{\left(1-\nu^2\right)^2}{\left(1+\nu\right)}w_{,xy}^2\right]dA$$

$$U=\frac{D}{2\left(1-\nu^2\right)}\int_A\left[\left(1-\nu^2\right)\left(w_{,xx}^2+w_{,yy}^2+2\nu w_{,xx}w_{,yy}+2\frac{\left(1-\nu^2\right)}{\left(1+\nu\right)}w_{,xy}^2\right)\right]dA$$

Finally:

$$U=\frac{D}{2}\int_A\left(w_{,xx}^2+w_{,yy}^2+2\nu w_{,xx}w_{,yy}+2\left(1-\nu\right)w_{,xy}^2\right)\,dA$$

Plate under in-plane bending

From the solution

$$w_{,xx}^2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^4\pi^4}{a^4}A_{mn}^2\sin^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)$$

$$w_{,yy}^2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{n^4\pi^4}{b^4}A_{mn}^2\sin^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)$$

$$w_{xx}w_{yy} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2\pi^2}{a^2}\frac{n^2\pi^2}{b^2}A_{mn}^2\sin^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)$$

$$w_{,xy}^2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2\pi^2}{a^2}\frac{n^2\pi^2}{b^2}A_{mn}^2\cos^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)$$

Plate under in-plane bending

Integrations:

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$

$$\int_0^a \sin^2\left(m\pi\frac{x}{a}\right)dx = \frac{1}{2}\int_0^a \left(1-\cos\left(2m\pi\frac{x}{a}\right)\right)dx$$

$$\int_0^a \sin^2\left(m\pi\frac{x}{a}\right)dx = \frac{a}{2}$$

$$\int_0^a\int_0^b w_{,xx}^2\,dxdy = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^4\pi^4}{a^4}A_{mn}^2\int_0^a\int_0^b \sin^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)dxdy =$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^4\pi^4}{a^4}\frac{ab}{4}A_{mn}^2$$

Plate under in-plane bending

Similarly:

$$\int_0^a\int_0^b w_{,xx}^2\,dxdy=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^4\pi^4}{a^4}\frac{ab}{4}A_{mn}^2$$

$$\int_0^a\int_0^b w_{,yy}^2\,dxdy=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{n^4\pi^4}{b^4}\frac{ab}{4}A_{mn}^2$$

$$\int_0^a\int_0^b w_{,xx}w_{yy}\,dxdy=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n^2\pi^4}{a^2b^2}\frac{ab}{4}A_{mn}^2$$

Plate under in-plane bending

Also:

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$

$$\Rightarrow \int_0^a \cos^2\left(m\pi\frac{x}{a}\right)dx = \frac{1}{2}\int_0^a \left(1-\cos\left(2m\pi\frac{x}{a}\right)\right)dx$$

$$\Rightarrow \int_0^a \cos^2\left(m\pi\frac{x}{a}\right)dx = \frac{a}{2}$$

$$\int_0^a\int_0^b w_{,xy}^2\,dxdy = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n^2\pi^4}{a^2b^2}A_{mn}^2\int_0^a\int_0^b \sin^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)dxdy =$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n^2\pi^4}{a^2b^2}\frac{ab}{4}A_{mn}^2$$

Plate under in-plane bending

$$U = \frac{D}{2}\int_A \left(w_{,xx}^2 + w_{,yy}^2 + 2\nu w_{,xx} w_{,yy} + 2(1-\nu) w_{,xy}^2 \right) dA =$$

$$= \frac{D}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^4}{a^4} + \frac{n^4}{b^4} + 2\nu\frac{m^2n^2}{a^2b^2} + 2(1-\nu)\frac{m^2n^2}{a^2b^2} \right)\frac{ab}{4}\pi^4 A_{mn}^2 =$$

$$= \frac{D}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^4}{a^4} + \frac{n^4}{b^4} + 2\frac{m^2n^2}{a^2b^2} \right)\frac{ab}{4}\pi^4 A_{mn}^2 =$$

$$= \frac{D}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \frac{ab}{4}\pi^4 A_{mn}^2$$

$$U = \frac{\pi^4}{8} abD \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 A_{mn}^2$$

Plate under in-plane bending

Potential energy of the in-plane resultant stresses

$$V = \int_A (N_x dy)\varepsilon_x^{NL} dx + \int_A (N_y dx)\varepsilon_y^{NL} dy + \int_A (N_s dx)\varepsilon_s^{NL} dy + \int_A (N_s dy)\varepsilon_s^{NL} dx$$

Non-linear strains:

$$\varepsilon_x^{NL} = -\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \qquad \varepsilon_y^{NL} = -\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad \varepsilon_s^{NL} = -\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

Resultant stresses:

$$N_x = N_0\left(1 - \beta\frac{y}{b}\right) \qquad N_y = N_s = 0$$

Plate under in-plane bending

Potential energy of the in-plane resultant stresses

$$V=\frac{N_0}{2}\int_A\left(1-\beta\frac{y}{b}\right)w_{,x}^2dxdy$$

but:

$$w_{,x}^2=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2\pi^2}{a^2}A_{mn}^2\sin^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)$$

Integrating:

$$V=\frac{N_0}{4}\left[\frac{ab}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^2\pi^2}{a^2}A_{mn}^2\right)-\frac{\beta a}{b}\sum_{m=1}^{\infty}\frac{m^2\pi^2}{a^2}\left(\frac{b}{4}\sum_{n=1}^{\infty}A_{mn}^2-\frac{8b^2}{n^2}\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\frac{A_{mn}A_{mn}ni}{\left(n^2-i^2\right)^2}\right)\right]$$

Plate under in-plane bending

Computing the equality: $U = V$

$$N_0 = \frac{\pi^4 D \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 A_{mn}^2}{\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^2\pi^2}{a^2}A_{mn}^2\right)-\frac{\beta}{2}\sum_{m=1}^{\infty}\frac{m^2\pi^2}{a^2}\left(\sum_{n=1}^{\infty}A_{mn}^2-\frac{32}{\pi^2}\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\frac{A_{mn}A_{mn}ni}{\left(n^2-i^2\right)^2}\right)\right]}$$

The coefficients A_{mn} are such that they minimize N_0

Plate under in-plane bending

Differentiating N_0 with respect to A_{mn} and equating to zero:

$$N_0\left[\left(\frac{m^2\pi^2}{a^2}A_{mn}\right)-\frac{\beta}{2}\frac{m^2\pi^2}{a^2}\left(A_{mn}-\frac{16}{\pi^2}\sum_{i=1}^{\infty}\frac{A_{mi}ni}{\left(n^2-i^2\right)}\right)\right]=$$

$$=\pi^4 D\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 A_{mn}^2$$

Plate under in-plane bending

From the previous equation:

$$A_{1n}\left[\left(1+\frac{n^2a^2}{b^2}\right)-N_{0,crit}\,\frac{a^2}{\pi^2 b}\left(1-\frac{\beta}{2}\right)\right]=8\beta N_{0,crit}\,\frac{a^2}{\pi^4 D}\sum_{i=1}^{\infty}\frac{A_{1i}\,ni}{\left(n^2-i^2\right)}$$

The above set of algebraic set of homogeneous equations may have a non-trivial solution yielding the buckling mode and the critical load

Plate under bending

$$\sigma_{cr} = \frac{k_b \pi^2 E}{12\left(1-\nu^2\right)}\left(\frac{t}{b}\right)^2$$

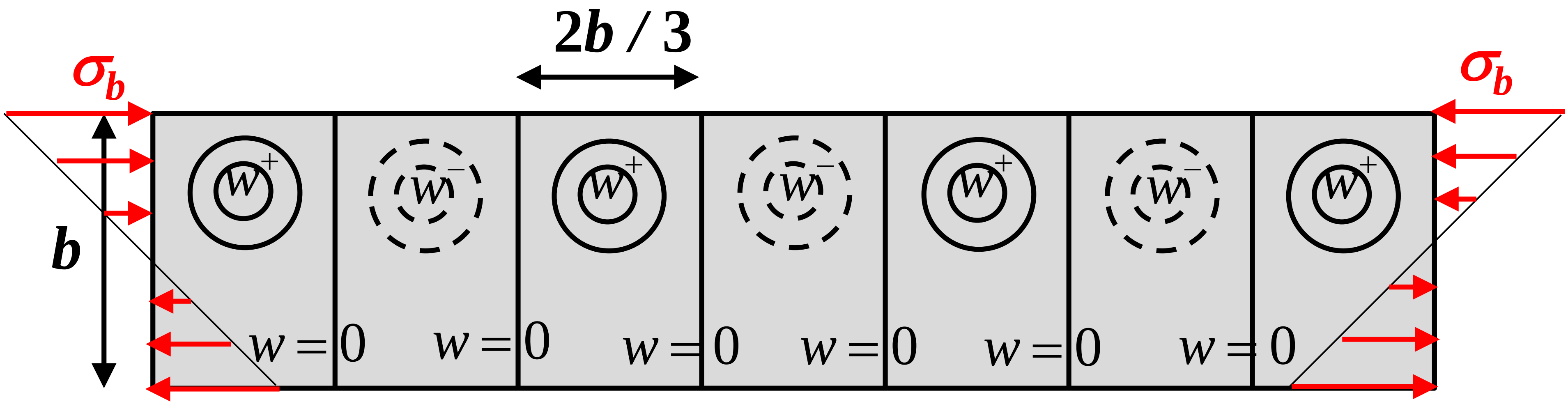
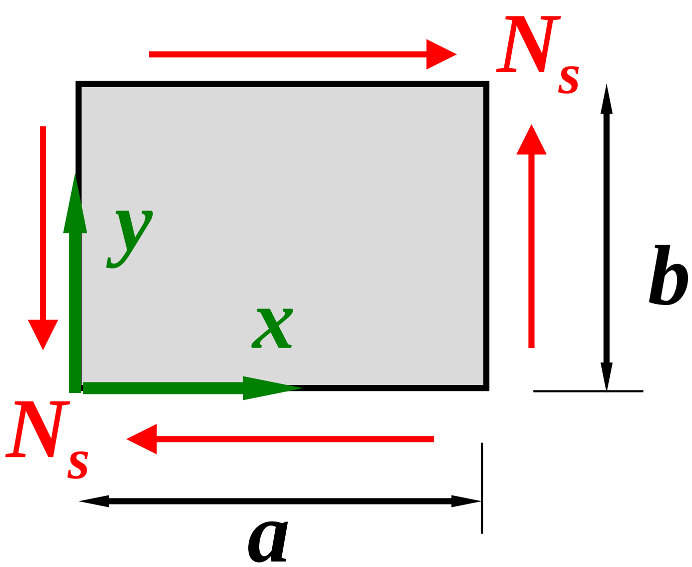


Plate under shear

Plate under shear



$$\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}+\frac{2N_s}{D}\frac{\partial^2 w}{\partial x\partial y}=0$$

$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$k_s=\frac{N_{xy}b^2}{D\pi^2}$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)+$$
$$\frac{N_s}{D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi x}{b}\right)=e(x,y)$$

Plate under shear

$$\int_0^a\int_0^b e(x,y)\sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right)dydx = 0$$

$$A_{mn}\left[m^2+n^2\left(\frac{a}{b}\right)^2\right]^2+\frac{32k_s}{\pi^2}\left(\frac{a}{b}\right)^3\sum_p^\infty\sum_q^\infty A_{pq}\frac{mnpq}{\left(p^2-m^2\right)\left(p^2-m^2\right)}=0$$

$$m,n = all$$

$$m+p = odd$$

$$n+q = odd$$

Plate under shear

Symmetric mode: A_{11} and A_{22} for $a = b$

$$\begin{vmatrix} \left[1+\left(\frac{a}{b}\right)^2\right]^2 & \frac{32k_s}{\pi^2}\left(\frac{a}{b}\right)^3\frac{4}{9} \\ \frac{32k_s}{\pi^2}\left(\frac{a}{b}\right)^3\frac{4}{9} & \left[4+4\left(\frac{a}{b}\right)^2\right]^2 \end{vmatrix} = \begin{vmatrix} 4 & \frac{64k_s}{9\pi^2} \\ \frac{64k_s}{9\pi^2} & 64 \end{vmatrix} = 0$$

$$k_s = \frac{(16)\left(9\pi^2\right)}{128} = 11.1$$

Plate under shear

Anti-symmetric mode: A_{11}, A_{13} and A_{22}

$$\begin{vmatrix} (1+1)^2 & 0 & \dfrac{128k_s}{9\pi^2} \\ 0 & (9+1)^2 & \dfrac{-384k_s}{15\pi^2} \\ \dfrac{128k_s}{9\pi^2} & \dfrac{-384k_s}{15\pi^2} & (4+4)^2 \end{vmatrix} = 0$$

$$k_s = \frac{\pi^2}{64} \frac{2\times 10\times 8}{\sqrt{\left(\dfrac{20}{9}\right)^2 + \left(\dfrac{12}{15}\right)^2}} = 10.45$$

Plate under shear

$$\tau_{cr} = \frac{\pi^2 k_s E}{12\left(1-\nu_e^2\right)}\left(\frac{t}{b}\right)^2$$

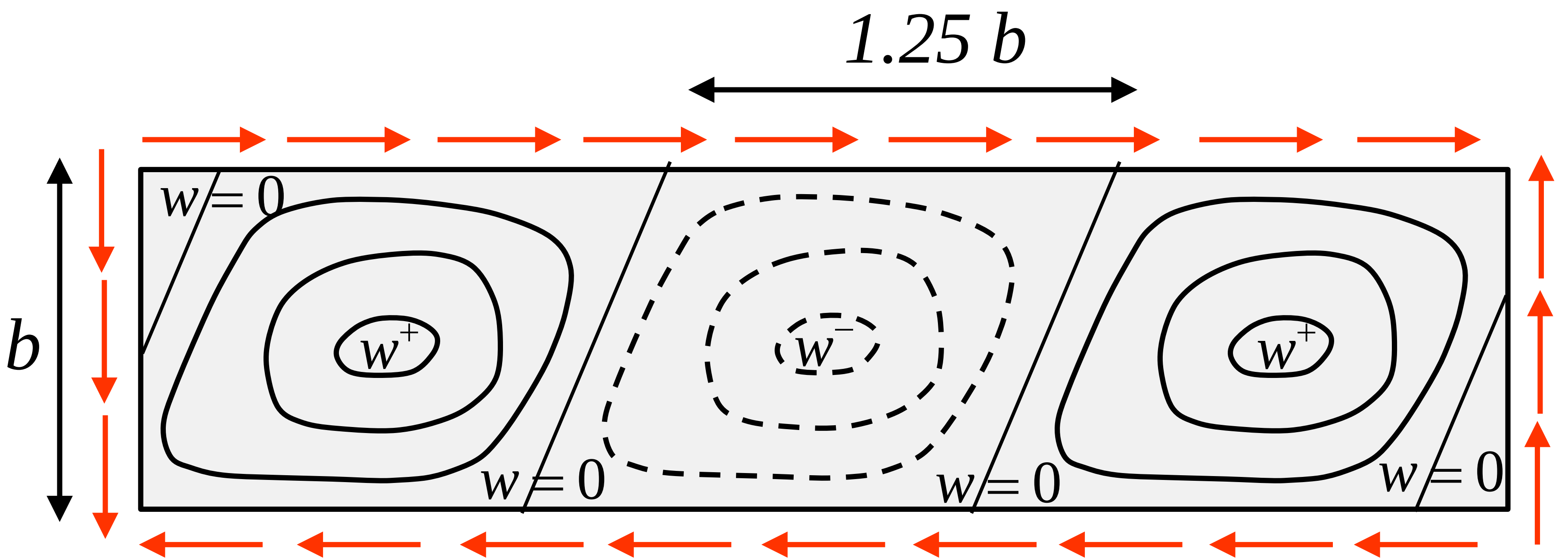


Plate under shear

Comments:

- For a square plate the buckling mode is symmetric $k_{s,crit} = 9.35$ (value obtained using 10 first terms)
- For the same case, $k_{s,crit} = 11.63$ for the first anti-symmetric mode
- For a > b:

$1 \le a/b < 2$ symmetric buckling mode

$2 \le a/b < 3.5$ anti-symmetric buckling mode

$3.5 \le a/b < \ldots$ symmetric buckling mode

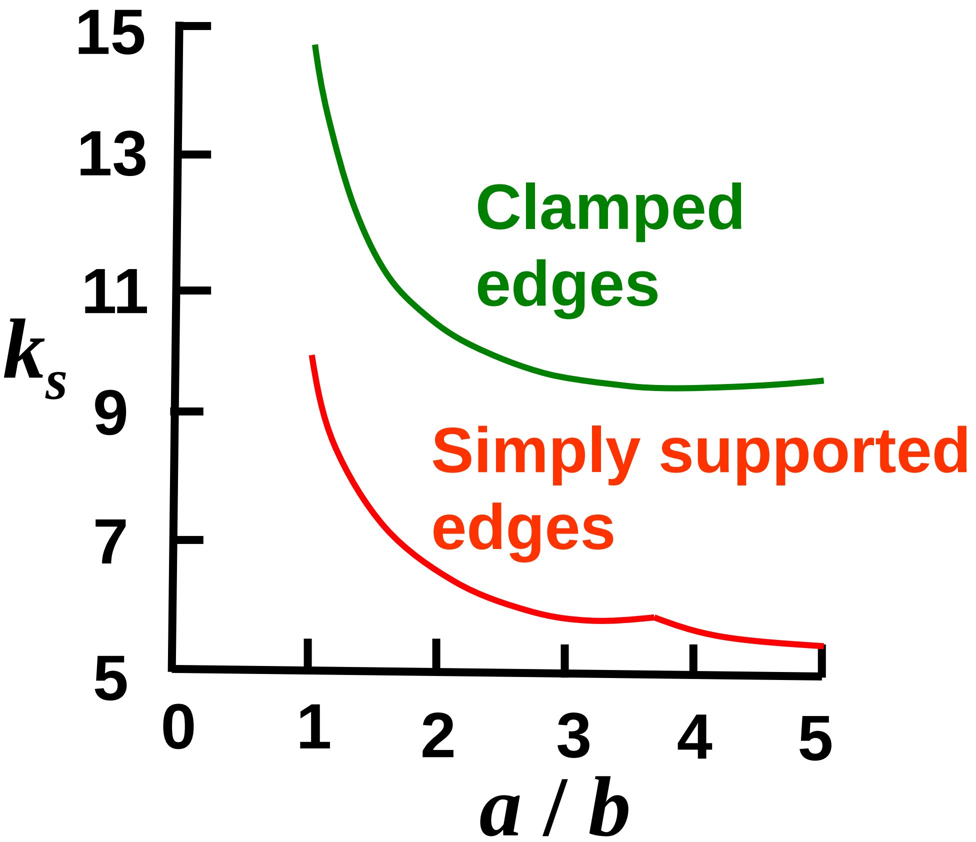
Plate under shear

Comments:

- As $a\,/\,b$ increases, it is difficult to distinguish between symmetric and anti-symmetric bucking modes
- For infinite plates the buckling coefficient does not depend on the symmetry conditions

Plate under shear

Effect of boundary conditions

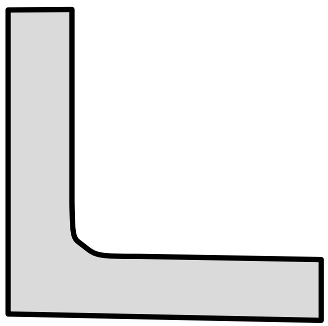


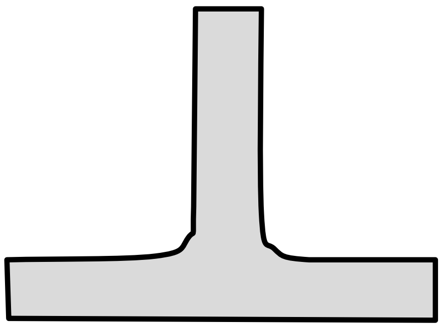
Local buckling

Local buckling

Complex shapes are more efficient than plates for buckling

Examples:





Angle **T-section**

These shapes buckle under different loading and boundary conditions

Local buckling

The unsupported shapes buckling are governed by beam buckling (typically, Euler beam)

$$\sigma_{cr} = \frac{k_w \pi^2 E}{12\left(1-\nu_e^2\right)}\left(\frac{t_w}{b_w}\right)^2$$

where:

k_w is a constant that depends on loading and boundary conditions

E modulus of elasticity

ν_e Poisson ratio

t_w thickness

b_w width

Crippling

Crippling

Crippling is characterized by a local distortion of the cross sectional shape

The initial distortion usually occurs at a load appreciably lower than the failing load with the more stable portions of the cross section continuing to take additional load while supporting the already buckled portions until complete collapse occurs

No satisfactory theory exists for the prediction of the average stress failure

Crippling

Therefore, design for crippling relies on test results or empirical methods

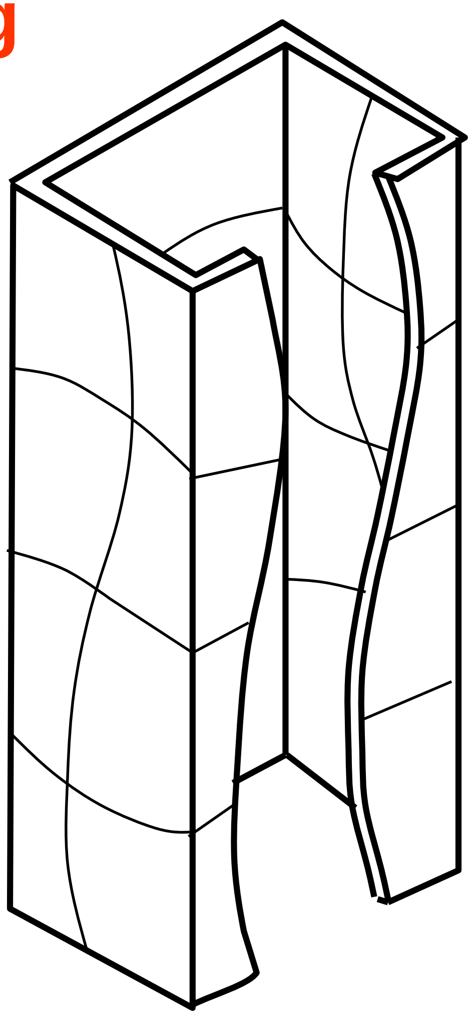
When the corners of a thin-walled section in compression are restrained against any lateral movement, the corner material can continue to be loaded even after local buckling has occurred in the section

The remaining material is largely ineffective in supporting additional loading above the local buckling load

Crippling

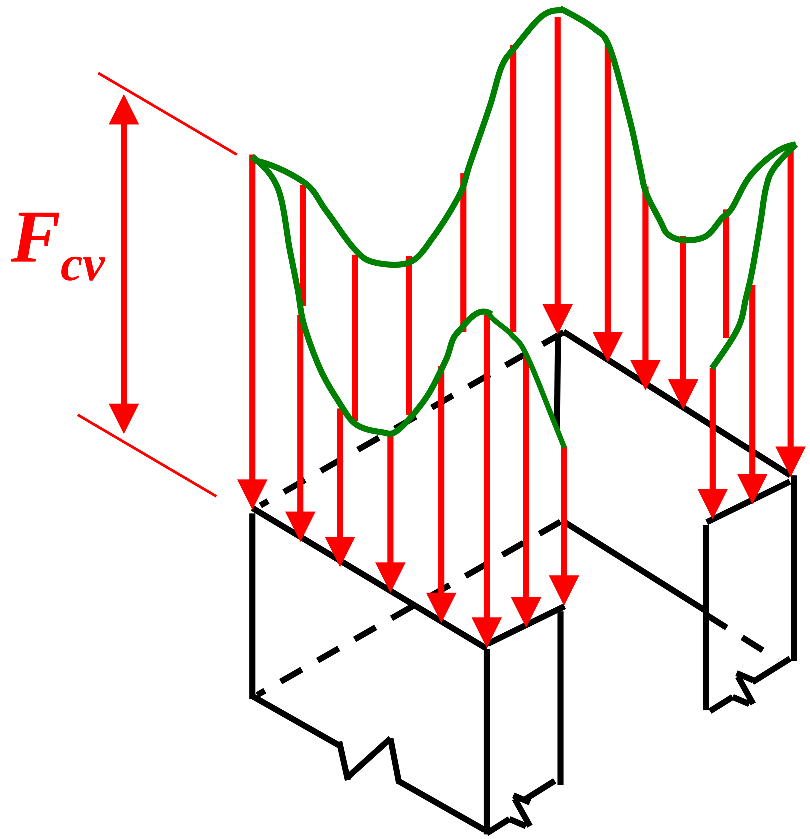
When the stress in the corners exceeds the yield stress, the section loses its ability to support any additional load and fails

The average stress on the section at the ultimate load is called the crippling stress, F_{cc}



Cross sectional distortion

Crippling



a change of stress distribution occurs after buckling

Stress distribution

Crippling

The crippling coefficients are available in the specialized literature in tables or figures

The analysis of crippling is very challenging. It involves large displacements, buckling, contact problems, plasticity (with non-linear constitutive equations) and a large variety of boundary conditions and loading