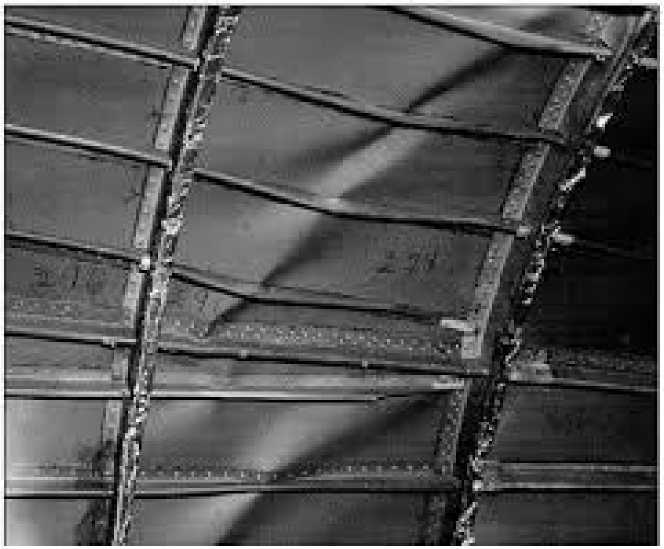
Stability

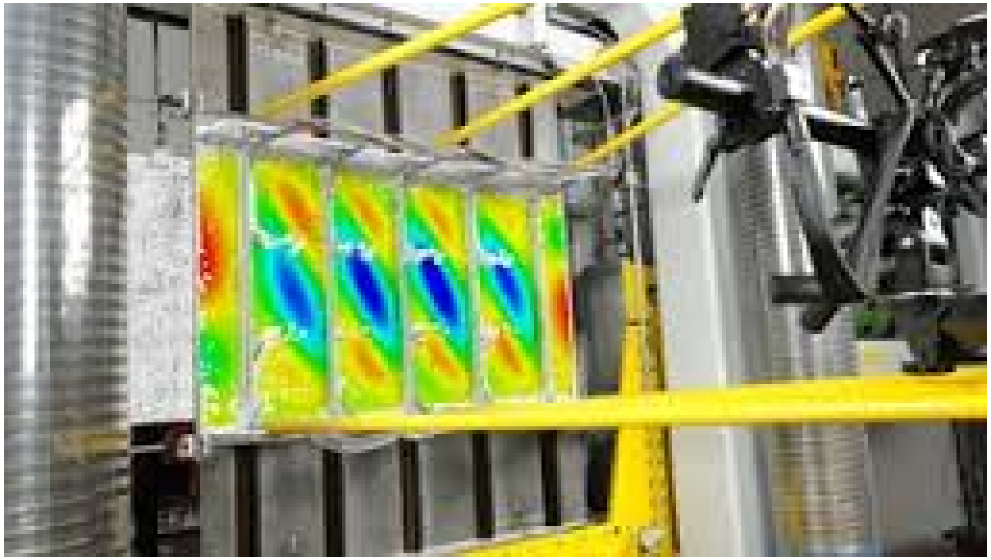
Lecture notes:

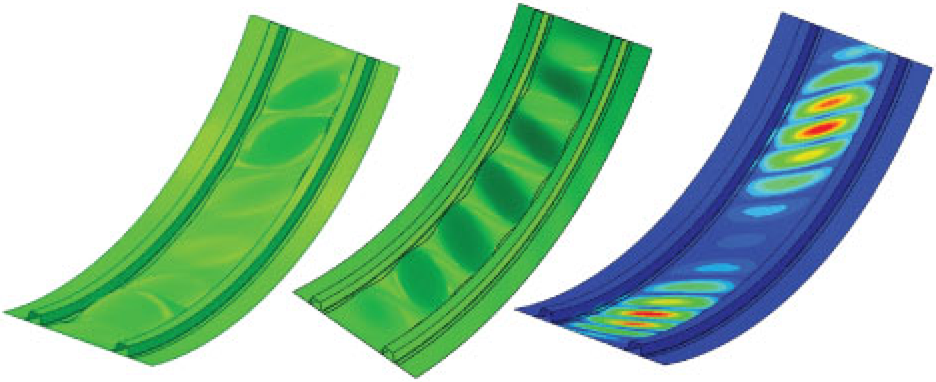
Prof. Sérgio Frascino Müller de Almeida

Beams

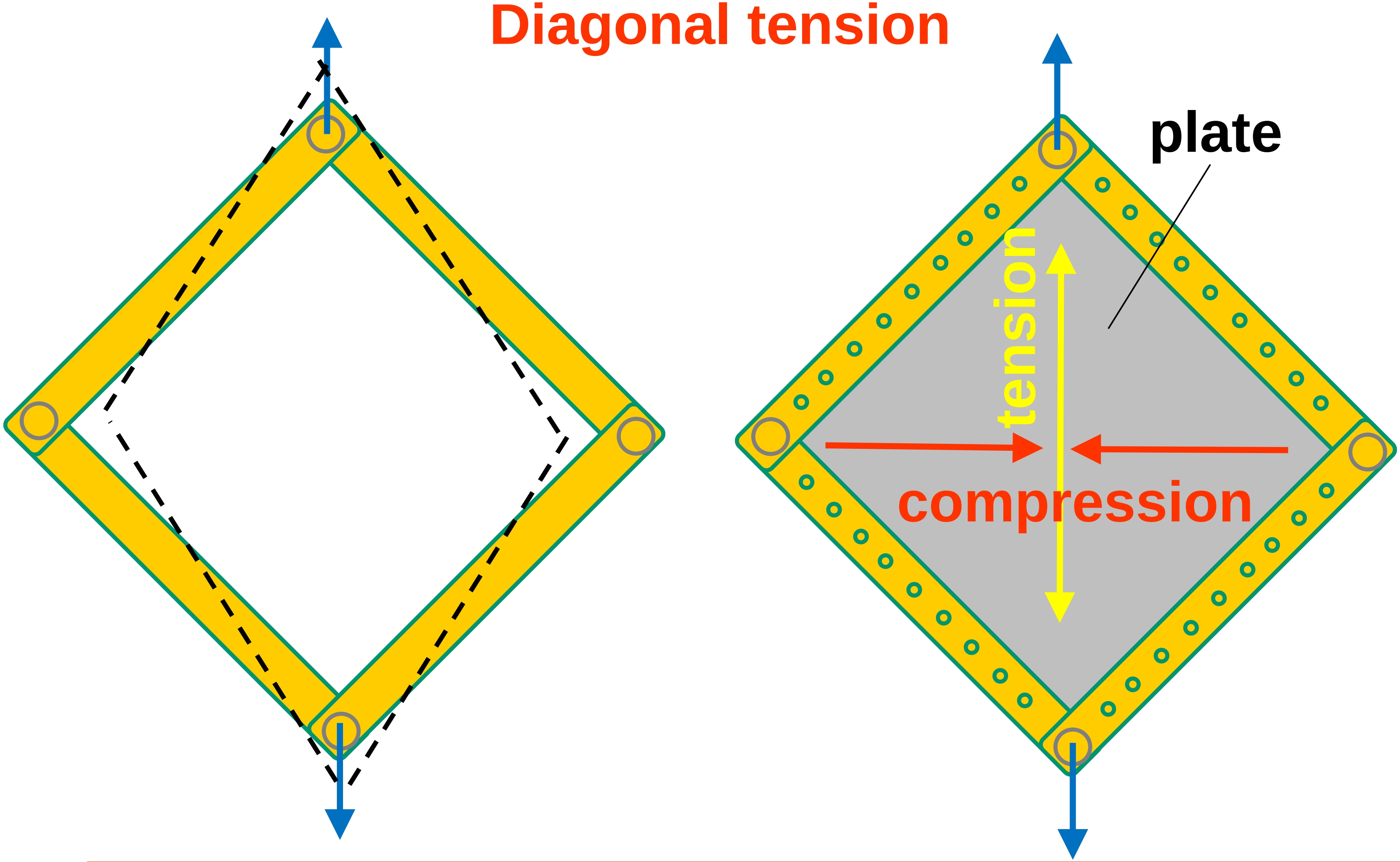
Fuselage buckling







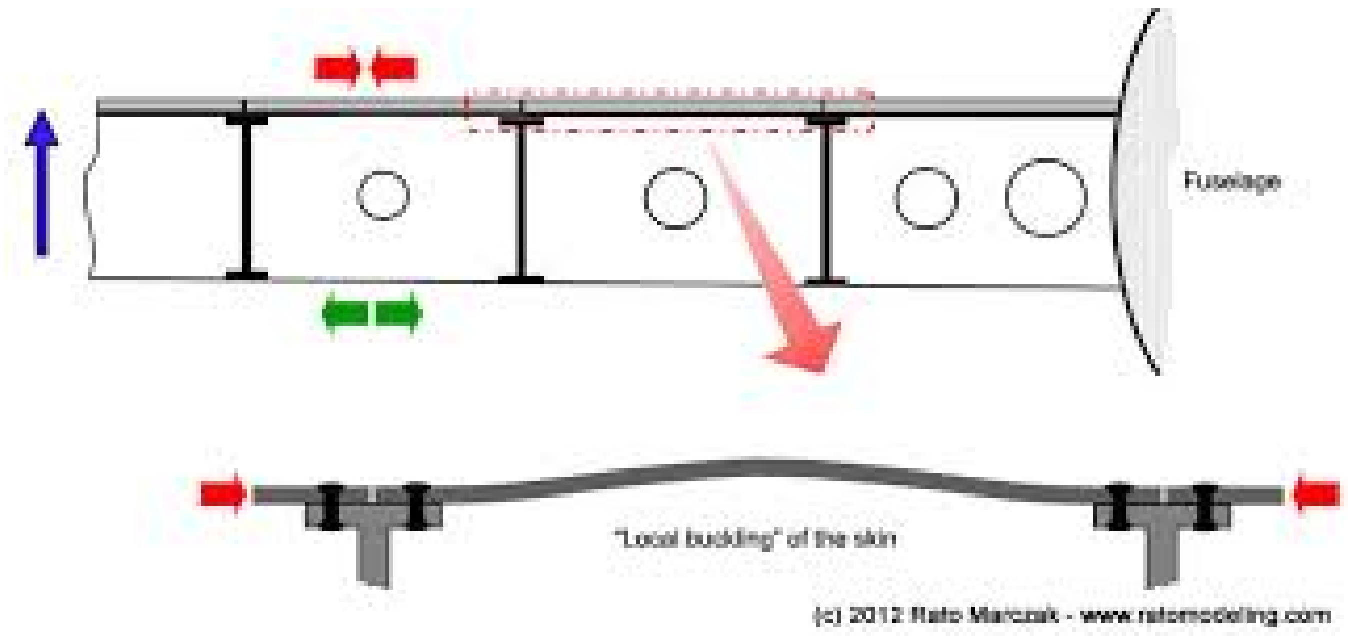
Diagonal tension



Shear buckling

Diagonal tension

Skin wing buckling



Web buckling

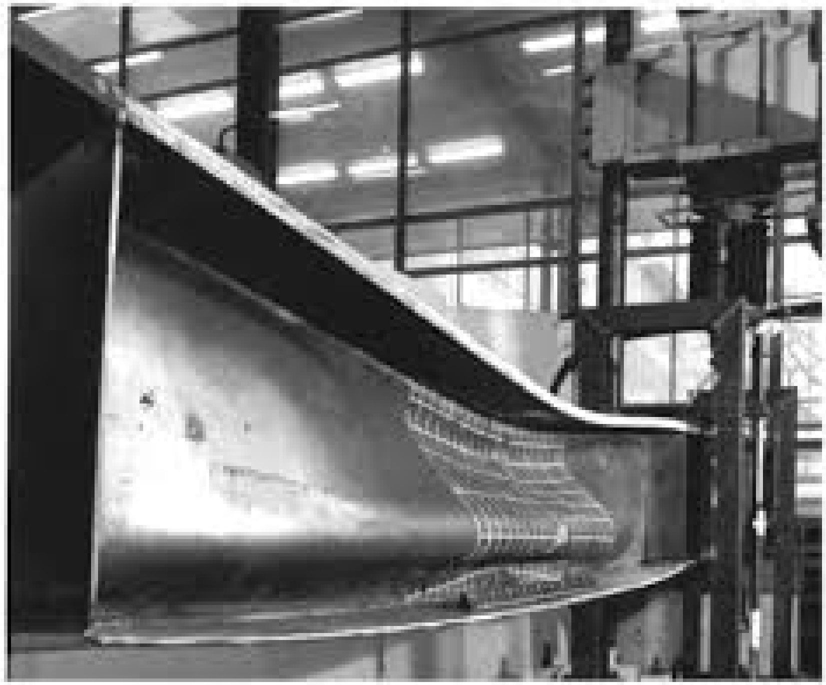
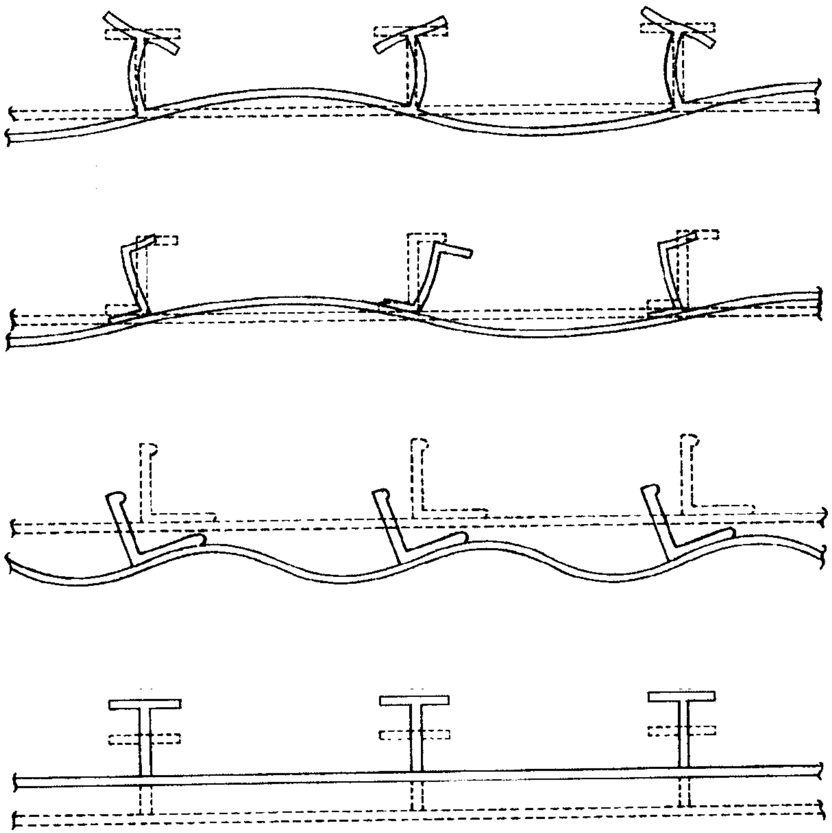
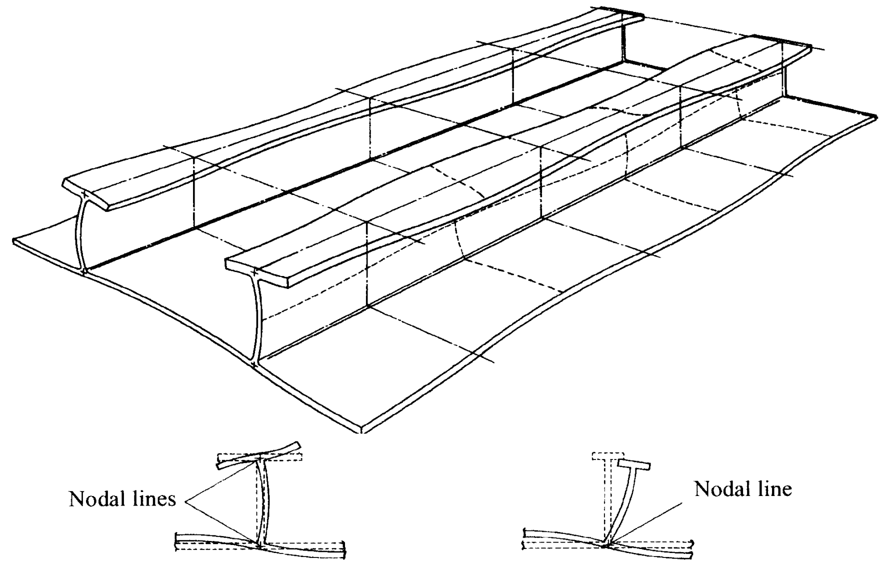


Figure 2. Vertical web buckling.

Reinforced panel buckling





Global buckling x local buckling

Global buckling is a buckling that affects the entire structure in a global way

Consequence: catastrophic failure

Local buckling is a buckling that affect a localized area of the structure (for example, a component); it may be tolerated

Consequence: drag increase
lift decrease
aeroelastic problems
local damage

Buckling x vibration

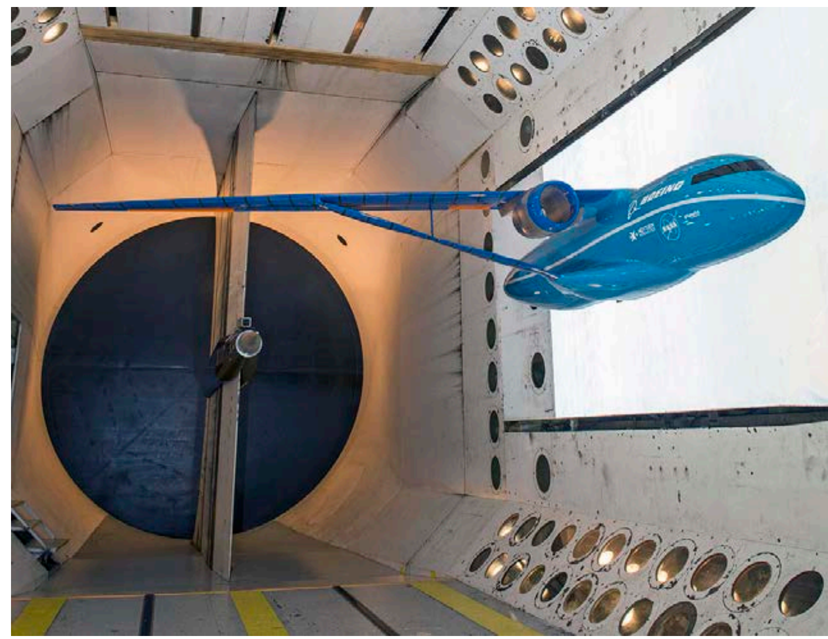
Buckling does affect the natural frequencies and mode shapes of the structure

This phenomenon is associated with aeroelastic behavior

Typically, the natural frequencies are reduced but some may increase

This phenomenon is very difficult to predict for complex structures

SUGAR aircraft (NASA)

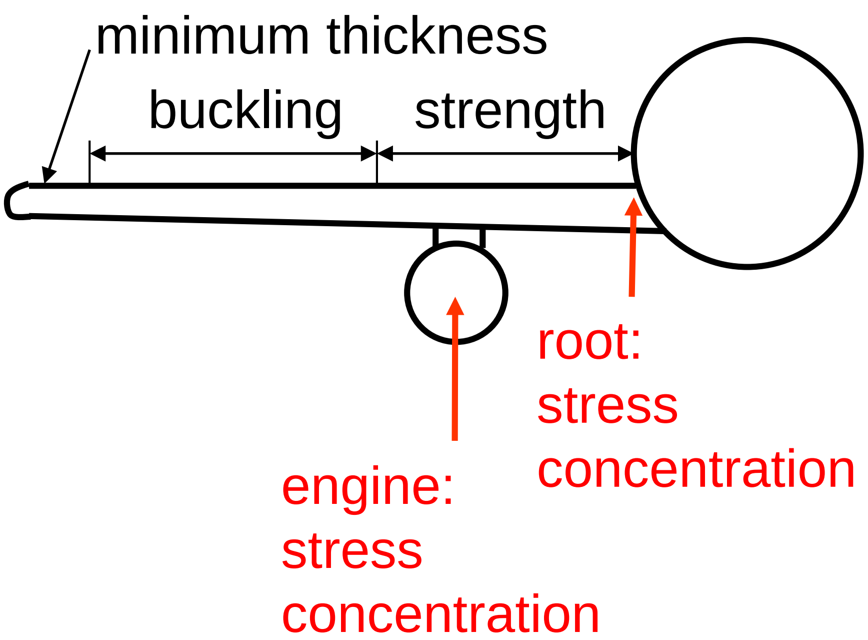


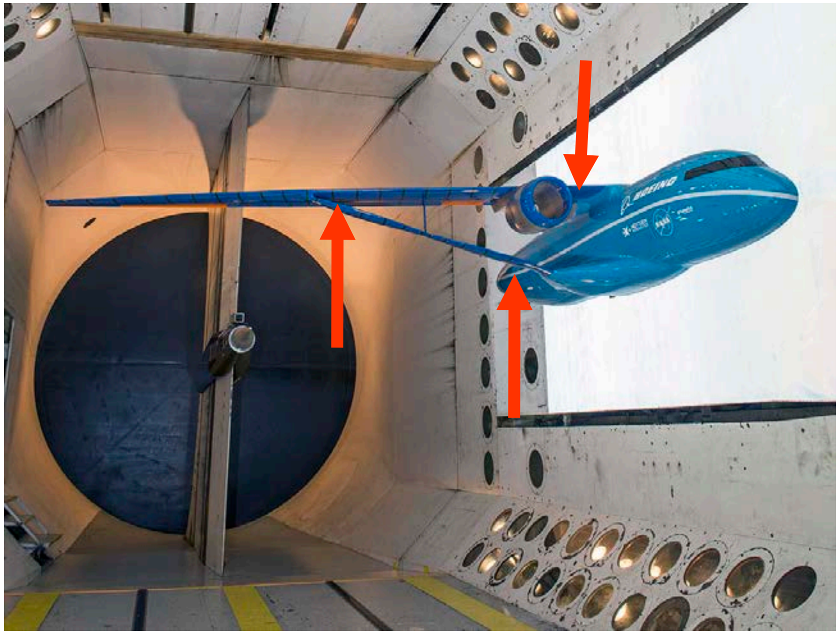
Critical design requirements for wings

- **Close to the root or stress concentration points**
    - **Strength**
- **Mid section of the span**
    - **Buckling**
- **Close to the tip**
    - **Minimum thickness**

Critical design requirements for wings

Buckling is relevant particularly for upper skin (compressive loads)





Critical design requirements for fuselage

Strength / fracture mechanics

Pressurization load

Skin

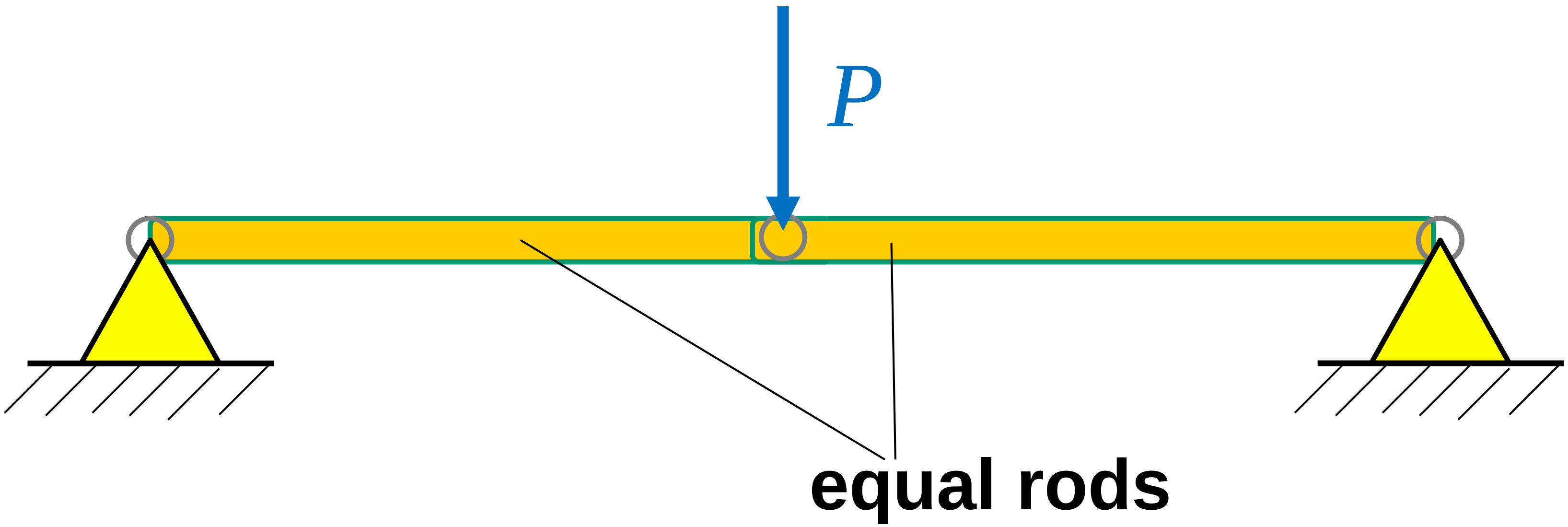
Buckling (torsional load is typically dominant - shear)

Ground test – Boeing 787

Ground test – Boeing 787

Strength failure at the root
Certification tests are required

Problem



rods:

length = L

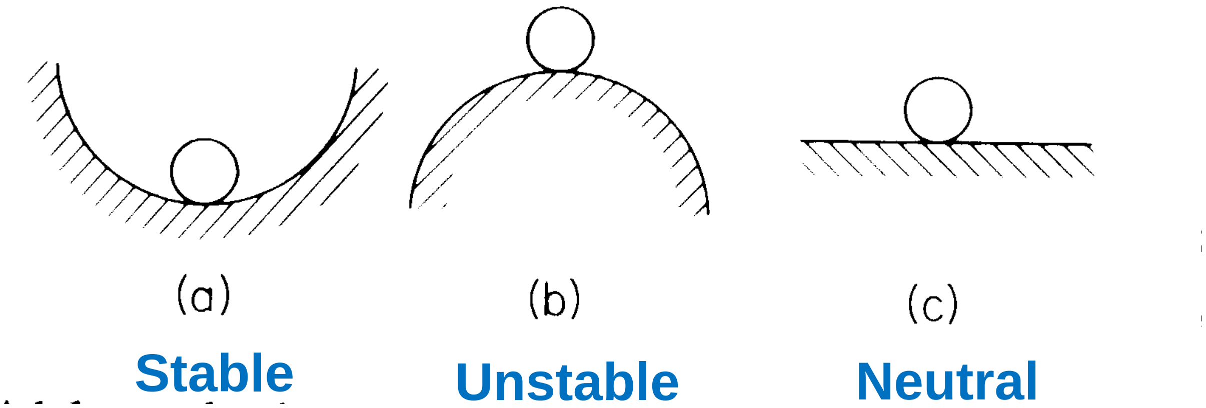
modulus of elasticity = E

area = A

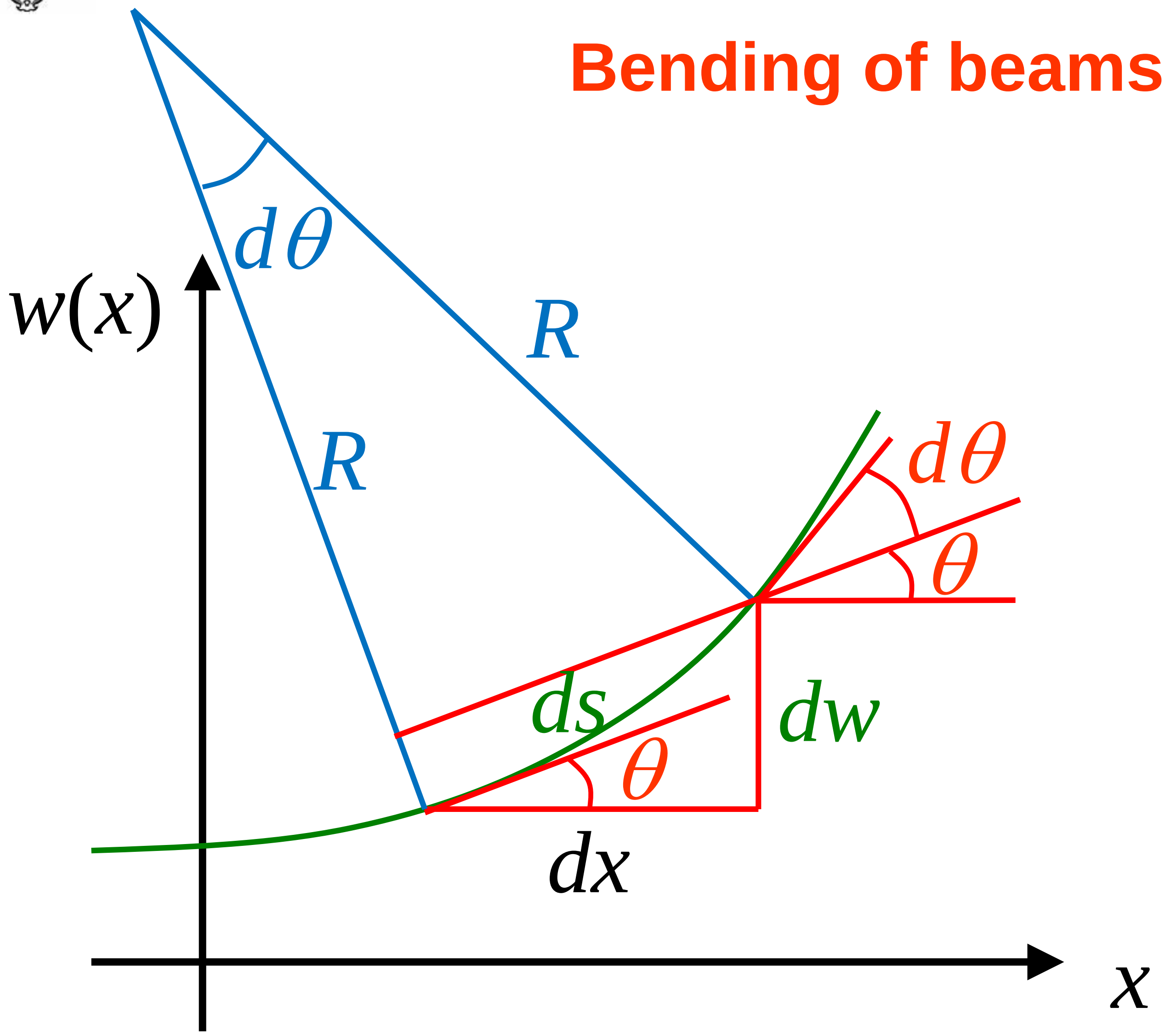
Compute the vertical displacements

Buckling of beams

Equilibrium



Bending of beams



$$\frac{1}{R} = \frac{d\theta}{ds} = \left(\frac{d\theta}{dx}\right)\frac{1}{\left(\frac{ds}{dx}\right)}$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dw}{dx}\right)^2}$$

$$\frac{dw}{dx} = \theta$$

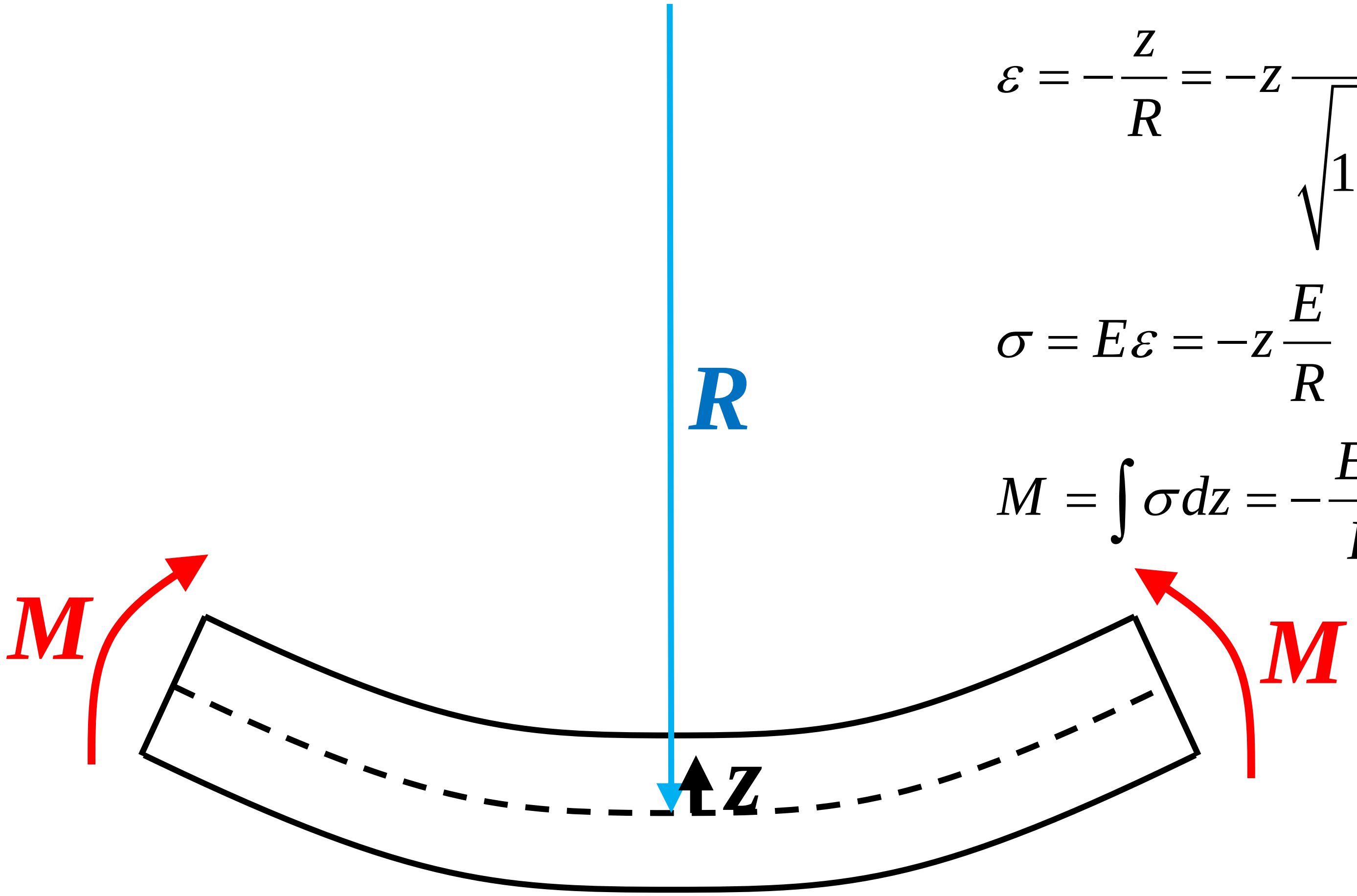
$$\frac{1}{R} = \frac{\frac{d^2w}{dx^2}}{\sqrt{1 + \left(\frac{dw}{dx}\right)^2}}$$

Bending of beams

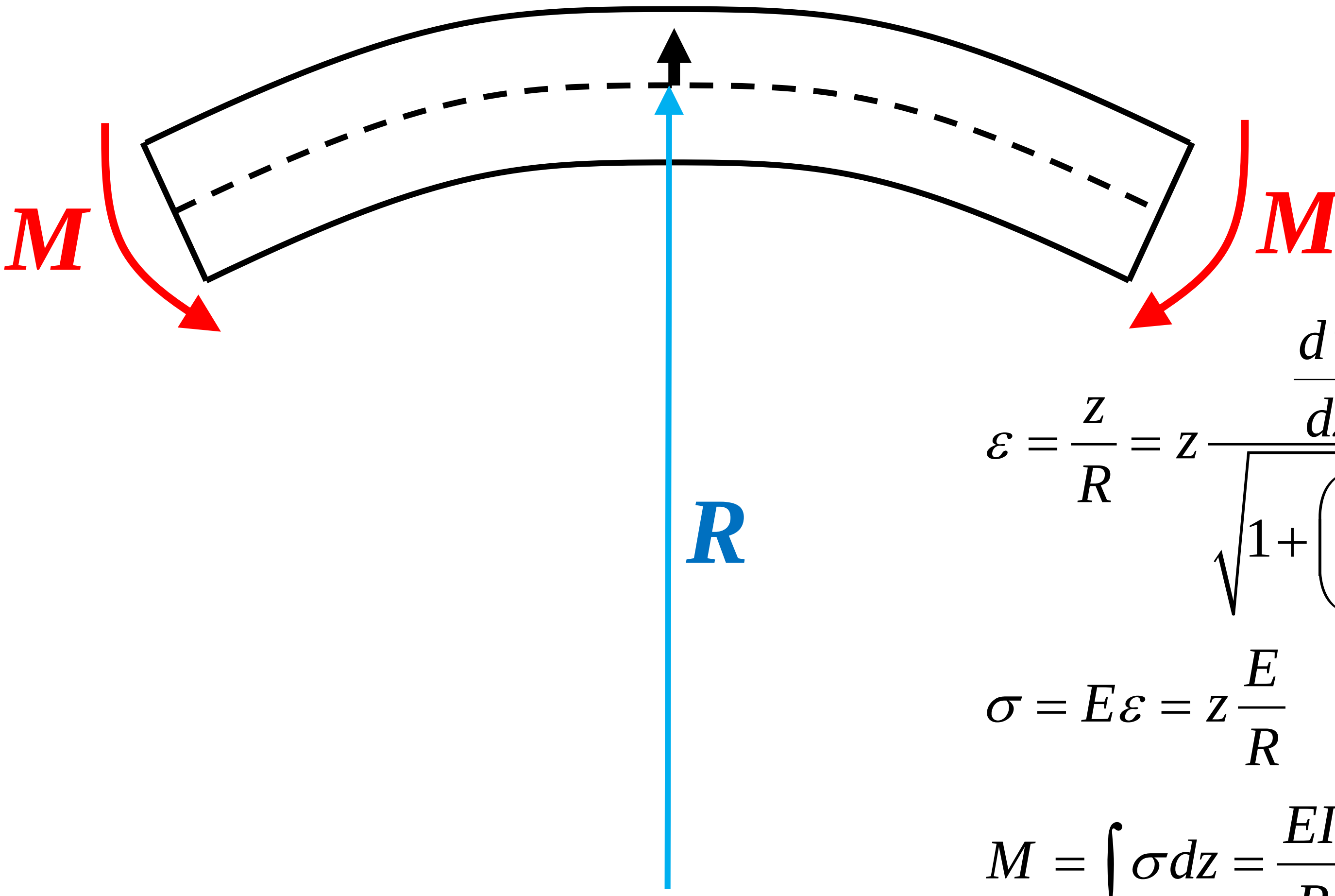
$$\varepsilon = -\frac{z}{R} = -z\frac{\dfrac{d^2w}{dx^2}}{\sqrt{1+\left(\dfrac{dw}{dx}\right)^2}} \simeq -z\frac{d^2w}{dx^2}$$

$$\sigma = E\varepsilon = -z\frac{E}{R}$$

$$M = \int \sigma dz = -\frac{EI}{R} \simeq -EI\frac{d^2w}{dx^2}$$



Bending of beams



$$\varepsilon = \frac{z}{R} = z\frac{\dfrac{d^2w}{dx^2}}{\sqrt{1+\left(\dfrac{dw}{dx}\right)^2}} \simeq z\frac{d^2w}{dx^2}$$

$$\sigma = E\varepsilon = z\frac{E}{R}$$

$$M = \int \sigma dz = \frac{EI}{R} \simeq EI\frac{d^2w}{dx^2}$$

Strains

$$\varepsilon_{xx} = \overbrace{\frac{\partial u}{\partial x}}^{\text{Linear}} + \overbrace{\frac{1}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right)}^{\text{Non-linear}}$$

$$\varepsilon_{xy} = \underbrace{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}}_{\text{Linear}} + \underbrace{\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)}_{\text{Non-linear}}$$

Strains

Von Kármán approximation

$$\varepsilon_{xx} = \overbrace{\frac{\partial u}{\partial x}}^{\text{Linear}} + \overbrace{\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2}^{\text{Non-linear}}$$

$$\varepsilon_{xy} = \underbrace{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}}_{\text{Linear}} + \underbrace{\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)}_{\text{Non-linear}}$$

Stress stiffening

- The stress stiffening phenomenon is caused by the non-linear strains
- It causes the increase of the transverse stiffness when a tensile load is applied in-plane
- If the in-plane stress is compressive, the transverse stiffness is reduced
- This phenomenon is the reason for buckling

Buckling of beams

Pre-buckling problem

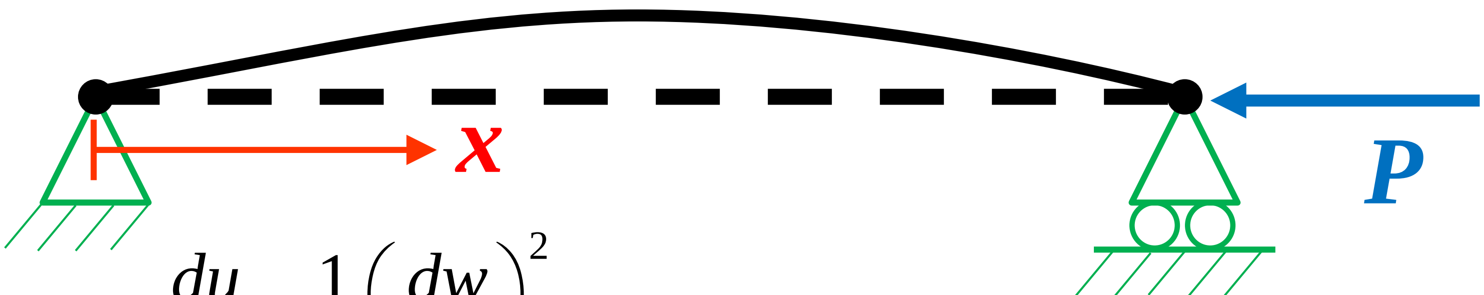
$$\varepsilon = \frac{\partial u}{\partial x}$$

un-deformed structure

$$w(x) = 0$$

Buckling of beams

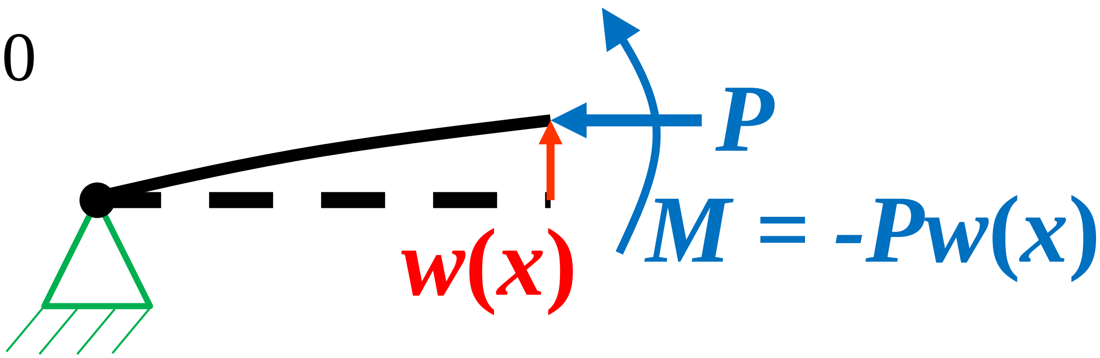
Buckling problem



$$\varepsilon = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2$$

deformed structure

$$w(x) \neq 0$$



Buckling of beams

Equilibrium equation:

$$M = \frac{EI}{R} \simeq EI \frac{d^2 w}{dx^2} = -Pw$$

$$EI \frac{d^2 w}{dx^2} + Pw = 0$$

$$k^2 = \frac{P}{EI} \quad \Rightarrow \quad \frac{d^2 w}{dx^2} + k^2 w = 0$$

Buckling of beams

General solution:

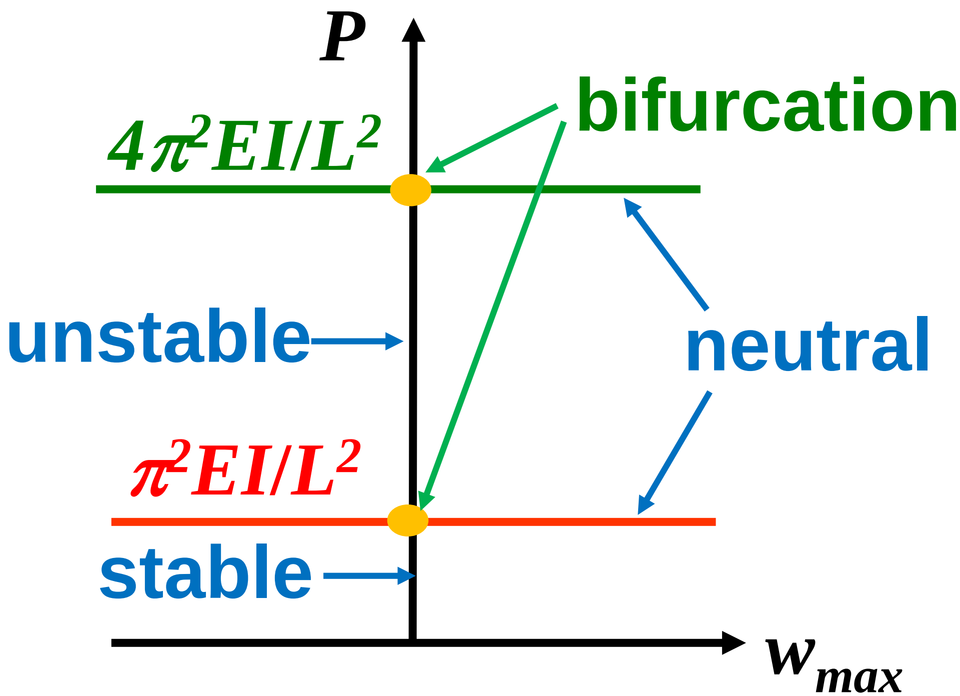
$$\frac{d^2w}{dx^2} + k^2 w = 0 \quad \Rightarrow \quad w(x) = A\sin(kx) + B\cos(kx)$$

For a simply supported beam:

$$M(0) = 0 \qquad w(0) = 0$$
$$M(L) = 0 \qquad w(L) = 0$$
$$\Rightarrow \quad B = 0, \quad \sin(kL) = 0 \quad \Rightarrow \quad kL = n\pi$$

Buckling of simply supported beams

$$kL = n\pi \quad \Rightarrow \quad \sqrt{\frac{P}{EI}}L = n\pi \quad \Rightarrow \quad P = n^2\frac{\pi^2 EI}{L^2}$$



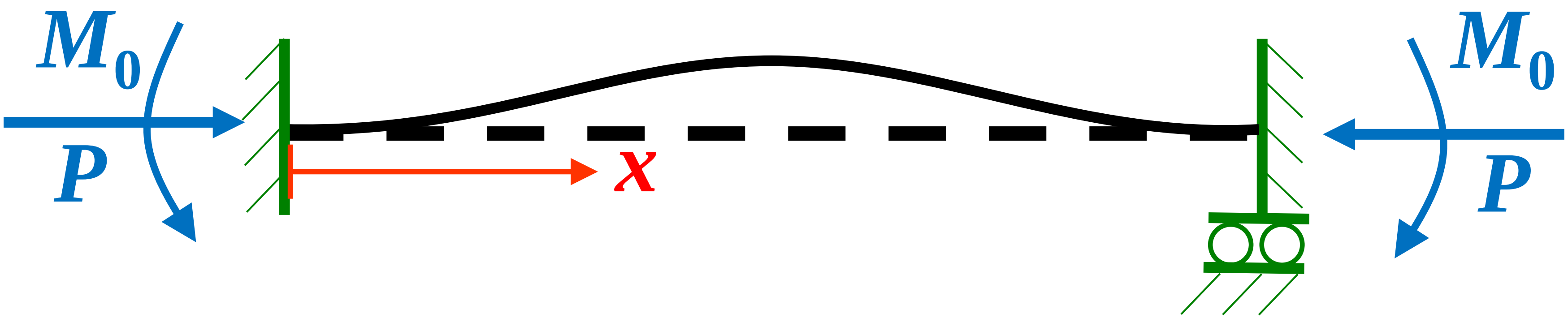
Critical load:

$$P_{crit} = \frac{\pi^2 EI}{L^2}$$

$$w(x) = A\sin(kx)$$

Buckling of beams

Clamped-clamped beam:



$$EI\frac{d^2w}{dx^2}+Pw=M_0 \quad \Rightarrow \quad \frac{d^2w}{dx^2}+k^2w=\frac{M_0}{EI}$$

$$w(x)=A\sin(kx)+B\cos(kx)+\frac{M_0}{P}$$

Buckling of beams

Boundary conditions:

$$\frac{dw}{dx}(0) = 0$$
$$\frac{dw}{dx}(L) = 0$$

$$A = 0$$

$$w(0) = 0$$
$$w(L) = 0$$

$$B\cos(0) + \frac{M_0}{P} = 0$$
$$B\cos(kL) + \frac{M_0}{P} = 0$$

Buckling of beams

Boundary conditions:

$$B\cos(0)+\frac{M_0}{P}=0$$

$$B\cos(kL)+\frac{M_0}{P}=0$$

$$B=-\frac{M_0}{P}$$

$$\frac{M_0}{P}\left(1-\cos(kL)\right)=0$$

$$\cos(kL)=1$$

$$kL=2n\pi$$

$$w(x)=\frac{M_0}{P}\left(1-\cos\left(kx\right)\right)$$

Buckling of beams

Critical load ($n = 1$):

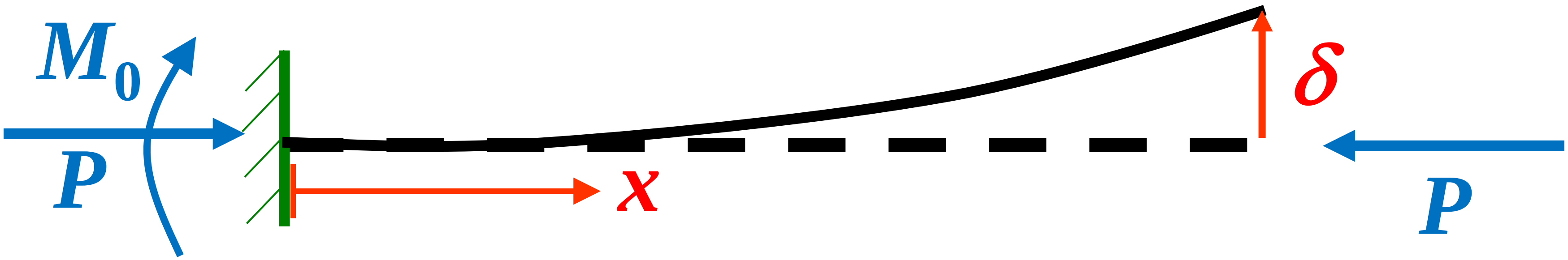
$$P_{crit} = 4\frac{\pi^2 EI}{L^2}$$

Four times bigger than the critical load for simply supported beams

It is equivalent to a simply supported beam of length $L/2$

Buckling of beams

Clamped-free beam:



$$EI\frac{d^2w}{dx^2}+Pw=M_0=P\delta \Rightarrow \frac{d^2w}{dx^2}+k^2w=k^2\delta$$

$$w(x)=A\sin(kx)+B\cos(kx)+\delta$$

Buckling of beams

Boundary conditions:

$$w'(0)=0 \qquad w(0)=0 \;\Rightarrow\; A=0 \qquad B=-\delta \;\Rightarrow\; w(x)=\delta\left(1-\cos(kx)\right)$$

$$w(L)=\delta \qquad w''(L)=0 \;\Rightarrow\; w\left(L\right)=\delta\left(1-\cos\left(kl\right)\right)=\delta$$

$$w''(L)=\delta k^2\cos(kL)=\frac{M\left(L\right)}{EI}=0$$

$$\Rightarrow\; \cos(kL)=0 \;\Rightarrow\; kL=\frac{\left(2n-1\right)\pi}{2}$$

Buckling of beams

Critical load ($n = 1$):

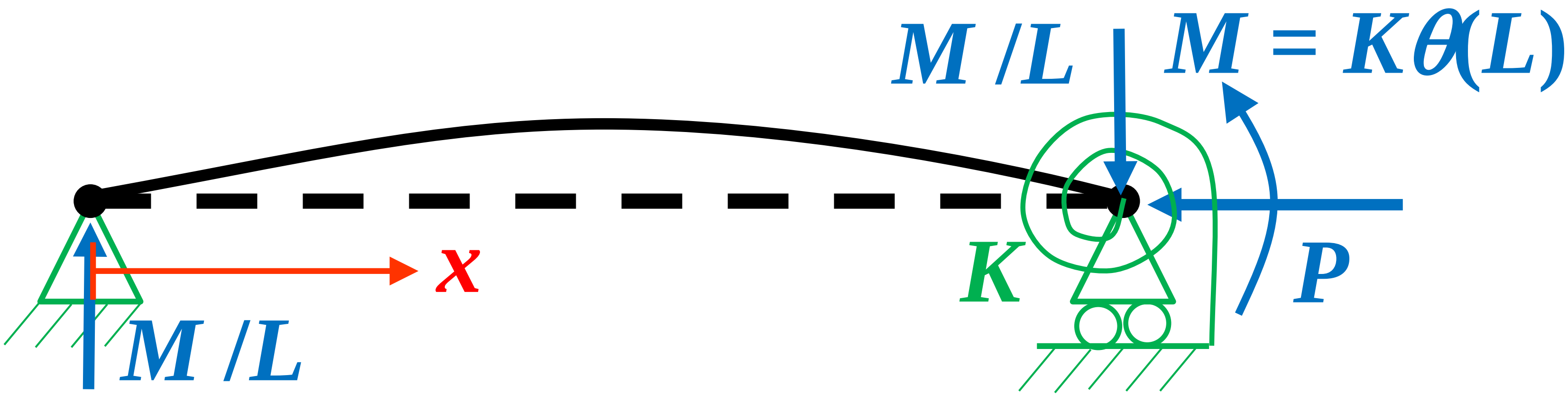
$$P_{crit} = \frac{\pi^2 EI}{4L^2}$$

Four times smaller than the critical load for simply supported beams

It is equivalent to a simply supported beam of length $2L$

Buckling of beams

Simply supported beam with elastic restraint:



$$EI\frac{d^2w}{dx^2} + Pw = \frac{Mx}{L} \quad \Rightarrow \quad \frac{d^2w}{dx^2} + k^2 w = \frac{Mx}{EIL}$$

$$w(x) = A\sin(kx) + B\cos(kx) + \frac{Mx}{PL}$$

Buckling of beams

Boundary conditions:

$$w(0)=0$$
$$w(L)=0$$

$$B=0$$
$$A=-\frac{M}{P\sin\left(kL\right)}$$

$$w(x)=-\frac{M}{P}\frac{\sin(kx)}{\sin\left(kL\right)}+\frac{Mx}{PL}=\frac{M}{P}\left(\frac{x}{L}-\frac{\sin(kx)}{\sin\left(kL\right)}\right)$$

$$\theta(L)=-\frac{dw}{dx}\left(L\right)=\frac{M}{P}\left(\frac{k}{\tan\left(kL\right)}-\frac{1}{L}\right)$$

Buckling of beams

Boundary conditions:

$$\frac{M}{K}=\theta(l)$$

$$\frac{M}{K}=-\frac{M}{kEI}\left(\frac{1}{kL}-\frac{1}{\tan(kL)}\right)$$

$$\frac{kEI}{K}=-\left(\frac{1}{kL}-\frac{1}{\tan(kL)}\right)$$

$$\frac{1}{\tan(kL)}=\frac{kEI}{K}+\frac{1}{kL}$$

$$\frac{1}{\tan(kL)}=\frac{k^2EIL+K}{KkL}$$

$$\tan(kL)=\frac{KkL}{k^2EIL+K}$$

Buckling of beams

Defining: $$\beta = \frac{KL}{EI}$$

$$\tan(kL) = \frac{KkL}{k^2 EIL + K} = \frac{\dfrac{KkL^2}{EI}}{(kL)^2 + \dfrac{KL}{EI}} = \frac{\beta kL}{(kL)^2 + \beta}$$

$$\tan(kL) = \frac{\beta kL}{(kL)^2 + \beta}$$

Non-linear equation to be solved for the eigenvalues

Buckling loads of beams

Dependence of beam critical buckling loads with some parameters:

- The buckling load decreases with the square of the length
- The buckling load increases with the cube of the thickness for rectangular cross sections
- The buckling load are strongly affected by the load distribution and the boundary conditions

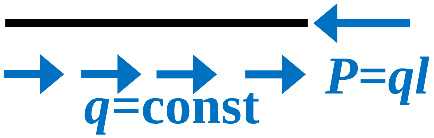
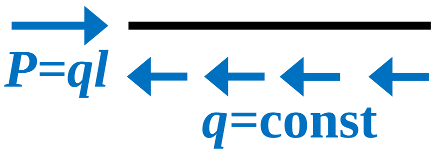
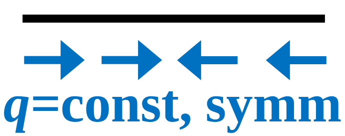
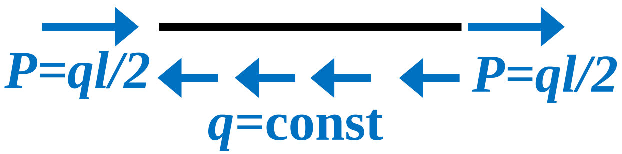
*** This is also valid for plates**

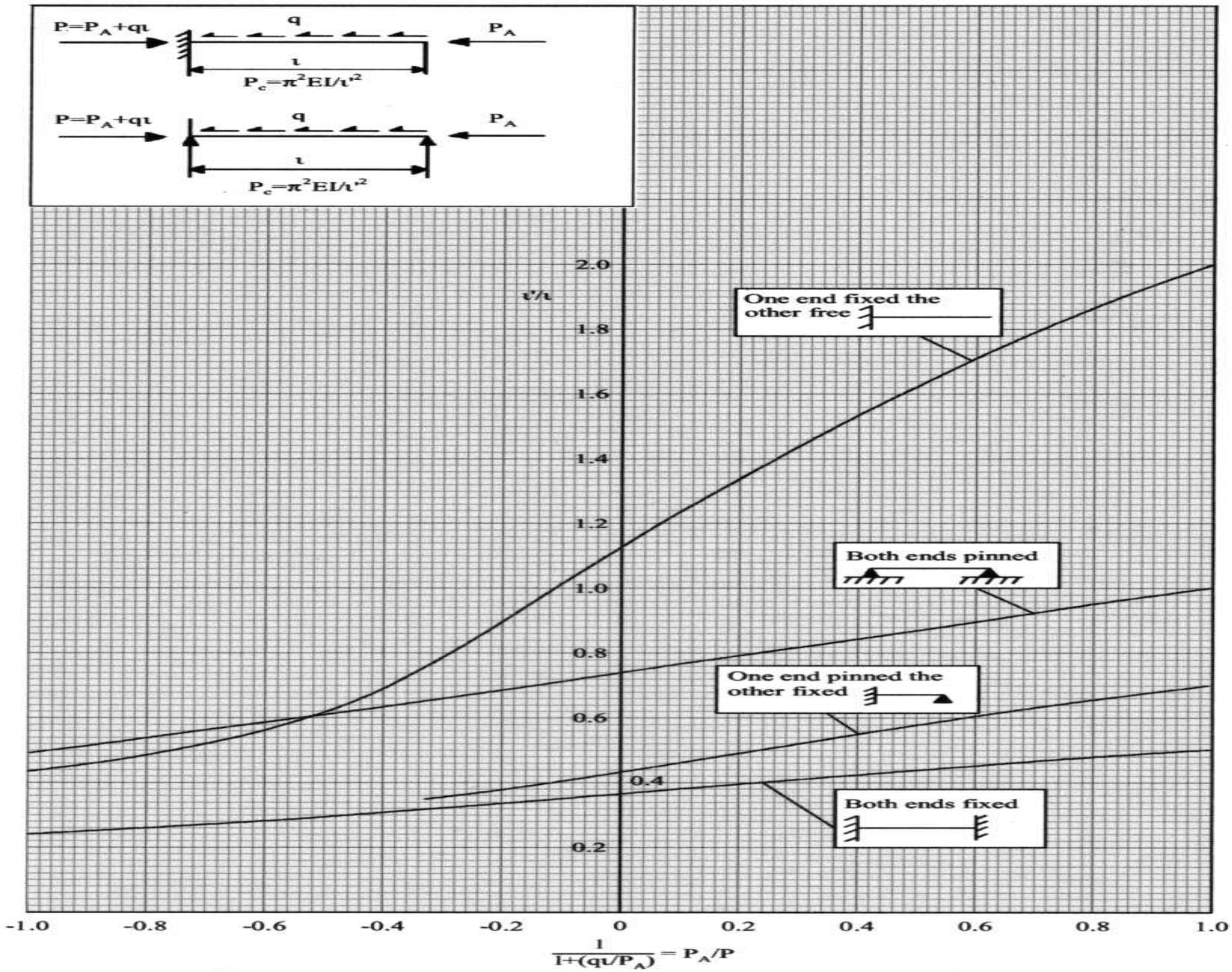
Effective length

The beam critical buckling load can be computed using the effective length as:

$$P_{crit} = \frac{\pi^2 EI}{L_{eff}^2}$$

Effective length

					
 P P	*2.0 L*	*1.0 L*	*1.0 L*	*0.7 L*	*0.5 L*
 P=ql q=const	*1.69 L*	-	*0.732 L*	*0.58 L*	*0.365 L*
 P=ql q=const	*1.12 L*	*0.72 L*	*0.732 L*	*0.43 L*	*0.365 L*
 q=const, symm	*1.43 L*	*0.84 L*	*0.57 L*	*0.45 L*	*0.36 L*
 P=ql/2 q=const P=ql/2	-	-	*0.49 L*	-	*0.24 L*


$P=P_A+q\iota$
q
P_A
ι
$P_c=\pi^2EI/\iota'^2$
$P=P_A+q\iota$
q
P_A
ι
$P_c=\pi^2EI/\iota'^2$
ι'/ι
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
One end fixed the other free
Both ends pinned
One end pinned the other fixed
Both ends fixed
-1.0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1.0
$\frac{1}{1+(q\iota/P_A)}=P_A/P$

Elastic restraint

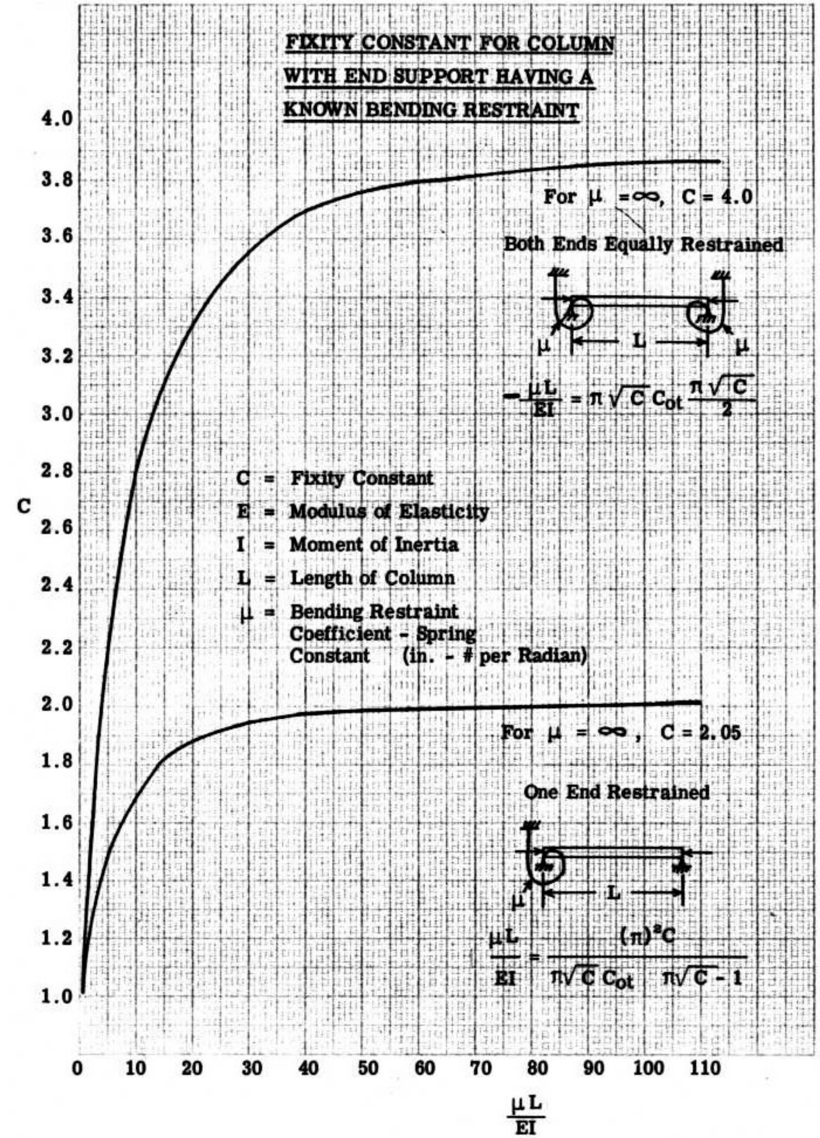
Rotation restraint at the ends:

a) one end

b) equal, at both ends

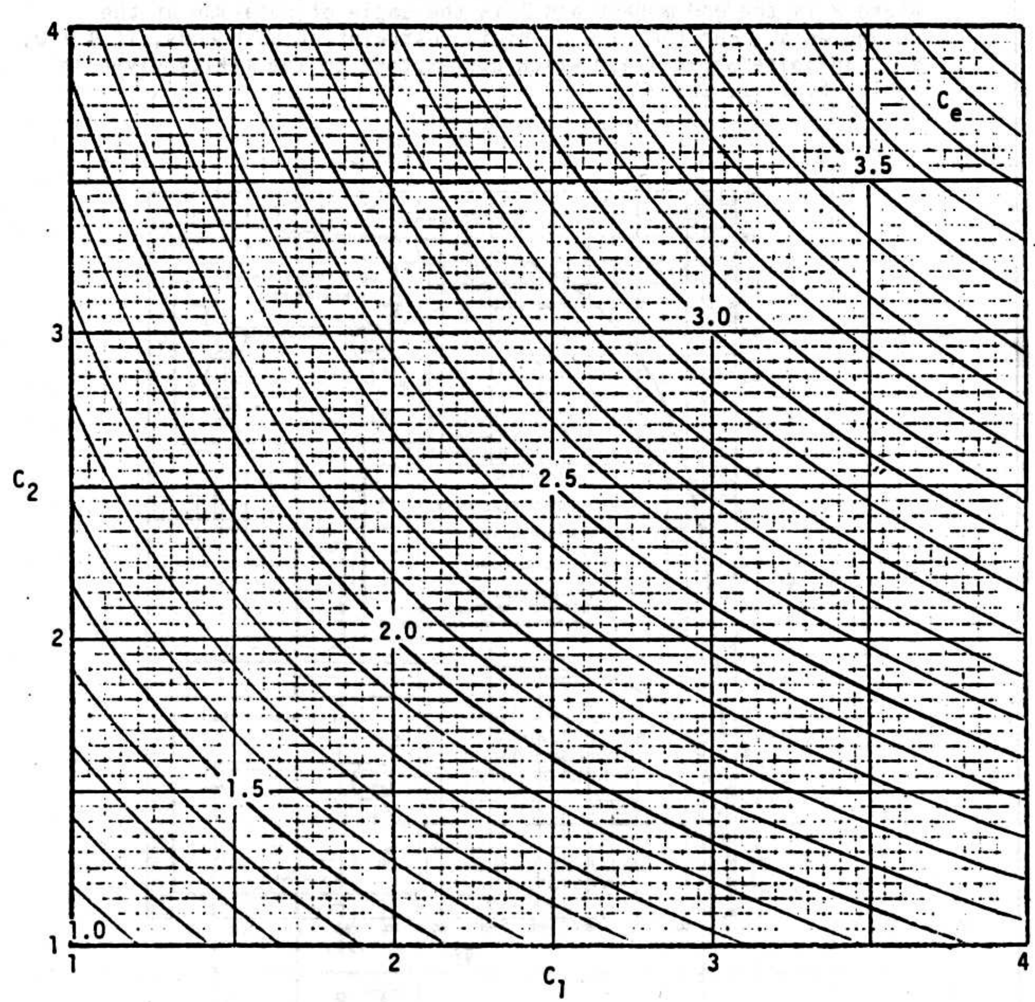
$$P_{crit} = \frac{\pi^2 EI}{L'^2} = \frac{c\pi^2 EI}{L^2}$$

$$c = \left(\frac{L}{L'}\right)^2$$



Elastic restraint

Different restrictions conditions at both ends



Strength x buckling load

- The buckling load increases with the cube of the thickness for rectangular cross sections
- The strength is typically proportional to the thickness; this is certainly valid for thin plates
- Therefore, at the root of a wing the strength is the dominant design criterion; near the tip, buckling is the dominant requirement

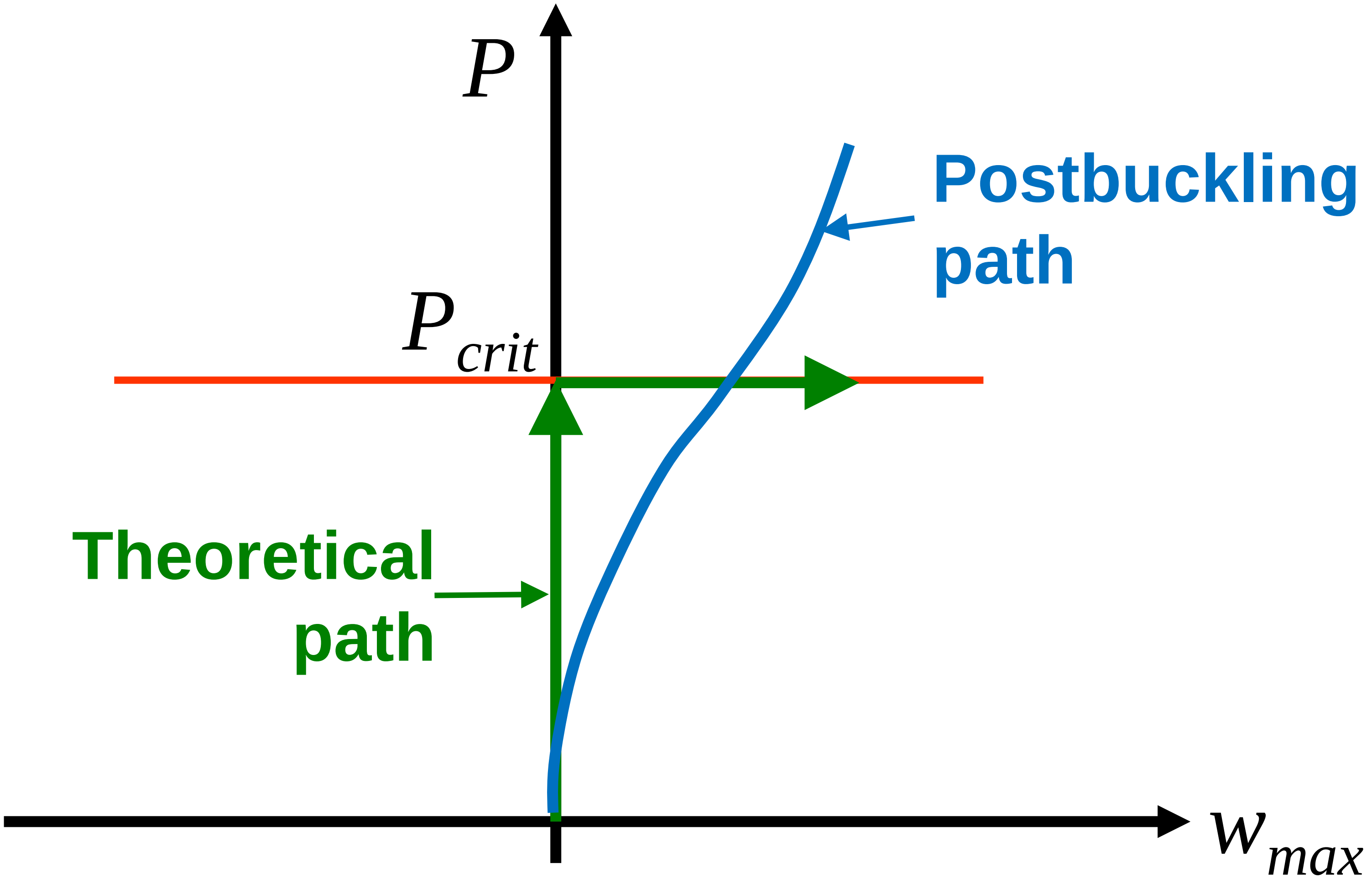
Postbuckling

Buckling x Postbuckling

The stable part of the out-of-plane undeformed configuration is never observed in practice due to:

- **Geometric imperfection**
- **Loading offset**

Buckling x Postbuckling



Postbuckling

Since postbuckling analyses involve solving a non-linear set of differential equations there could be more than one solution (typically this is the case)

Due to some external perturbation, one solution may jump into another solution (snap through buckling)

This is very hard to predict numerically

Postbuckling

Force control: the applied force is maintained as programmed; force control typically causes catastrophic failure

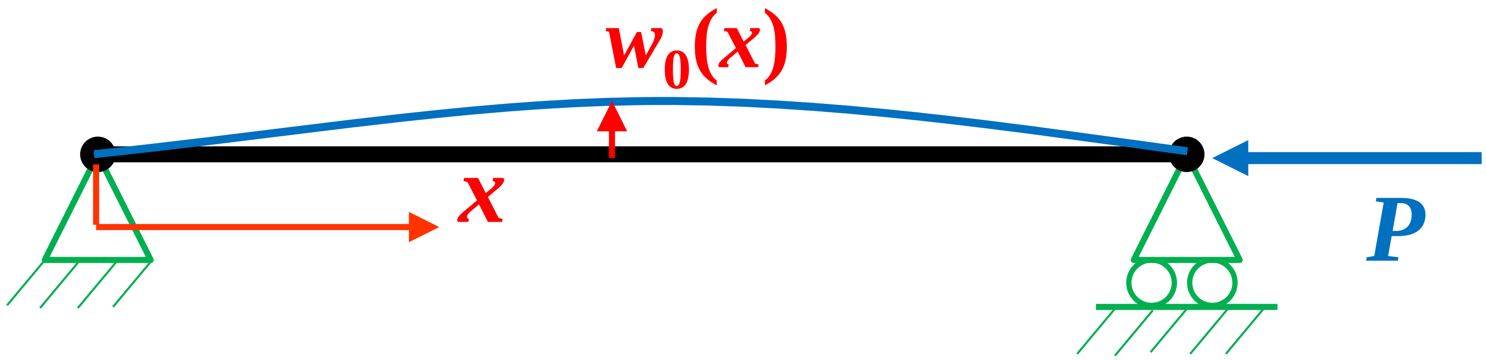
Displacement control: the displacement is varied as programmed

Aerodynamic loads are typically force control

Beam with imperfections

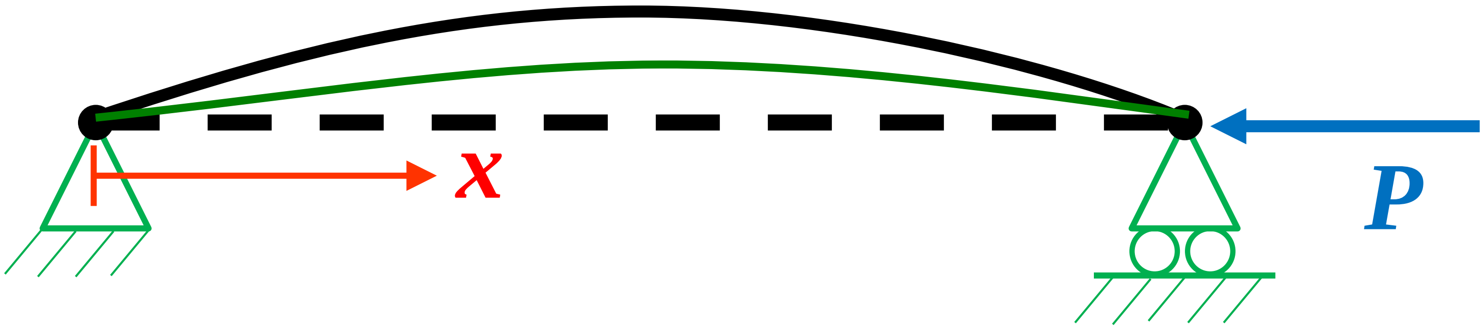
Beam with imperfections

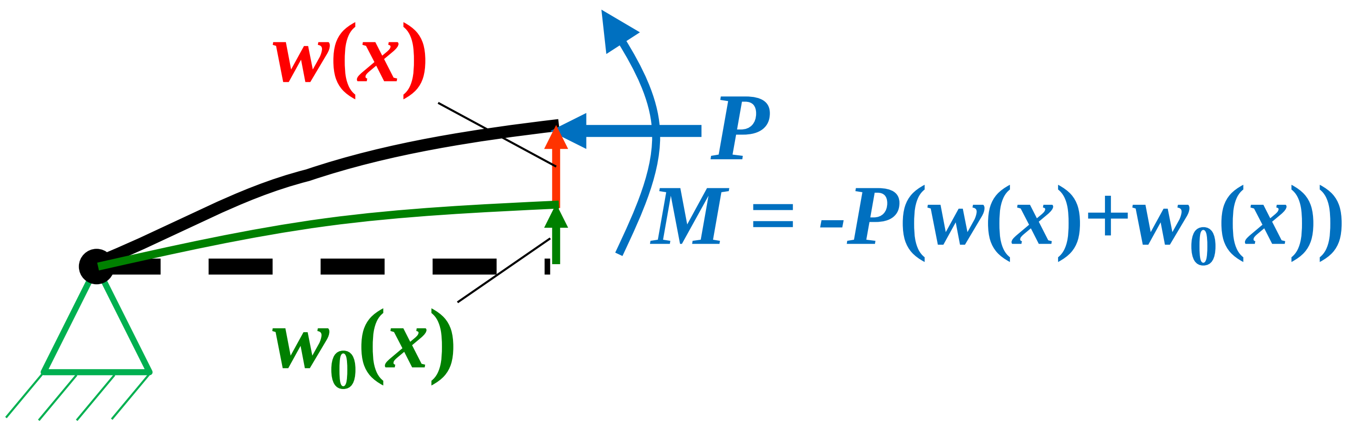
The beam is assumed to have an initial imperfection $w_0(x)$



Beam with imperfections

Buckling problem





Beam with imperfections

Equilibrium equation:

$$M = \frac{EI}{R} \simeq EI\frac{d^2w}{dx^2} = -Pw$$

$$EI\frac{d^2w}{dx^2} + Pw = 0$$

$$k^2 = \frac{P}{EI} \quad \Rightarrow \quad \frac{d^2w}{dx^2} + k^2w = -k^2w_0$$

Beam with imperfections

$$w_h(x) = A\sin(kx) + B\cos(kx)$$

$$w_0(x) = \sum_{n=1}^{\infty} w_n \sin\left(n\pi\frac{x}{l}\right) \Rightarrow w_m(x) = \frac{2}{l}\sum_{n=1}^{\infty} w_0(x)\sin\left(n\pi\frac{x}{l}\right)$$

$$\frac{d^2w}{dx^2} + k^2 w = -k^2\sum_{n=1}^{\infty} w_n \sin\left(n\pi\frac{x}{l}\right)$$

$$w_p(x) = -k^2\sum_{n=1}^{\infty} c_n \sin\left(n\pi\frac{x}{l}\right) \Rightarrow c_n = \frac{w_n k^2}{\left(\frac{n\pi}{l}\right)^2 - k^2}$$

Beam with imperfections

$$c_n = \frac{w_n k^2}{\left(\frac{n\pi}{l}\right)^2 - k^2} = \frac{w_n}{\frac{1}{k^2}\left(\frac{n\pi}{l}\right)^2 - 1}$$

$$k^2 = \frac{P}{EI}$$

$$P_{Euler} = \pi^2 \frac{EI}{l^2}$$

$$\Rightarrow \quad c_n = \frac{w_n}{\frac{EI}{P}\left(\frac{n\pi}{l}\right)^2 - 1} = \frac{w_n}{n^2 \frac{P_{Euler}}{P} - 1}$$

Beam with imperfections

$$w(x) = w_h(x) + w_p(x)$$

$$w(x) = A\sin(kx) + B\cos(kx) + \sum_{n=1}^{\infty} \frac{w_n}{n^2 \frac{P_{Euler}}{P} - 1} \sin\left(n\pi \frac{x}{l}\right)$$

$$A_n == \frac{1}{n^2 \frac{P_{Euler}}{P} - 1}$$

$$A = B = 0$$

$$w(x) = \sum_{n=1}^{\infty} A_n \sin\left(n\pi \frac{x}{l}\right)$$

$$P = P_{Euler}$$

$$n = 1$$

Euler solution

Beam with imperfections

$$w_T(x) = w(x) + w_0(x)$$

$$w(x) = \sum_{n=1}^{\infty} A_n w_n \sin\left(n\pi \frac{x}{l}\right) + \sum_{n=1}^{\infty} w_n \sin\left(n\pi \frac{x}{l}\right)$$

$$\Rightarrow \quad w(x) = \sum_{n=1}^{\infty} (1 + A_n) \sin\left(n\pi \frac{x}{l}\right)$$

Beam with imperfections