

Energy and momentum in special relativity

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The special relativistic expressions for momentum and energy are obtained by requiring their conservation in a totally inelastic variant of the Lewis–Tolman symmetric collision. The resulting analysis is simpler and more straightforward than the usual textbook treatments of relativistic dynamics. © 2008 American Association of Physics Teachers.

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I. INTRODUCTION

The Lewis–Tolman symmetric collision was introduced by Lewis and Tolman in 1909¹ as a thought experiment useful for obtaining the expression

$$m(v) = m\gamma(v), \quad (1)$$

for the “relativistic mass” $m(v)$ in terms of the rest mass m and $\gamma(v) \equiv \sqrt{1 - v^2/c^2}$. With the assumption that the momentum $m(v)\vec{v}$ is conserved in such a collision, they concluded that the relativistic mass varies with speed according to Eq. (1). This collision has been used by many authors to derive expressions for the relativistic mass and relativistic momentum.^{2–13} We follow the same basic approach (although we do not make use of the notion of relativistic mass—in this paper mass means rest mass) and extend it to obtain an expression for relativistic energy. We assume that both the momentum and energy of an object are proportional to the object’s mass, and that momentum is proportional to the object’s velocity. That is, we assume that

$$\vec{p} = m\vec{v}f(v), \quad (2a)$$

$$E = mc^2g(v), \quad (2b)$$

where $f(v)$ and $g(v)$ are dimensionless functions of the object’s speed. The factor c^2 in E ensures that $g(v)$ is dimensionless. We make the additional assumption that $f(v)$ and $g(v)$ are continuous and smooth functions for $0 \leq v < c$. Finally, we assume that the relativistic momentum and energy have appropriate non-relativistic limiting values. Specifically,

$$f(0) = 1, \quad (3)$$

so that Eq. (2a) is consistent with the usual nonrelativistic expression for momentum, and

$$g(v) = g(0) + \frac{1}{2}\left(\frac{v}{c}\right)^2 + O(v^3), \quad (4)$$

so that the energy

$$E = mc^2g(0) + \frac{1}{2}mv^2 + O(v^3) \quad (5)$$

contains the usual nonrelativistic kinetic energy. We make no separate assumption about the value of $g(0)$. The value of $g(0)$ and thus the relation $E_0 = mc^2$ relating the rest energy to mass will be consequences of our reasoning. We assume that

both relativistic momentum and energy, as calculated in any Lorentz frame, are conserved in collisions.

As a final ingredient, we will make use of the relativistic velocity transformation

$$v_{\parallel} = \frac{u + v'_{\parallel}}{1 + uv'_{\parallel}/c^2}, \quad (6a)$$

$$v_{\perp} = \frac{v'_{\perp}}{\gamma(u)(1 + uv'_{\parallel}/c^2)}, \quad (6b)$$

which relates velocities as observed from two frames S and S' where S' moves with velocity \vec{u} relative to S . Here v_{\parallel} and v'_{\parallel} are velocity components parallel to \vec{u} , and v_{\perp} and v'_{\perp} are velocity components perpendicular to \vec{u} , as observed from S and S' , respectively.

II. THE LEWIS–TOLMAN SYMMETRIC COLLISION

A totally inelastic variant of the Lewis–Tolman symmetric collision is shown in Fig. 1. Two identical particles, each of mass m , collide and coalesce to form an object of mass M . The same collision is pictured from two points of view: in Fig. 1(a) as observed in frame S in which particle 1 moves in the $-y$ direction with speed w , and in Fig. 1(b) as observed in frame S' in which particle 2 moves in the $+y$ direction with the same speed w . The frames S' and S are related by a boost of speed u (with $u < c$) in the x direction as shown. The velocities of particle 2 in frame S and of particle 1 in frame S' are found by using the Lorentz velocity transformation formulas.

One can imagine that such a collision could be brought about by use of two identical particle launchers, one at rest in S and pointing in the $-y$ direction, and one at rest in S' and pointing in the $+y$ direction. If the particles are launched at the proper instants they will collide.

The symmetry of the collision constrains the velocity of the final-state particle. This particle will have no y velocity because the launchers and initial particles are identical.¹⁴ So this particle must move along the x axis, and its speed U must be the same as observed in S as from S' .¹⁵ The velocity transformation law constrains U according to

$$U = \frac{u - U}{1 - uU/c^2}. \quad (7)$$

The solution for U in terms of u is

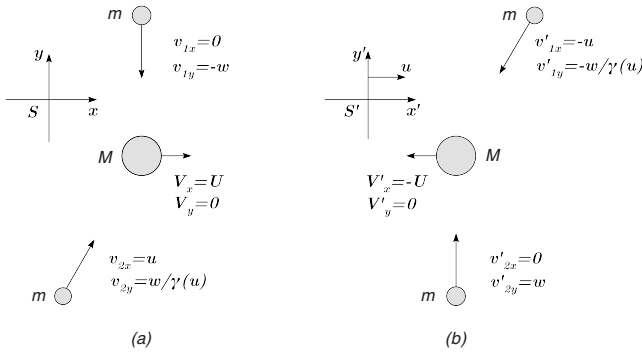


Fig. 1. The totally inelastic Lewis–Tolman symmetric collision in which two identical particles of mass m collide and coalesce to form a particle of mass M . (a) In frame S particle 1 approaches in the $-y$ direction with speed w . (b) In frame S' particle 2 approaches in the $+y$ direction with the same speed w . Frame S' moves in the x direction with velocity u with respect to frame S .

$$U = \frac{c^2}{u} (1 - \sqrt{1 - u^2/c^2}) = \frac{u}{1 + \sqrt{1 - u^2/c^2}}. \quad (8)$$

The requirement of momentum conservation in the y direction leads to the expression for relativistic momentum. Conservation of y momentum in frame S implies that

$$-mwf(w) + m\left(\frac{w}{\gamma(u)}\right)f(\sqrt{u^2 + w^2/\gamma^2(u)}) = 0, \quad (9)$$

which must hold for all values of u and w . We divide by mw and rearrange terms to find

$$f(\sqrt{u^2 + w^2/\gamma^2(u)}) = \gamma(u)f(w). \quad (10)$$

Because Eq. (10) holds for all $w > 0$,¹⁶ we can take the $w \rightarrow 0$ limit and use the assumed continuity of $f(w)$ along with the value $f(0) = 1$ to arrive at the standard result

$$f(u) = \gamma(u). \quad (11)$$

It is instructive to show that

$$\gamma(\sqrt{u^2 + w^2/\gamma^2(u)}) = \gamma(u)\gamma(w), \quad (12)$$

so that Eqs. (9) and (10) hold for all values of u and w and not just in the $w \rightarrow 0$ limit. We note that the final particle mass M does not enter into our considerations because M has no y velocity.

The requirements of x momentum conservation and energy conservation lead to the expression for relativistic energy. Conservation of x momentum (in frame S) gives

$$muf(\sqrt{u^2 + w^2/\gamma^2(u)}) = MUf(U). \quad (13)$$

We use Eqs. (11) and (12) in Eq. (13) to express the final particle mass as

$$M = m\gamma(w)\frac{u\gamma(u)}{U\gamma(U)}. \quad (14)$$

Conservation of energy (in frame S) gives

$$mc^2g(w) + mc^2g(\sqrt{u^2 + w^2/\gamma^2(u)}) = Mc^2g(U). \quad (15)$$

We eliminate M by use of Eq. (14) to find

$$g(w) + g(\sqrt{u^2 + w^2/\gamma^2(u)}) = g(U)\gamma(w)\frac{u\gamma(u)}{U\gamma(U)}. \quad (16)$$

The functional relation for g given in Eq. (16) must hold for all u and w . Specifically, we take the limit $u \rightarrow 0$ and divide by two to find

$$g(w) = g(0)\gamma(w), \quad (17)$$

because $U \rightarrow 0$ and $u/U \rightarrow 2$ as $u \rightarrow 0$. We can identify the value of $g(0)$ by expanding both sides of Eq. (17) to order w^2 . The known expansion of g is given in Eq. (4), and $\gamma(w) = (1 - w^2/c^2)^{-1/2} = 1 + w^2/2c^2 + \dots$, and we find

$$g(0) + \frac{1}{2}\frac{w^2}{c^2} + \dots = g(0)\left(1 + \frac{1}{2}\frac{w^2}{c^2} + \dots\right). \quad (18)$$

Upon matching the coefficients of w^2 in Eq. (18) we see that $g(0) = 1$, and so

$$g(v) = \gamma(v). \quad (19)$$

By making use of Eq. (8) and Eq. (12), we can verify that the functional relation Eq. (16) is solved by Eq. (19) for all values of u and w .

Our consideration of the $u \rightarrow 0$ limit of Eq. (16) was useful for two reasons: it led us to the solution Eq. (19) without having to guess the solution or have prior knowledge, and it showed that Eq. (19) is the only possible solution for Eq. (16).

III. DISCUSSION

We have shown that the Lewis–Tolman symmetrical collision can be used to deduce the relativistic energy formula $E = mc^2\gamma(v)$ and the expression $\vec{p} = m\vec{v}\gamma(v)$ for relativistic momentum. In standard treatments the Lewis–Tolman collision is used for momentum alone, and the energy is handled differently. A unified approach such as the one given here has the advantage of treating the closely related concepts of momentum and energy in a parallel fashion. The intimate connection between momentum and energy is also central to the four-vector approach, but our method more directly illustrates the conservation of these quantities in collisions. Once the Lewis–Tolman collision has been described in a discussion of relativistic momentum, the expression for relativistic energy can be easily obtained as well.

Many treatments of relativistic momentum use the concept of relativistic mass $m(v)$ and define a conserved momentum $\vec{p} = m(v)\vec{v}$. This definition is equivalent to our Eq. (2a) with $m(v) = mf(v)$. Many authors make the additional assumption that the relativistic mass itself is conserved in collisions. With this additional assumption, the expression $m(v) = m\gamma(v)$ can be obtained efficiently from a simple longitudinal collision.^{5,17–22} In fact, conservation of relativistic mass is not an independent assumption: it can be derived from momentum conservation for a longitudinal collision as observed in a frame with an infinitesimal transverse velocity.²³ Our view is that the use of relativistic mass detracts from the deep connection between momentum and energy that is evident in the parallel conservation laws and particularly in the spacetime based four-vector formalism by unnecessarily introducing yet another velocity-dependent conserved quantity. Various views on the topic of relativistic mass are found in Refs. 24–27.

We have made the explicit assumption that the energy of an object is proportional to its mass, where the mass parameter represents inertia and is measured by the usual nonrelativistic techniques. More general assumptions can be made,^{28–30} and it is found that the energy can have the more general form $E = \alpha\gamma(v) + q$, where α is a constant, v the speed, and q a conserved scalar. The choice $q = 0$ is simplest and is consistent with observation.

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¹⁴Symmetry and the principle of relativity constrain the velocity of the final

state particle M . If M had a nonvanishing y velocity, it would move either in the direction of the particle launched vertically in frame S (if $V_y < 0$), or in the direction of the particle launched vertically in frame S' (if $V_y > 0$). A nonzero value of V_y would indicate the presence of a preferred frame, either S or S' , contradicting the principle of relativity.

¹⁵The equality of $|V_x|$ and $|V'_x|$ in Fig. 1 is also a consequence of symmetry and the principle of relativity, because if $|V_x| \neq |V'_x|$, say $|V_x| > |V'_x|$, then we could distinguish frame S from frame S' in a way that would violate the principle of relativity. The velocity transformation Eq. (6a) with $v_{||} = V_x = U$ and $v'_{||} = V'_x = -U$ gives Eq. (7).

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