

Appendix D

Equations of Motion in Cylindrical Coordinates

The equations of motion of an incompressible newtonian fluid with constant μ , k , and c_p are given here in cylindrical coordinates (r, θ, z) , which are related to cartesian coordinates (x, y, z) as in Fig. 4.2:

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z \quad (\text{D.1})$$

The velocity components are v_r , v_θ , and v_z . Here are the equations:

Continuity:

$$\frac{1}{r} \frac{\partial}{\partial r}(r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0 \quad (\text{D.2})$$

Convective time derivative:

$$\mathbf{V} \cdot \nabla = v_r \frac{\partial}{\partial r} + \frac{1}{r} v_\theta \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z} \quad (\text{D.3})$$

Laplacian operator:

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \quad (\text{D.4})$$

The r -momentum equation:

$$\frac{\partial v_r}{\partial t} + (\mathbf{V} \cdot \nabla) v_r - \frac{1}{r} v_\theta^2 = -\frac{1}{\rho} \frac{\partial p}{\partial r} + g_r + \nu \left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right) \quad (\text{D.5})$$

The θ -momentum equation:

$$\frac{\partial v_\theta}{\partial t} + (\mathbf{V} \cdot \nabla) v_\theta + \frac{1}{r} v_r v_\theta = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + g_\theta + \nu \left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right) \quad (\text{D.6})$$

The z -momentum equation:

$$\frac{\partial v_z}{\partial t} + (\mathbf{V} \cdot \nabla) v_z = -\frac{1}{\rho} \frac{\partial p}{\partial z} + g_z + \nu \nabla^2 v_z \quad (\text{D.7})$$

The energy equation:

$$\rho c_p \left[\frac{\partial T}{\partial t} + (\mathbf{V} \cdot \nabla) T \right] = k \nabla^2 T + \mu [2(\epsilon_{rr}^2 + \epsilon_{\theta\theta}^2 + \epsilon_{zz}^2) + \epsilon_{\theta z}^2 + \epsilon_{rz}^2 + \epsilon_{r\theta}^2] \quad (\text{D.8})$$

where

$$\begin{aligned} \epsilon_{rr} &= \frac{\partial v_r}{\partial r} & \epsilon_{\theta\theta} &= \frac{1}{r} \left(\frac{\partial v_\theta}{\partial \theta} + v_r \right) \\ \epsilon_{zz} &= \frac{\partial v_z}{\partial z} & \epsilon_{\theta z} &= \frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \\ \epsilon_{rz} &= \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} & \epsilon_{r\theta} &= \frac{1}{r} \left(\frac{\partial v_r}{\partial \theta} - v_\theta \right) + \frac{\partial v_\theta}{\partial r} \end{aligned} \quad (\text{D.9})$$

Viscous stress components:

$$\begin{aligned} \tau_{rr} &= 2\mu\epsilon_{rr} & \tau_{\theta\theta} &= 2\mu\epsilon_{\theta\theta} & \tau_{zz} &= 2\mu\epsilon_{zz} \\ \tau_{r\theta} &= \mu\epsilon_{r\theta} & \tau_{\theta z} &= \mu\epsilon_{\theta z} & \tau_{rz} &= \mu\epsilon_{rz} \end{aligned} \quad (\text{D.10})$$

Angular velocity components:

$$\begin{aligned} 2\omega_r &= \frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \\ 2\omega_\theta &= \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \\ 2\omega_z &= \frac{1}{r} \frac{\partial}{\partial r} (rv_\theta) - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \end{aligned} \quad (\text{D.11})$$