2 Theoretical Foundations

This chapter introduces variable structure systems and their application for the design of sliding mode controllers. The goal is to build a general understanding of the theoretical framework for sliding mode techniques that is also later used in the design of SMOs.

2.1 An Overview of Variable Structure Systems

2.1.1 Brief History

The theory of variable structure systems was born from the work of Emelyanov [16] and several co-researchers in the former Soviet Union during the late 1960's. However, as all of the research material was published in Russian, the ideas surrounding variable structure systems only became easily available in the mid 1970's, after the publication of articles in English [33].

The methodology of variable structure systems was then quickly recognized for its potential as a robust control method, that is, to project controllers capable of guaranteeing stability and performance even in the presence of mismatches between the real system and its mathematical model. Since then, variable structure systems were successfully implemented in many fields of control engineering such as electrical motors [34], chemical processes [7], robotic arms [19], helicopters [37] and power plants [8].

2.1.2 Variable Structure Systems

Variable structure systems (VSSs) are a particular class of nonlinear systems of the form

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{f_1}(\mathbf{x}, t), \text{ for } \mathbf{x} \in \mathcal{X}_1 \\ \mathbf{f_2}(\mathbf{x}, t), \text{ for } \mathbf{x} \in \mathcal{X}_2 \\ \dots \\ \mathbf{f_k}(\mathbf{x}, t), \text{ for } \mathbf{x} \in \mathcal{X}_k, \end{cases} \tag{2.1}$$

where $\mathbf{x} := (x_1, x_2, \dots, x_n)^\top \in \mathbb{R}^n$ is the state vector, $t \in \mathbb{R}_+$ is the time variable and $\mathbf{f_i} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a continuous vector function with

$$\mathbf{f_i}(\mathbf{x}, t) := \big(f_{i,1}(\mathbf{x}, t), f_{i,2}(\mathbf{x}, t), \dots, f_{i,n}(\mathbf{x}, t)\big)^\top, \forall i \in \{1, 2, \dots, k\}.$$

The name variable structure system comes from the piece-wise nature of the system function. As one can see in Equation (2.1), for each one of the defined regions of the state space $(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_k)$ the dynamics of the system are described by a different continuous nonlinear vector function $\left(\mathbf{f_1}(\mathbf{x}, t), \mathbf{f_2}(\mathbf{x}, t), \ldots, \mathbf{f_k}(\mathbf{x}, t)\right)$, and at the boundaries of these regions, their dynamics switch abruptly. More information about variable structure systems can be found in [25], [12], [22], [40], [30] and [23].

2.1.3 Analysis Technique

Given the piece-wise nature of VSSs, the most reasonable way to represent such systems is with the state space representation given in Equation (2.1). Consequently, traditional control theory methods that are based on transfer functions, such as the root locus and Bode-diagrams, cannot be used to analyse such systems. Thus, one of the most common techniques to analyse VSSs involves interpreting their associated phase portrait, which can be constructed with the following the procedure:

- First consider each subsystem $\dot{\mathbf{x}} = \mathbf{f_i}(\mathbf{x}), \forall i \in \{1, \ldots, k\}$ separately, and draw its individual phase portrait for $\mathbf{x} \in \mathbb{R}^n$.
- Now consider the partitions $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_k$ and draw on a new phase portrait the boundaries of each partition.
- Finally, draw in each partition the corresponding section of the individual phase portrait associated with this partition.

Example 2.1. *To illustrate the analysis technique defined above, the following example is given. Consider an $n = 2$ dimensional VSS with $k = 2$ linear structures:*

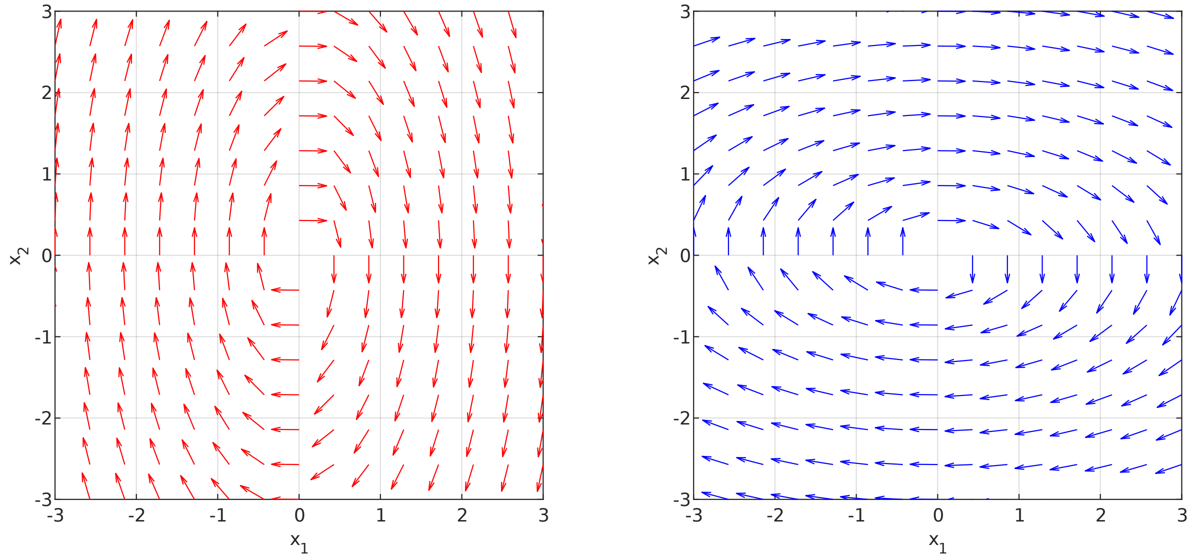
$$\dot{\mathbf{x}} = \begin{cases} \mathbf{A_1}\,\mathbf{x}, & \text{for } x_1 x_2 \geq 0 \\ \mathbf{A_2}\,\mathbf{x}, & \text{for } x_1 x_2 < 0, \end{cases} \tag{2.2}$$

where

$$\mathbf{A_1} = \begin{bmatrix} 0 & 1 \\ -4 & -0.01 \end{bmatrix} \quad \text{and} \quad \mathbf{A_2} = \begin{bmatrix} 0 & 1 \\ -0.3 & -0.01 \end{bmatrix}.$$

The first step of the procedure is to draw the phase portraits of each subsystem separately, as shown in Figure 2.1. The second step is to partition the state space, as one can see in the background of Figure 2.2, where $\mathcal{X}_1 = \{\mathbf{x} \in \mathbb{R}^n : x_1 x_2 \geq 0\}$ is colored in red and $\mathcal{X}_2 = \{\mathbf{x} \in \mathbb{R}^n : x_1 x_2 < 0\}$ is colored in blue.

The final step is to draw in each partition the section of the associated phase portrait, the resulting combined phase portrait that describes the VSS of Equation (2.2) *is shown in Figure 2.2. Other examples similar to this one can be viewed in [33].*



(a) Phase Portrait for Structure A_1

(b) Phase Portrait for Structure A_2

Figure 2.1: Individual Phase Portraits

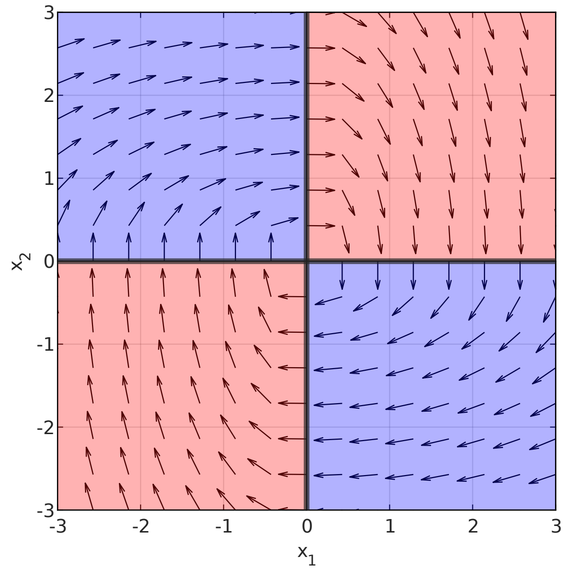


Figure 2.2: Combined Phase Portrait

2.1.4 Existence and Uniqueness

Another adaptation that had to be done in the field of VSSs, are the theorems concerning the existence and uniqueness of solutions to systems of differential equations such as Equation (2.1). As the system function is not continuous, it is clearly also not Lipschitz continuous, and therefore the classical Picard–Lindelöf theorem cannot be used.

To solve this issue, a new theory about differential equations with discontinuous right-hand side was developed. The idea involves constructing a solution as the "average" of the solutions that are obtained by approaching the point of discontinuity from different directions, this new solution is then called a Filipov Solution [17].

2.1.5 Variable Structure Control

The theory of VSSs can also be used as a method to design nonlinear controllers. This is known as variable structure control (VSC), and it is usually implemented for the control of linear systems, although it can also be applied to some types of nonlinear systems.

VSC is characterized by two elements: a set of continuous control structures and a switching logic, as shown in Figure 2.3.

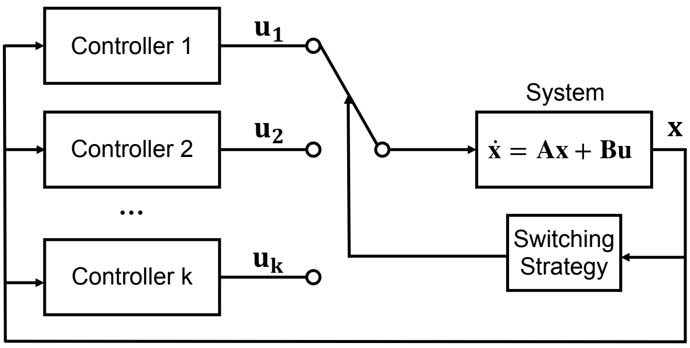


Figure 2.3: VSC General Model [1]

The switching logic chooses which one of the control structures is applied to the system, based on the current state of the system. Combining those two concepts, this elaborate control law can be expressed in a single discontinuous control function as

$$\mathbf{u}(\mathbf{x}, t) = \begin{cases} \mathbf{u_1}(\mathbf{x}, t), \text{ for } \mathbf{x} \in \mathcal{X}_1 \\ \mathbf{u_2}(\mathbf{x}, t), \text{ for } \mathbf{x} \in \mathcal{X}_2 \\ \dots \\ \mathbf{u_k}(\mathbf{x}, t), \text{ for } \mathbf{x} \in \mathcal{X}_k, \end{cases} \tag{2.3}$$

where $\mathbf{u_i}(\mathbf{x}), \forall i \in \{1, \dots, k\}$ is a continuous control function.

By applying the control law defined in Equation (2.3) in the dynamics of the linear system Σ, it is clear that

$$\dot{\mathbf{x}}(t) = \begin{cases} \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u_1}(\mathbf{x},t), & \text{for}\,\mathbf{x} \in \mathcal{X}_1 \\ \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u_2}(\mathbf{x},t), & \text{for}\,\mathbf{x} \in \mathcal{X}_2 \\ \dots \\ \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u_k}(\mathbf{x},t), & \text{for}\,\mathbf{x} \in \mathcal{X}_k \end{cases} \implies \dot{\mathbf{x}} = \begin{cases} \mathbf{f_1}(\mathbf{x},\mathbf{u}), & \text{for}\,\mathbf{x} \in \mathcal{X}_1 \\ \mathbf{f_2}(\mathbf{x},\mathbf{u}), & \text{for}\,\mathbf{x} \in \mathcal{X}_2 \\ \dots \\ \mathbf{f_k}(\mathbf{x},\mathbf{u}), & \text{for}\,\mathbf{x} \in \mathcal{X}_k, \end{cases} \tag{2.4}$$

and therefore the resulting closed loop system is a VSS, that can be subsequently analyzed with VSS techniques.

One possible VSC design method is to use state feedback controllers to place the poles of each subsystem, and then choose the regions of the state space in which each controller is applied. This two step process can be described as the shaping of the trajectories of each subsystem and then the combination of those trajectories into one phase plane.

This design freedom of VSC is what give its status as a powerful control technique. VSC were also shown to yield better performance than simple linear controllers, and at the same time being easier to implement than time optimal controllers [1]. The content of this section was motivated by the following sources: [42], [38], [12] and [27].

2.2 Introduction to Sliding Mode Control

Sliding Mode Control (SMC) is the main form of implementation of VSC. The goal of SMC is to design a VSS in which the trajectories are induced to slide along a section of the state space. For an in-depth introduction about sliding mode controllers, the following books are advice [27], [24], [6], [12], [22] and [40].

An example of the phase portrait of a VSS devised with SMC techniques is shown in Figure 2.4. One can see that the trajectories of the system are forced to approach and stay constricted to the section of the state space denoted in red.

2.2.1 Main Concepts and Definitions

This section establishes the main concepts of the theory of conventional sliding mode controllers. Elaborate implementations of SMC such as higher order, integral and super-twisting sliding mode controller design can be viewed in [27] and [40].

Because SMC is a particular type of VSC, the characterization of SMC not only involves the description of the subspace in which the sliding takes place, but also involves the specification of the switching logic and the set of continuous control structures. In order to accomplish this, the following definitions are made:

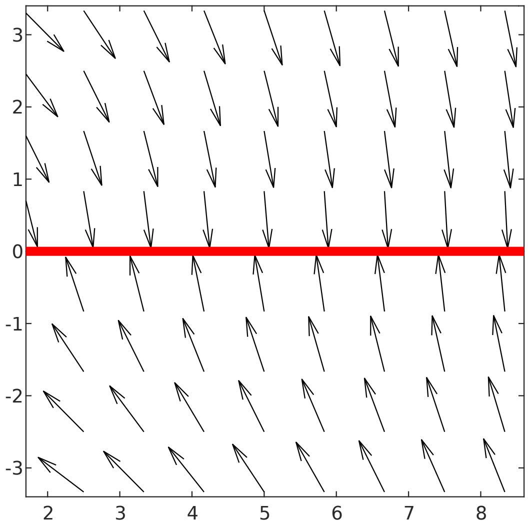


Figure 2.4: Depiction of Sliding Mode

Definition 2.1 (Sliding Variable)**.** *The sliding variable $\boldsymbol{\sigma}$, sometimes also called switching function, is defined as the linear vector function $\boldsymbol{\sigma}(\mathbf{x}) = \big(\sigma_1(\mathbf{x}), \sigma_2(\mathbf{x}), \dots, \sigma_m(\mathbf{x})\big)^\top : \mathbb{R}^n \to \mathbb{R}^m$. It is used as the basis for the definitions of the switching logic and sliding surface.*

Definition 2.2 (Sliding Surface)**.** *The sliding surface $\mathcal{S}$, also called sliding hyperplane, is defined as the subspace $\mathcal{S} := \{\mathbf{x} \in \mathbb{R}^n : \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0}\}$. It is the section of the state space to which the trajectories of the VSS system are supposed to converge. The sliding surface $\mathcal{S}$ can also be described as $\mathcal{S} = \bigcap_{i=1}^{m} \mathcal{S}_i$, with $\mathcal{S}_i := \{\mathbf{x} \in \mathbb{R}^n : \sigma_i = 0\}, \forall i \in \{1, \dots, m\}$.*

The switching logic of SMC is characterized by the sign of each component of the sliding variable $\sigma_i : \mathbb{R}^n \to \mathbb{R}$, and it is therefore associated with the following partitioning of the state space:

$$
\begin{aligned}
&\mathcal{X}_1 := \{\mathbf{x} \in \mathbb{R}^n : \sigma_1(\mathbf{x}) > 0, \sigma_2(\mathbf{x}) > 0, \dots, \sigma_m(\mathbf{x}) > 0\},\\
&\mathcal{X}_2 := \{\mathbf{x} \in \mathbb{R}^n : \sigma_1(\mathbf{x}) < 0, \sigma_2(\mathbf{x}) > 0, \dots, \sigma_m(\mathbf{x}) > 0\},\\
&\dots,\\
&\mathcal{X}_{2^m} := \{\mathbf{x} \in \mathbb{R}^n : \sigma_1(\mathbf{x}) < 0, \sigma_2(\mathbf{x}) < 0, \dots, \sigma_m(\mathbf{x}) < 0\}.
\end{aligned}
$$

Lastly, the control structure is given by $k = 2^m$ state-feedback controllers:

$$
\mathbf{u}\big(\mathbf{x}(t)\big) = \begin{cases}
\mathbf{u}_1(\mathbf{x}(t)) : \mathbb{R}^n \to \mathbb{R}^m, \text{ for } \mathbf{x} \in \mathcal{X}_1\\
\mathbf{u}_2(\mathbf{x}(t)) : \mathbb{R}^n \to \mathbb{R}^m, \text{ for } \mathbf{x} \in \mathcal{X}_2\\
\dots\\
\mathbf{u}_{2^m}(\mathbf{x}(t)) : \mathbb{R}^n \to \mathbb{R}^m, \text{ for } \mathbf{x} \in \mathcal{X}_{2^m}.
\end{cases}
$$

2.2.2 Existence and Finite Time Rechability

After the presentation of the basic components characterizing SMC, the following section addresses the mathematical theorems that constitute the theoretical foundations of SMC. The most important aspects that are analysed in this section include the conditions for which the system trajectories converge to the sliding surface and the speed of this convergence.

Theorem 2.1 (Existence and finite time reachability). *If the dynamics of the system guarantee that $\boldsymbol{\sigma}^\top(\mathrm{x})\,\dot{\boldsymbol{\sigma}}(\mathrm{x}) \leq -\mu'||\boldsymbol{\sigma}(\mathrm{x})||$ then the trajectories of the system reach the sliding surface $\mathcal{S}$ in finite time and remain there thereafter.*

The following proof of Theorem 2.1 is based on the work of [35]. It is done in two parts, with each part presenting an adaptation of the classical Lyapunov stability theorem. The first part is the proof of existence of the sliding mode; it uses a generalization about the application of Lyapunov functions to not only points (the usual stationary points, or equilibrium points), but also to hyperplanes in the state space. The second part is the finite time rechability to the sliding surface; it employs a stronger restriction on the time-derivative of Lyapunov functions that allows for convergence rates faster than asymptotic, including even finite time convergence.

Proof (Existence). To show that the trajectories of the system converge to the sliding surface $\mathcal{S}$, consider the following Lyapunov function candidate:

$$V\big(\boldsymbol{\sigma}(\mathrm{x})\big) = \frac{1}{2}\boldsymbol{\sigma}^\top(\mathrm{x})\,\boldsymbol{\sigma}(\mathrm{x}) = \frac{1}{2}||\boldsymbol{\sigma}(\mathrm{x})||^2. \tag{2.5}$$

As $\boldsymbol{\sigma}(\mathrm{x}) = \mathbf{0}$ defines the sliding surface $\mathcal{S}$, it is intuitive to think about $||\boldsymbol{\sigma}(\mathrm{x})||$ as representing the "distance" that the current state x is from $\mathcal{S}$. The goal of this proof is to show that Equation (2.5) is in fact a valid Lyapunov function, which means that as the time passes this "distance" becomes smaller until reaching zero, and staying so thereafter.

First, the basic requisites are shown to hold:

(R1). $\boldsymbol{\sigma}(\mathrm{x}) = \mathbf{0} \Leftrightarrow V(\mathbf{0}) = 0, \forall \mathbf{x} \in \{\mathbf{x} \in \mathbb{R}^n : \boldsymbol{\sigma}(\mathrm{x}) = \mathbf{0}\}$,
(R2). $V(\mathrm{x}) = \frac{||\boldsymbol{\sigma}(\mathrm{x})||^2}{2} > 0, \forall \mathbf{x} \notin \{\mathbf{x} \in \mathbb{R}^n : \boldsymbol{\sigma}(\mathrm{x}) = \mathbf{0}\}$.

Thus, by the Lyapunov stability theorem, a sufficient condition for existence is that

$$\dot{V}(\mathbf{x}) \leq 0, \tag{2.6}$$

which in the context of sliding mode can be written in terms of the Lyapunov function candidate defined in Equation (2.5) as

$$\boldsymbol{\sigma}^\top\,\dot{\boldsymbol{\sigma}} \leq 0. \tag{2.7}$$

Therefore, if the condition $\boldsymbol{\sigma}^\top\,\dot{\boldsymbol{\sigma}} \leq 0$ is met, then the trajectories of the system convergence asymptotically to the set $\{\mathbf{x} \in \mathbb{R}^n : V(\mathbf{x}) = 0\} = \{\mathbf{x} \in \mathbb{R}^n : \boldsymbol{\sigma}(\mathbf{x}) = 0\} = S$. ■

Now, to show that the sliding surface is reachable in finite time, one needs to bound the time derivative of the Lyapunov function strongly away from zero. This idea comes from the fact that by forcing the time derivative to be negative but large in magnitude, it is possible to achieve faster convergence rates than when the derivative is negative but small in value, almost vanishing, with the convergence being at most asymptotic.

Theorem 2.2 (Finite time convergence of Lyapunov Functions)**.** *To obtain finite time convergence, the third Lyapunov condition needs to be strengthened to* $\dot{V}(\mathbf{x}) \leq -\mu\sqrt{V(\mathbf{x})}$, *where* $\mu > 0$.

Proof (Finite time convergence of Lyapunov Functions). To prove that this strengthened condition does in fact lead to finite time convergence, the comparison lemma can be used. Let $W(\mathbf{x}) = 2\sqrt{V(\mathbf{x})}$, by the chain rule it is obtained that

$$\frac{dW}{dt} = \frac{dW}{dV}\frac{dV}{dt} = 2\frac{V^{-1/2}}{2}\dot{V}(\mathbf{x}) = \frac{\dot{V}(\mathbf{x})}{\sqrt{V(\mathbf{x})}}.$$

Thus, the above defined condition in Theorem 2.2 can then be simplified to

$$\dot{V}(\mathbf{x}) \leq -\mu\sqrt{V(\mathbf{x})} \implies \frac{\dot{V}(\mathbf{x})}{\sqrt{V(\mathbf{x})}} \leq -\mu \implies \dot{W}(\mathbf{x}) \leq -\mu.$$

Therefore, when the following differential equation is considered

$$\begin{aligned} \dot{z}(t) &= -\mu \\ z(0) &= z_0, \end{aligned}$$

with solution

$$z(t) = z_0 - \mu t, \forall t \geq 0,$$

by the comparison lemma, it must be the case that the similarly defined problem

$$\begin{aligned} \dot{W}(t) &\leq -\mu \\ W(0) &= W_0, \end{aligned}$$

has to have a solution, such that

$$W(\mathbf{x}(t)) \leq W_0 - \mu t, \forall t \geq 0.$$

Substituting the definition of $W(\mathbf{x})$ in the last equation leads to

$$2\sqrt{V(\mathbf{x}(t))} \leq 2\sqrt{V(\mathbf{x}(0))} - \mu t, \forall t \geq 0,$$

but, as $\sqrt{V} \geq 0$, it is easy to see that $\sqrt{V}$ must reach $\sqrt{V(t_\mathrm{r})} = 0$ for a finite time $t_\mathrm{r} \leq 2\frac{\sqrt{V(\mathbf{x}(0))}}{\mu}$, and so does $V(\mathbf{x})$ as well. Therefore it has been proven that the modification of the Lyapunov condition given in Equation (2.6) to the condition in Theorem 2.2 implies that finite time convergence is achieved. ■

Proof (Existence and finite time reachability). The condition given in Theorem 2.2 can then be re-written in terms of the Lyapunov candidate given in Equation (2.5) as

$$\boldsymbol{\sigma}(\mathbf{x})^\top \dot{\boldsymbol{\sigma}}(\mathbf{x}) \leq -\mu \frac{||\boldsymbol{\sigma}(\mathbf{x})||}{\sqrt{2}},$$

with $\mu > 0$, and by defining $\mu' := \frac{\mu}{\sqrt{2}}$, the above equation can be simplified to

$$\boldsymbol{\sigma}(\mathbf{x})^\top \dot{\boldsymbol{\sigma}}(\mathbf{x}) \leq -\mu' ||\boldsymbol{\sigma}(\mathbf{x})||. \tag{2.8}$$

■

Corollary 2.2.1 (Special case: $m = 1$). *Considering the case when the sliding variable defined in Equation* (2.1) *is a one dimensional function, a simpler form of Condition* (2.8) *can be derived as*

$$\sigma\dot{\sigma} \leq -\mu'|\sigma| \implies \operatorname{sign}(\sigma)\dot{\sigma} \leq -\mu'. \tag{2.9}$$

where sign *denotes the sign function, defined as*

$$\operatorname{sign}(\sigma) := \begin{cases} 1, \text{ for } \sigma > 0 \\ 0, \text{ for } \sigma = 0 \\ -1, \text{ for } \sigma < 0. \end{cases} \tag{2.10}$$

Additionally, Equation (2.9) *can be separated into the following conditions:*

$$\begin{aligned} &\operatorname{sign}(\dot{\sigma}) \neq \operatorname{sign}(\sigma), \\ &|\dot{\sigma}| \geq \mu' > 0, \end{aligned} \tag{2.11}$$

which can be interpreted as: the first condition $\operatorname{sign}(\dot{\sigma}) \neq \operatorname{sign}(\sigma)$ *guarantees that the trajectories of the system move towards the sliding surface; and the second condition* $|\dot{\sigma}| \geq \mu' > 0$ *guarantees that the convergence rate to the sliding surface always has a lower bound greater than zero.*

Corollary 2.2.2 (Region of attraction). *The Lyapunov stability theorem also allows the determination of the largest invariant set for which the sliding surface* $\mathcal{S}$ *is reachable. This is called the region of attraction* $\mathcal{A}$ *and it is given by*

$$\mathcal{A} := \{\mathbf{x} \in \mathbb{R}^n : \boldsymbol{\sigma}^\top \dot{\boldsymbol{\sigma}} \leq 0\}.$$

If the initial conditions of the system belong to this subspace, then the Lyapunov candidate of Equation (2.5) *is in fact a Lyapunov function and the trajectories of the system stay inside this subspace and move towards the sliding surface.*

Moreover, if Condition (2.8) *is satisfied, then it is also guaranteed that the sliding surface will be attained in finite time, because* $\mathcal{F} := \{\mathbf{x} \in \mathbb{R}^n : \boldsymbol{\sigma}^\top \dot{\boldsymbol{\sigma}} < -\mu'||\boldsymbol{\sigma}(\mathbf{x})||\} \subseteq \mathcal{A}$*, and therefore*

$$\mathbf{x_0} \in \mathcal{A} \implies \mathbf{x}(t) \in \mathcal{F}, \forall t > t_1 \implies \mathbf{x}(t) \in \mathcal{S}, \forall t > t_2, \text{ with } t_2 > t_1 \in \mathbb{R}_+.$$

Having established the condition for finite time convergence, the follow definitions are motivated:

Definition 2.3 (Sliding Mode)**.** *It is said that the system is in sliding mode, or that the sliding phase was reached, when the states of the systems enter and remain in the sliding surface $\mathcal{S}$. The interpretation that the trajectories of the system are sliding on this surface, gives the method its name.*

Definition 2.4 (Reaching Phase)**.** *It is said that the system is in the reaching phase, before the system enters the sliding mode. Furthermore, the time that it takes for the system to reach the sliding phase is called the reaching time t_{r}.*

2.2.3 Basic Design Procedure

Now that the basic structure and theoretical foundation of sliding mode techniques are introduced, this section describes a basic procedure to design SMCs. Consider the following linear system as the system to be controlled:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t), \end{aligned} \tag{2.12}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} \in \mathbb{R}^m$ is the input vector, $\mathbf{y} \in \mathbb{R}^p$ is the output vector, $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the system matrix, $\mathbf{B} \in \mathbb{R}^{n\times m}$ is the input matrix and $\mathbf{C} \in \mathbb{R}^{p\times n}$ is the output matrix.

Note that the system has no feedthrough matrix, and throughout this section it is assumed without loss of generality that $\mathbf{B}$ is a full rank matrix. This means that each one of the inputs u_i has a linearly independent effect on the system.

Selection of the Sliding Surface

The first step in the design of SMCs is the selection of a subspace from the state space to be the sliding surface $\mathcal{S}$. Let $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ be the associated sliding variable, that can then be generically represented via a matrix $S \in \mathbb{R}^{m\times n}$ as

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{S}\mathbf{x} = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \dots & & & \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} s_{11}x_1 + s_{12}x_2 + \dots + s_{1n}x_n \\ s_{21}x_1 + s_{22}x_2 + \dots + s_{2n}x_n \\ \dots \\ s_{m1}x_1 + s_{m2}x_2 + \dots + s_{mn}x_n \end{bmatrix} := \begin{bmatrix} \sigma_1(x) \\ \sigma_2(x) \\ \dots \\ \sigma_m(x) \end{bmatrix}. \tag{2.13}$$

The sliding surface $\mathcal{S}$ can then be considered as the nullspace of the linear transformation associated with the matrix $S \in \mathbb{R}^{m\times n}$, that is $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Sx} = 0\}$, which can also be represented as the set of solutions of the following homogeneous linear system:

$$\begin{cases} s_{11}x_1 + s_{12}x_2 + \ldots + s_{1n}x_n = 0 \\ s_{21}x_1 + s_{22}x_2 + \ldots + s_{2n}x_n = 0 \\ \ldots \\ s_{m1}x_1 + s_{m2}x_2 + \ldots + s_{mn}x_n = 0. \end{cases} \tag{2.14}$$

Given that generally $n > m$, from linear algebra it is known that $0 \leq \text{rank}(\mathbf{S}) \leq m$. Assuming that $\text{rank}(\mathbf{S}) = m$, which is a reasonable assumption, because this is to say that none of the hyperplane can be expressed as a linear combination of the others (i.e., all hyperplanes $\sigma_i(\mathbf{x}) = 0, \forall i \in \{1, \ldots, m\}$ are linearly independent) one can then perform Gaussian elimination to solve System (2.14) and get $\mathbf{x_h} \in \mathbb{R}^n$ with $n - m$ degrees of freedom as a solution.

Therefore, the general n-dim solution of the System (2.12) now only has $n - m$ degrees of freedom, because m of the original degrees of freedom are locked by the algebraic equation $\mathbf{Sx} = \mathbf{0}$. This means that the trajectories $\mathbf{x}(t)$ that could live inside of the whole $\mathbb{R}^n$ are being forced to converge to trajectories $\mathbf{x_h}(t)$ that live inside of $\mathcal{S} \subseteq \mathbb{R}^n$.

Thus, by selecting the sliding variable $\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{Sx}$, the designer is choosing the surface to which the dynamics of the System (2.12) will be reduced. This restriction of the trajectories of the system, to be confined inside an a priori chosen surface is seen as advantageous, because just like a marble rolling down a pre-established path, the designer is able to choose a surface on which the system trajectories exhibit a desirable behaviour.

Once the system reaches this surface, the general nonlinear behaviour of VSS systems gets simplified to the reduced order linear dynamics of the system inside the sliding surface, which are easier to analyse.

Calculation of the control gains

The second step in the design of SMCs is to find the appropriate feedback-law that changes the dynamics of the system to ensure the trajectories of the system reach and stay on the specified sliding surface. To do this, an approach similar to control Lyapunov functions is used.

The Lyapunov function of Equation (2.5) is now considered as a control Lyapunov function, and the goal is to find a control signal $\mathbf{u}$ that makes this Lyapunov function satisfy Condition (2.8). Given the assumptions that both $\mathbf{B}$ and $\mathbf{S}$ are full rank matrices, from linear algebra it is known that $(\mathbf{SB})^{-1}$ exists. The following theorem can then be formulated:

Theorem 2.3 (Sliding mode control effort). *If $\rho > 0$, then the following feedback law guarantees the existence, reachability and holdability to the sliding surface $\mathcal{S}$:*

$$\mathbf{u}(\mathbf{x}) = (\mathbf{SB})^{-1}\left(-\mathbf{SAx} - \rho\frac{\mathbf{Sx}}{||\mathbf{Sx}||}\right). \tag{2.15}$$

Proof. (Sliding mode control effort). From the definition of the sliding variable given in Equation (2.13) and the linear system of Equation (2.12), it is clear that $\mathbf{u}$ has a direct impact on $\dot{\boldsymbol{\sigma}}$, as

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{S}\mathbf{x} \implies \dot{\boldsymbol{\sigma}}(\mathbf{x}) = \mathbf{S}\dot{\mathbf{x}} = \mathbf{S}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) = \mathbf{S}\mathbf{A}\mathbf{x} + \mathbf{S}\mathbf{B}\mathbf{u}. \tag{2.16}$$

Calculating $\boldsymbol{\sigma}^\top\dot{\boldsymbol{\sigma}}$ with the control function of Equation (2.15) substituted in Equation (2.16) results in

$$\boldsymbol{\sigma}^\top\dot{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^\top(\mathbf{S}\mathbf{A}\mathbf{x} + \mathbf{S}\mathbf{B}\mathbf{u}) = \boldsymbol{\sigma}^\top\left(\mathbf{S}\mathbf{A}\mathbf{x} + \mathbf{S}\mathbf{B}(\mathbf{S}\mathbf{B})^{-1}\left(-\mathbf{S}\mathbf{A}\mathbf{x} - \rho\frac{\mathbf{S}\mathbf{x}}{||\mathbf{S}\mathbf{x}||}\right)\right),$$

simplifying the expression and using again definition of the sliding variable given in Equation (2.13) leads to

$$\boldsymbol{\sigma}^\top\dot{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^\top\left(-\rho\frac{\boldsymbol{\sigma}}{||\boldsymbol{\sigma}||}\right) = -\rho\frac{\boldsymbol{\sigma}^\top\boldsymbol{\sigma}}{||\boldsymbol{\sigma}||} = -\rho||\boldsymbol{\sigma}||.$$

Because of the condition that $\rho > 0$, there does exists an $\mu' > 0$ such that $\rho \geq \mu' > 0$. Therefore, the control signal of Equation (2.15) does indeed satisfies Condition (2.8). ■

Remark 2.3.1 (On the control signal)**.** *Intuitively, the sliding mode controller acts like a stiff pressure that uses each component u_i of the input to push the states of the system inside the respective $\sigma_i = 0$ subspace, with the culmination of these being the trajectories of the system restricted to the sliding surface $\mathcal{S}$.*

Also note that, the control signal is discontinuous along $\mathbf{S}\mathbf{x} = \mathbf{0}$, because as the trajectories cross the sliding surface $\mathcal{S}$, it needs to abruptly change value to be able to guarantee that the time-derivative $\dot{\boldsymbol{\sigma}}$ is firmly bounded away from zero.

Remark 2.3.2 (On the design procedure)**.** *It is important to pay attention to the order in which things are fixed. By first selecting the sliding surface, it is already established how the system trajectories behave in the sliding phase. This can raise some doubts, because the solution of the system equations is defined before one actually solves them. However, the trick is to notice that it is possible to find an input signal that forces this desirable solution to become the actual solution. Therefore, in a sense, first the solution is chosen, and then the input that produces this solution is calculated.*

Remark 2.3.3 (Special case: $m = 1$)**.** *For the particular case in which the linear System* (2.12) *has only one input signal, the above derived feedback-law can be simplified. Notice that for $m = 1$ the sliding variable σ becomes one dimensional, and then the vector treatment of this variable can now be simplified to a scalar treatment; the Euclidian vector norm becomes the absolute value, and this leads to*

$$u(\mathbf{x}) = -(\mathbf{S}\mathbf{B})^{-1}\mathbf{S}\mathbf{A}\mathbf{x} - \rho(\mathbf{S}\mathbf{B})^{-1}\text{sign}(\mathbf{S}\mathbf{x}), \tag{2.17}$$

2.2.4 The Equivalent Control Method

Another way of analysing SMC is with the equivalent control method. By applying the input of Equation (2.15) to the System (2.12), the sliding surface $\mathcal{S}$ is reached after a time t_{r}. The sliding phase then begins; the trajectories of the system stay inside the sliding surface $\mathcal{S}$ and the order of the system is reduced to $n-m$. The equations that govern the dynamics of the reduced system can then be derived from the fact that

$$\boldsymbol{\sigma}(\mathbf{x}_{\mathrm{h}}) = \mathbf{0} \implies \dot{\boldsymbol{\sigma}}(\mathbf{x}_{\mathrm{h}}) = \mathbf{0}, \forall t \geq t_{\mathrm{r}}. \tag{2.18}$$

From the definition of the sliding variable in Equation (2.13) and the System (2.12), it follows that

$$\dot{\boldsymbol{\sigma}}(\mathbf{x}_{\mathrm{h}}) = \mathbf{0} \implies \mathbf{S}\dot{\mathbf{x}}_{\mathrm{h}} = \mathbf{0} \implies \mathbf{S}(\mathbf{A}\mathbf{x}_{\mathrm{h}} + \mathbf{B}\mathbf{u}_{\mathrm{eq}}) = \mathbf{0}, \tag{2.19}$$

where $\mathbf{u}_{\mathrm{eq}}$ is the equivalent control and it represents the control action necessary to maintain an ideal sliding motion, given that the system has already entered the sliding phase. By manipulating Equation (2.19), the equivalent control can be calculated as

$$\mathbf{S}\mathbf{A}\mathbf{x}_{\mathrm{h}} + \mathbf{S}\mathbf{B}\mathbf{u}_{\mathrm{eq}} = \mathbf{0} \implies \mathbf{u}_{\mathrm{eq}} = -(\mathbf{S}\mathbf{B})^{-1}\mathbf{S}\mathbf{A}\mathbf{x}_{\mathrm{h}}, \tag{2.20}$$

which is a simple state feedback controller. Substituting the derived formula for the equivalent control of Equation (2.20) in the System (2.12) yields

$$\dot{\mathbf{x}}_{\mathrm{h}} = \mathbf{A}\mathbf{x}_{\mathrm{h}} + \mathbf{B}(-(\mathbf{S}\mathbf{B})^{-1}\mathbf{S}\mathbf{A}\mathbf{x}_{\mathrm{h}}) \implies \dot{\mathbf{x}}_{\mathrm{h}} = (\mathbf{I} - \mathbf{B}(\mathbf{S}\mathbf{B})^{-1}\mathbf{S})\mathbf{A}\mathbf{x}_{\mathrm{h}}. \tag{2.21}$$

This equation represents the reduced dynamics of the system, once it enters the sliding mode. From linear algebra, one could prove that $\mathrm{rank}((\mathbf{I} - \mathbf{B}(\mathbf{S}\mathbf{B})^{-1}\mathbf{S})\mathbf{A}) = n - m$, which reflects the fact that those are indeed the dynamics for the state vector $\mathbf{x}_{\mathrm{h}}$ with $n-m$ degrees of freedom. In other words, this is the result of combining the imposed dynamics of the sliding variable with the system internal dynamics.

Note that, as expected, the behaviour of the system in sliding mode is linear, and so the system can be analysed with linear control techniques. This fact is often used to tune the choice of the sliding surface $\mathcal{S}$. One can also see that this result is indeed coherent with the fact that $\dot{\boldsymbol{\sigma}}(\mathbf{x}_{\mathrm{h}}) = \mathbf{0}$, as substituting Equation (2.21) in Equation (2.18) results in

$$\dot{\boldsymbol{\sigma}}(\mathbf{x}_{\mathrm{h}}) = \mathbf{S}\dot{\mathbf{x}}_{\mathrm{h}} = \mathbf{S}(\mathbf{I} - \mathbf{B}(\mathbf{S}\mathbf{B})^{-1}\mathbf{S})\mathbf{A}\mathbf{x}_{\mathrm{h}} = (\mathbf{S} - \mathbf{S})\mathbf{A}\mathbf{x}_{\mathrm{h}} = \mathbf{0}.$$

Remark 2.3.4 (Comments about the equivalent control)**.** *It is important to keep in mind that the actual control that is being applied to the system is the discontinuous control signal defined in Equation* (2.15)*. The equivalent control of Equation* (2.20) *can be thought of as the "average" control effort that acts in the system, once it finds itself in the sliding phase.*

In fact, when analysing the actual control signal of Equation (2.15)*, one can see that it can be divided into two parts:*

$$\mathbf{u}(\mathbf{x}) = -(\mathbf{S}\mathbf{B})^{-1}\mathbf{S}\mathbf{A}\mathbf{x} - \rho(\mathbf{S}\mathbf{B})^{-1}\frac{\mathbf{S}\mathbf{x}}{||\mathbf{S}\mathbf{x}||} = \mathbf{u}_{\mathrm{eq}} - \rho(\mathbf{S}\mathbf{B})^{-1}\frac{\mathbf{S}\mathbf{x}}{||\mathbf{S}\mathbf{x}||}, \tag{2.22}$$

with the first part being the equivalent control, and the second part being the control action that is necessary to keep the system inside of the sliding surface $\mathcal{S}$.

Because the second part is usually a high frequency switching signal, it can be filtered out by a low pass filter. By doing so, it is then possible to estimate the equivalent control signal from the actual control law.

2.2.5 Insensitivity Properties and Matching Conditions

This section introduces the robustness properties of SMC and discusses the conditions that are needed for such properties to hold. Consider now that System (2.12) is modified to

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) + \mathbf{B}\,\boldsymbol{\xi}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t), \end{aligned} \tag{2.23}$$

where $\boldsymbol{\xi} : \mathbb{R} \rightarrow \mathbb{R}^m$ represents a disturbance. It is assumed that $||\boldsymbol{\xi}(t)|| \leq \alpha(t), \forall t \in \mathbb{R}$, this means that $\boldsymbol{\xi}$ is an unknown bounded vector function, with a known bound α. Notice that it is also assumed that the uncertainty distribution matrix is equal to the input matrix $\mathbf{B}$. This could be considered as an uncertainty present in the control channel, and it is called a matched uncertainty.

By selecting the same sliding surface $\mathcal{S}$ as before in Equation (2.13), the design of the feedback control can be done with the same procedure. From the definition of the sliding variable given in Equation (2.13) and the linear system of Equation (2.23), one can see that

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{S}\mathbf{x} \implies \dot{\boldsymbol{\sigma}}(\mathbf{x}) = \mathbf{S}\dot{\mathbf{x}} \implies \dot{\boldsymbol{\sigma}}(\mathbf{x}) = \mathbf{S}(\mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u}+\mathbf{B}\boldsymbol{\xi}) = \mathbf{S}\mathbf{A}\mathbf{x}+\mathbf{S}\mathbf{B}\mathbf{u}+\mathbf{S}\mathbf{B}\boldsymbol{\xi}. \tag{2.24}$$

It is then easy to see that a control signal of the form as in Equation (2.15) solves the problem again. Substituting the proposed control signal in Equation (2.24) and using the definition of the sliding variable of Equation (2.13) yields

$$\begin{aligned} \boldsymbol{\sigma}^\top \dot{\boldsymbol{\sigma}} &= \boldsymbol{\sigma}^\top \left(\mathbf{S}\mathbf{A}\mathbf{x} + \mathbf{S}\mathbf{B}(\mathbf{u}+\boldsymbol{\xi})\right) = \boldsymbol{\sigma}^\top \left(\mathbf{S}\mathbf{A}\mathbf{x} + \mathbf{S}\mathbf{B}\left((\mathbf{S}\mathbf{B})^{-1}\left(-\mathbf{S}\mathbf{A}\mathbf{x} - \rho\frac{\boldsymbol{\sigma}}{||\boldsymbol{\sigma}||}\right)+\boldsymbol{\xi}\right)\right) \\ &= \boldsymbol{\sigma}^\top \left(-\rho\frac{\boldsymbol{\sigma}}{||\boldsymbol{\sigma}||} + \mathbf{S}\mathbf{B}\boldsymbol{\xi}\right) = -\rho||\boldsymbol{\sigma}|| + \boldsymbol{\sigma}^\top\mathbf{S}\mathbf{B}\boldsymbol{\xi}, \end{aligned}$$

using the sub-multiplicative properties of matrix norms and the bound $||\boldsymbol{\xi}|| \leq \alpha$ one gets

$$\boldsymbol{\sigma}^\top \dot{\boldsymbol{\sigma}} \leq -\rho||\boldsymbol{\sigma}|| + ||\boldsymbol{\sigma}^\top\mathbf{S}\mathbf{B}\boldsymbol{\xi}|| \leq -\rho||\boldsymbol{\sigma}|| + ||\boldsymbol{\sigma}||||\mathbf{S}\mathbf{B}||||\boldsymbol{\xi}|| \leq -\rho||\boldsymbol{\sigma}|| + ||\boldsymbol{\sigma}||||\mathbf{S}\mathbf{B}||\alpha,$$

and by defining $\rho' := \rho - ||\mathbf{S}\mathbf{B}||\alpha$,

$$\boldsymbol{\sigma}^\top \dot{\boldsymbol{\sigma}} \leq -\rho'||\boldsymbol{\sigma}||.$$

Therefore, if $\rho' > 0$ then there exists an $\mu' > 0$ with $\rho' \geq \mu' > 0$, which means that Condition (2.8) is satisfied.

This result illustrates that as long as the control gain is large enough the sliding mode is maintained as if there was no disturbance. However, it is important to notice that the gain ρ', which is responsible to guarantee finite time convergence to the sliding surface $\mathcal{S}$, is smaller than the original gain ρ, because a portion of $||\mathbf{SB}||\alpha$ is used to compensate the disturbance. This result is summarized in the following theorem:

Theorem 2.4 (Sliding mode control effort for systems with matched disturbance)**.** *If $\rho > ||\mathbf{SB}||\alpha$ then the feedback-law*

$$\mathbf{u} = (\mathbf{SB})^{-1}\left(-\mathbf{SAx} - \big(\rho' + ||\mathbf{SB}||\alpha\big)\frac{\mathbf{Sx}}{||\mathbf{Sx}||}\right), \tag{2.25}$$

with $\rho = \rho' + ||\mathbf{SB}||\alpha$, guarantees the existence, reachability and holdability to the sliding surface $\mathcal{S}$ even in the presence of a matched disturbance.

After determining the actual control, the equivalent control can also be calculated similarly as before by looking at dynamics of the sliding variable and using Equations (2.13) and (2.23), it follows that

$$\begin{aligned} &\dot{\boldsymbol{\sigma}}(\mathbf{x}_\mathrm{h}) = \mathbf{0} \implies \mathbf{S}\dot{\mathbf{x}}_\mathrm{h} = \mathbf{0} \implies \mathbf{S}(\mathbf{A}\mathbf{x}_\mathrm{h} + \mathbf{B}\mathbf{u}_\mathrm{eq} + \mathbf{B}\boldsymbol{\xi}) = \mathbf{0} \\ &\implies \mathbf{SA}\mathbf{x}_\mathrm{h} + \mathbf{SB}\mathbf{u}_\mathrm{eq} + \mathbf{SB}\boldsymbol{\xi} = \mathbf{0} \implies \mathbf{u}_\mathrm{eq} = -(\mathbf{SB})^{-1}\mathbf{SA}\mathbf{x}_\mathrm{h} - \boldsymbol{\xi}. \end{aligned} \tag{2.26}$$

By substituting the Equivalent Control Formula (2.26) in the uncertain system dynamics of Equation (2.23), the reduced dynamics are obtained as

$$\dot{\mathbf{x}}_\mathrm{h} = \mathbf{A}\mathbf{x}_\mathrm{h} + \mathbf{B}(-(\mathbf{SB})^{-1}\mathbf{SA}\mathbf{x}_\mathrm{h} - \boldsymbol{\xi}) + \mathbf{B}\boldsymbol{\xi} \implies \dot{\mathbf{x}}_\mathrm{h} = (\mathbf{I} - \mathbf{B}(\mathbf{SB})^{-1}\mathbf{S})\mathbf{A}\mathbf{x}_\mathrm{h}.$$

Note that the uncertainty $\boldsymbol{\xi}$ vanishes, and the reduced dynamics are the same as in the case without uncertainties. This result demonstrates the insensitivity properties that SMCs present.

The equivalent control signal can be estimated with a low pass filter, as commented in Remark 2.3.4. In doing so, it is also possible to obtain an approximation of the uncertainty $\boldsymbol{\xi}$, as

$$\boldsymbol{\xi}_\mathrm{approx} = -(\mathbf{SB})^{-1}\mathbf{SA}\mathbf{x}_\mathrm{h} - \mathbf{u}_\mathrm{eq}, \tag{2.27}$$

assuming that the model of the system is known.

Remark 2.4.1 (Comments about insensitivity properties and matching conditions)**.** *This invariance to bounded perturbations entering the system through the control channel can also be transferred to parametric uncertainties as long as they are considered matched.*

If the uncertainties are not matched, the SMC is able to compensate the matched part and still guarantees the rechability of the trajectories from the system to the sliding surface $\mathcal{S}$, but the ensuing reduced dynamics are affected by the unmatched part of the uncertainties, and thus there is no guarantee that resulting system is asymptotically stable.

2.2.6 The Chattering Problem

Definition and causes of the Chattering Problem

So far the topic of SMC was discussed theoretically, but real implementations of SMC can only approximate this theoretical behaviour. In fact, to be able to truly reject the effects of matched disturbances and to achieve perfect insensitivity, the frequency of the switching control signal would need to be infinite, which is not physically realisable.

Therefore, in real implementations of SMC, instead of the state vector being exactly restricted to the sliding surface $\mathcal{S}$, the trajectories of the system actually zigzag in a neighborhood of the sliding surface $\mathcal{S}$ as one can see in Figure 2.5.

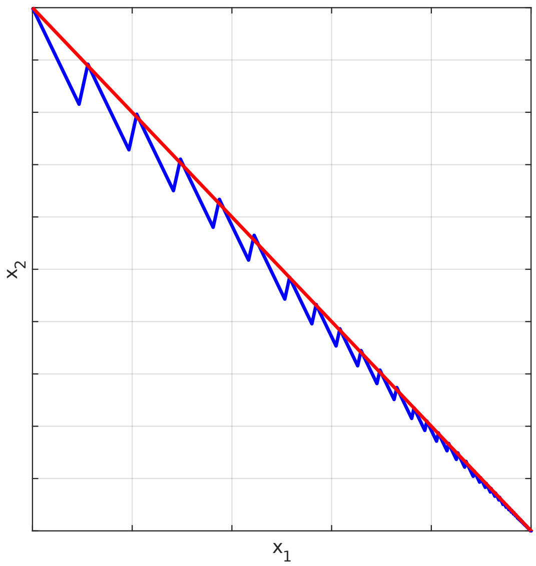


Figure 2.5: Comparison between Real and Theoretical Implementations of Sliding Mode

This low-amplitude, high-frequency zigzag movement when the system is in the sliding phase, as seen in the blue trajectory of Figure 2.5, is then defined as chattering. The red trajectory of Figure 2.5 depicts the theoretical behaviour that the system should present, as given by the reduced dynamics of Equation (2.21).

Another cause of the chattering problem is the digital implementation of SMC. Here, the computer sampling period must exceed the frequency of the control signal to accurately capture the rapid control actions that maintain trajectories within the sliding surface $\mathcal{S}$. However, achieving this can be challenging due to the typically high frequency of the switching signal. Other factors such as uncertainties, unmodeled dynamics and delays in the control actuators can also cause chattering.

Solutions to the Chattering Problem

The chattering behaviour is to be seen as very undesirable as it can lead to energy loss, plant damage, and excitation of unmodeled dynamics [21]. It is therefore a problem that needs to be avoided.

One obvious solution to the chattering problem is to substitute the discontinuous high frequency control signal with a continuous signal. One approach called Quasi-Sliding-Mode is characterized by the approximation of the sign function from Equation (2.10) by a $\mathrm{sigmoid}$ function. The $\mathrm{sigmoid}$ function is defined as

$$\mathrm{sigmoid}(\sigma) := \frac{\sigma}{|\sigma| + \epsilon}, \tag{2.28}$$

where $\epsilon \in \mathbb{R}_+$. It is easy to see that, for small values of ϵ the $\mathrm{sigmoid}$ function can be used as a good approximation for the sign function, because:

$$\lim_{\epsilon \to 0} \frac{\sigma}{|\sigma| + \epsilon} = \mathrm{sign}(\sigma).$$

However, there is a trade-off to be considered: to maintain ideal performance, a small as possible ϵ is needed; but to ensure a smooth control signal, ϵ needs to be sufficiently large.

By applying the $\mathrm{sigmoid}$ function, the resulting control signal is continuous and no chattering occurs, but as the name Quasi-Sliding-Mode suggests, this is no longer a true sliding mode; the sliding variable no longer converges to zero in finite time, and therefore the theoretical accuracy and robustness are lost. Nevertheless, it is the case that in many practical applications of SMC, even if the equality $\boldsymbol{\sigma} = \mathbf{0}$ only approximately holds, the system still behave approximately as if it was in sliding mode, hence the system continues to be insensitive to perturbations.

Other approaches that seek better solutions to the chattering problem also include approaches referred to as higher order sliding mode. They represent a generalization of the classical SMC methods presented in Section 2.2. In higher order SMC, the control signal acts on higher order time derivatives of the sliding variable, and the sliding surface is then defined as the hypersurface in which the lower order derivatives vanish.

There are also other first order sliding mode solutions to the chattering problem such as adaptive SMC [41], which uses an online estimation of the uncertainties to construct a continuous control law. Methods for a digital implementation of SMC are a current area of research [18], [5], [39] and [3].

2.2.7 Example

To demonstrate the theorems and formulas that were developed in this chapter, an example is given. Consider the SISO linear time invariant system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,u(t) \\ y(t) &= \mathbf{c}^\top\,\mathbf{x}(t), \end{aligned} \tag{2.29}$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{c}^\top = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The goal is to design a SMC that stabilizes this system around the origin. By choosing the sliding variable

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{S}\mathbf{x} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 + x_2, \tag{2.30}$$

the associated homogeneous system of linear equations (2.14) is simply

$$x_1 + x_2 = 0, \tag{2.31}$$

the homogeneous solution can then be given as

$$\mathbf{x}_\mathrm{h} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix}.$$

The resulting compensated dynamics of the system can be calculated with Equation (2.21), resulting in

$$\begin{bmatrix} -\dot{x}_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} \implies \begin{bmatrix} -\dot{x}_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_2 \end{bmatrix}, \tag{2.32}$$

and as expected, the reduced dynamics matrix has rank 1, because one degree of freedom was lost in Equation (2.31). Combining the independent equations of System (2.32) with the equations of System (2.31) yields

$$\begin{aligned} \dot{x}_2 &= -x_2 \\ x_1 &= -x_2, \end{aligned}$$

which are very desirable dynamics, as one can see by solving these equations. The trajectories of the system for $t > t_\mathrm{r}$ are given by

$$\begin{aligned} x_1(t) &= -e^{-t}\, x_2(0) \\ x_2(t) &= e^{-t}\, x_2(0), \end{aligned}$$

which present an exponential convergence to the origin.

The control effort can then be calculated with Equation (2.17) to be

$$u(x) = -x_2 - \rho\, \mathrm{sign}(x_1 + x_2). \tag{2.33}$$

Note that this feedback-law also guarantees that Condition (2.11) is satisfied. To determine the value of the control gain ρ, the reaching time t_r can be used as a figure of merit. Recall that Theorem 2.3 gives the condition that $\rho > 0$, and as seen in Theorem 2.2, the reaching time can be calculated as

$$t_\mathrm{r} \leq 2\frac{\sqrt{V(\mathbf{x}_0)}}{\mu} = 2\frac{\sqrt{V(\mathbf{x}_0)}}{\mu'\sqrt{2}} = 2\frac{\sqrt{V(\mathbf{x}_0)}}{\rho\sqrt{2}},$$

by choosing $\rho = 2$, it is obtained that

$$t_{\mathrm{r}} \leq 2\frac{\sqrt{\frac{(-2)^2+(-2)^2}{2}}}{2\sqrt{2}} = 2\mathrm{s},$$

which is considered satisfactory for this example. The control signal can then be re-written as

$$u(x) = \begin{cases} -x_2 - 2, \text{ for } x_1 + x_2 \geq 0 \\ -x_2 + 2, \text{ for } x_1 + x_2 < 0. \end{cases} \tag{2.34}$$

By substituting Equation (2.34) in Equation (2.29), similarly as presented in Equation (2.4), the resulting VSS is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} x_2 \\ -x_2 - 2 \end{bmatrix}, \text{ for } x_1 + x_2 \geq 0 \\ \\ \begin{bmatrix} x_2 \\ -x_2 + 2 \end{bmatrix}, \text{ for } x_1 + x_2 < 0. \end{cases} \tag{2.35}$$

Using the VSS analysis technique described in Section 2.1.3 the combined phase portrait of above VSS is drawn in Figure 2.6a. Moreover, by simulating the system for an initial condition $\mathbf{x_0} = (-2, -2)^\top$, the trajectory in Figure 2.6b is obtained.

One can then also plot the time evolution of the system states and sliding variable, as seen in Figure 2.7. Note that the sliding variable does in fact converge to zero in finite time, with the reaching time being indeed $t_{\mathrm{r}} = 2\mathrm{s}$. It is also clear that once the system enters sliding mode, that is, for $t > t_{\mathrm{r}}$, the state variables converge asymptotically to the origin.

Consider now that a matched disturbance is added to System (2.29), yielding

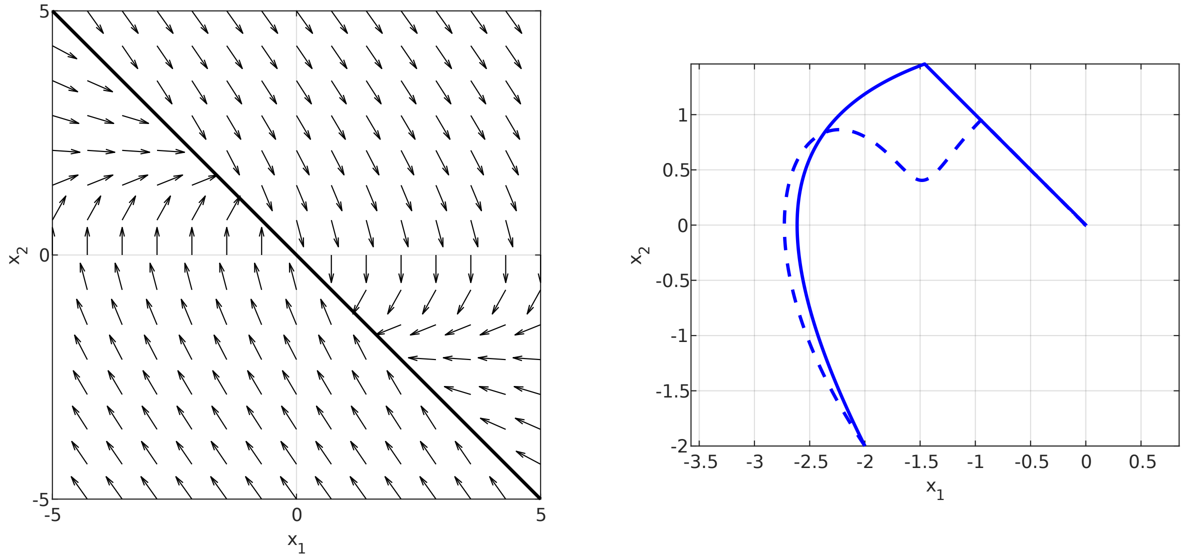
$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,u(t) + \mathbf{b}\,\xi(t) \\ y(t) &= \mathbf{c}^\top\,\mathbf{x}(t), \end{aligned} \tag{2.36}$$

with $|\xi(t)| \leq \alpha(t) = 1, \forall t \in \mathbb{R}$, for simulation purposes, it is considered that $\xi(t) = \sin(2t)$.

The system is then simulated for the same initial conditions and same gain $\rho = 2$. The system states are plotted with dashed lines in Figure 2.7a, the sliding variable in Figure 2.7b and the trajectory in Figure 2.6b.

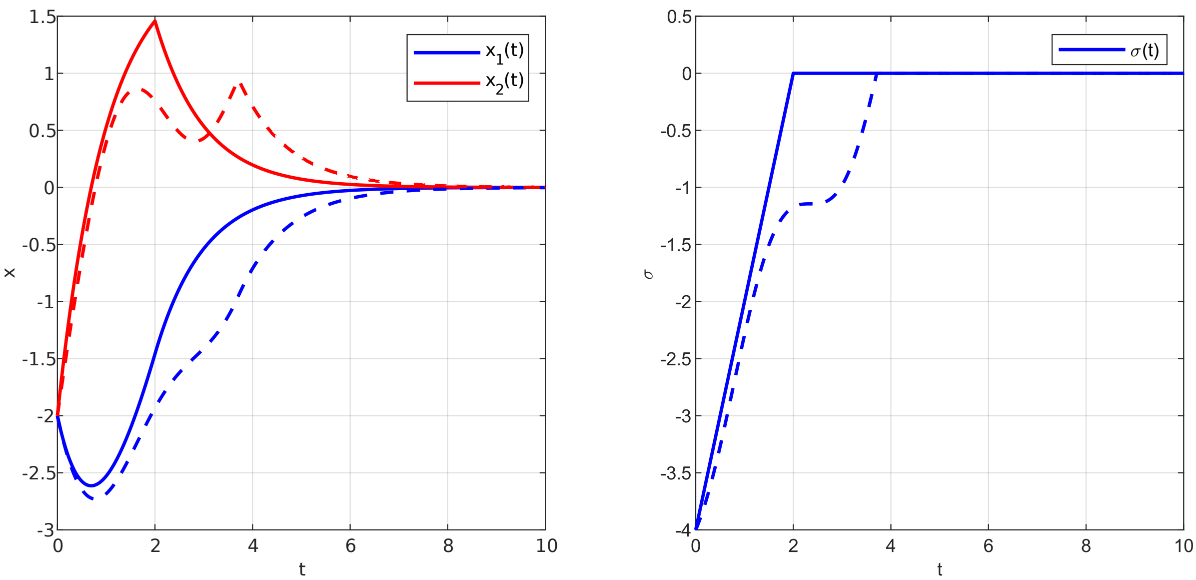
Note that, although being in the presence of a disturbance, the gain $\rho = 2$ satisfies condition $\rho \geq ||\mathbf{SB}||\alpha = 1$ from Theorem 2.4, and therefore the control effort is able to again guarantee finite time convergence of the sliding variable to zero, and asymptotic convergence of the system states to the origin. However, the reduction of the total gain ρ to the actual gain $\rho' = \rho - \alpha = 2 - 1 = 1$ results in a increase of the reaching time to $t_{\mathrm{r}} \leq 4$, which can be viewed in Figures 2.7a and 2.7b.

Figure 2.8 showcases the difference between the actual discontinuous control effort and the equivalent control. As mentioned in Remark 2.3.4 the equivalent control is obtained by a low pass filtering of the actual control signal. In these example, a simple first order transfer



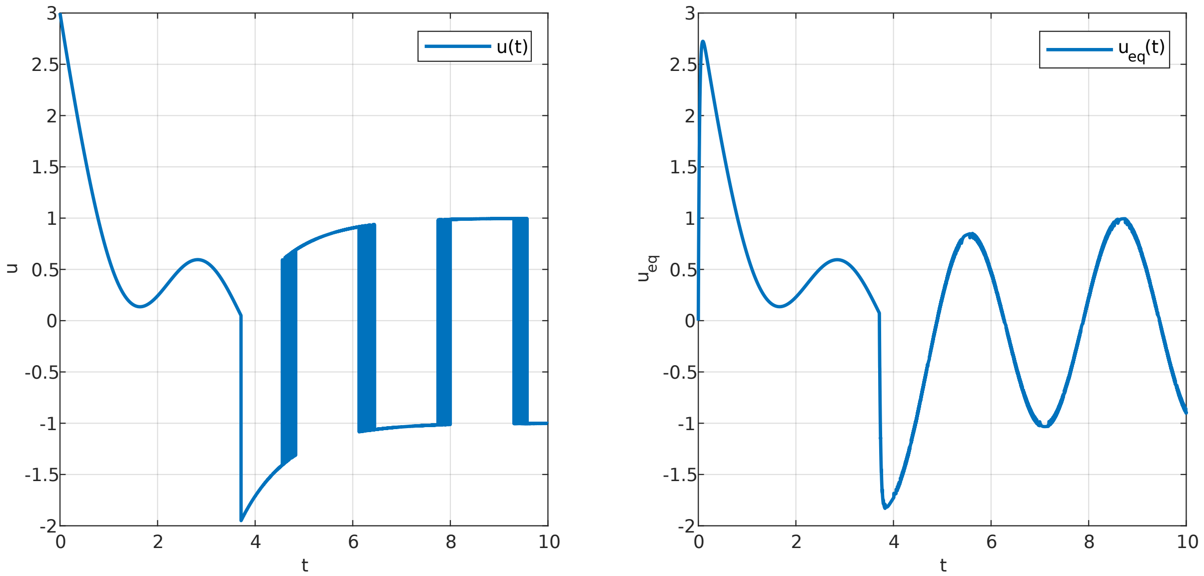
(a) Complete Phase Portrait (b) Example Trajectories

Figure 2.6: Phase Portraits



(a) Time Evolution of the System States (b) Time Evolution of the Sliding Variable

Figure 2.7: Time Evolution Plots



(a) Time Evolution of the Actual Control Signal (b) Time Evolution of the Equivalent Control

Figure 2.8: Control Efforts

function $G(s) = \frac{1}{\tau s+1}$ with a time constant of $\tau = 0.025$, was chosen as the low pass filter.

Using Equation (2.27) one can calculate an estimate for the disturbance. Figure 2.9 illustrates the comparison between the actual simulated disturbance and the estimated disturbance. One can see that after the sliding phase begins, the disturbance is accurately approximated.

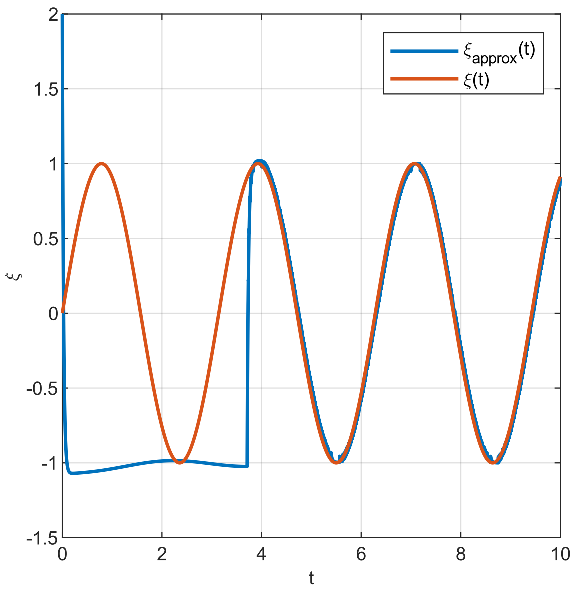


Figure 2.9: Time Evolution of the Disturbance

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