Appendix 4A

Net Present Value: First Principles of Finance

In this appendix, we show the theoretical underpinnings of the net present value rule. We first show how individuals make intertemporal consumption choices, and then we explain the net present value (NPV) rule. The appendix should appeal to students who like a theoretical model. Those of you who can accept the NPV analysis contained in Chapter 4 can skip to Chapter 5.

4A.1 Making Consumption Choices over Time

Figure 4A.1 illustrates the situation faced by a representative individual in the financial market. This person is assumed to have an income of \$50,000 this year and an income of \$60,000 next year. The market allows him not only to consume \$50,000 worth of goods this year and \$60,000 next year, but also to borrow and lend at the equilibrium interest rate.

The line *AB* in Figure 4A.1 shows all of the consumption possibilities open to the person through borrowing or lending, and the shaded area contains all of the feasible choices. Let's look at this figure more closely to see exactly why points in the shaded area are available.

We will use the letter r to denote the interest rate—the equilibrium rate—in this market. The rate is risk-free because we assume that no default can take place. Look at point A on the vertical axis of Figure 4A.1. Point A is a height of:

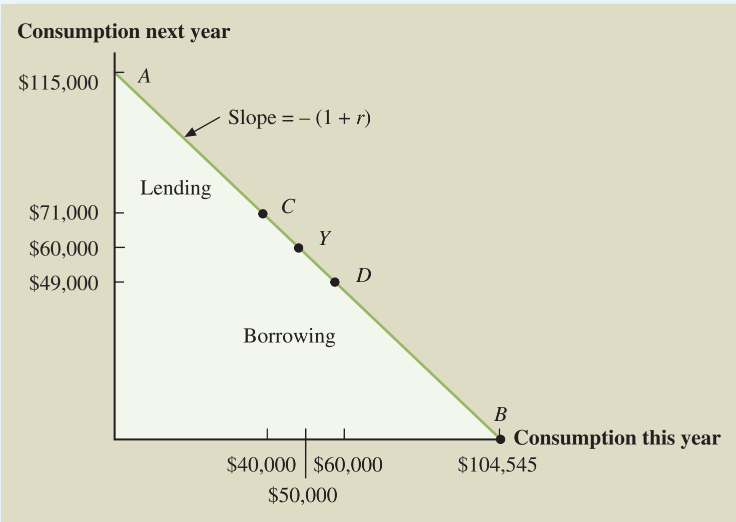
$$A = \$60{,}000 + [\$50{,}000 \times (1 + r)]$$

For example, if the rate of interest is 10 percent, then point A would be:

$$\begin{aligned} A &= \$60{,}000 + [\$50{,}000 \times (1 + 0.1)] \\ &= \$60{,}000 + \$55{,}000 \\ &= \$115{,}000 \end{aligned}$$

Point A is the maximum amount of wealth that this person can spend in the second year. He gets to point A by lending the full income that is available this year, \$50,000, and consuming none of it. In the second year, then, he will have the second year's income

Figure 4A.1
Intertemporal Consumption Opportunities



of $60,000 plus the proceeds from the loan that he made in the first year, $55,000, for a total of $115,000.

Now let's take a look at point *B*. Point *B* is a distance of:

$$B = \$50{,}000 + [\$60{,}000/(1 + r)]$$

along the horizontal axis. If the interest rate is 10 percent, point *B* will be:

$$\begin{aligned} B &= \$50{,}000 + [\$60{,}000/(1 + 0.1)] \\ &= \$50{,}000 + \$54{,}545 \\ &= \$104{,}545 \end{aligned}$$

(We have rounded off to the nearest dollar.)

Why do we divide next year's income of $60,000 by (1 + *r*), or 1.1 in the preceding computation? Point *B* represents the maximum amount available for this person to consume this year. To achieve that maximum he would borrow as much as possible and repay the loan from the income, $60,000, that he was going to receive next year. Because $60,000 will be available to repay the loan next year, we are asking how much he could borrow this year at an interest rate of *r* and still be able to repay the loan. The answer is:

$$\$60{,}000/(1 + r)$$

because if he borrows this amount, he must repay it next year with interest. Thus, next year he must repay:

$$[\$60{,}000/(1 + r)] \times (1 + r) = \$60{,}000$$

no matter what the interest rate, *r,* is. In our example we found that he could borrow $54,545 and, sure enough:

$$\$54{,}545 \times 1.1 = \$60{,}000$$

(after rounding off to the nearest dollar).

Furthermore, by borrowing and lending different amounts the person can achieve any point on the line *AB*. For example, point *C* is a point where he has chosen to lend $10,000 of today's income. This means that at point *C* he will have:

$$\text{Consumption this year at point } C = \$50{,}000 - \$10{,}000 = \$40{,}000$$

and

$$\text{Consumption next year at point } C = \$60{,}000 + [\$10{,}000 \times (1 + r)] = \$71{,}000$$

when the interest rate is 10 percent.

Similarly, at point *D* the individual has decided to borrow $10,000 and repay the loan next year. At point *D*:

$$\text{Consumption this year at point } D = \$50{,}000 + \$10{,}000 = \$60{,}000$$

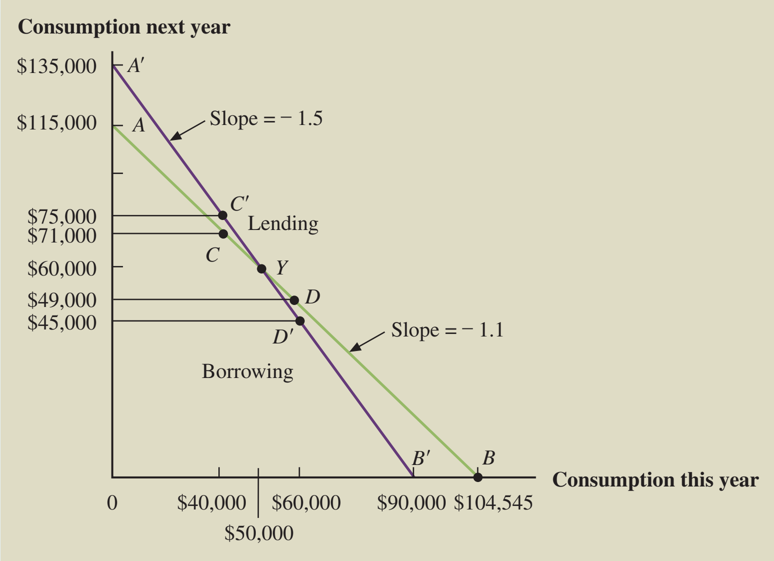
and:

$$\text{Consumption next year at point } D = \$60{,}000 - [\$10{,}000 \times (1 + r)] = \$49{,}000$$

at an interest rate of 10 percent.

In fact, this person can consume at any point on the line *AB*. This line has a slope of $-(1 + r)$, which means that for each dollar that is added to the *x* coordinate along the line, (1 + *r*) dollars are subtracted from the *y* coordinate. Moving along the line from point *A,* the initial point of $50,000 this year and $60,000 next year, toward point *B* gives the person more consumption today and less next year. In other words, moving toward point *B*

Figure 4A.2
The Effect of Different Interest Rates on Consumption Opportunities



is borrowing. Similarly, moving up toward point *A,* he is consuming less today and more next year and is lending. The line is a straight line because the individual has no effect on the interest rate. This is one of the assumptions of perfectly competitive financial markets.

Where will the person actually be? The answer to that question depends on the individual's tastes and personal situation, just as it did before there was a market. If the person is impatient, he might wish to borrow money at a point such as *D.* If he is patient, he might wish to lend some of this year's income and enjoy more consumption next year at, for example, a point such as *C.*[1]

Notice that whether we think of someone as patient or impatient depends on the interest rate he or she faces in the market. Suppose that our individual was impatient and chose to borrow $10,000 and move to point *D.* Now suppose that we raise the interest rate to 20 percent or even 50 percent. Suddenly our impatient person may become very patient and might prefer to lend some of this year's income to take advantage of the very high interest rate. The general result is depicted in Figure 4A.2. We can see that lending at point *C′* yields much greater future income and consumption possibilities than before.[2]

4A.2 Making Investment Choices

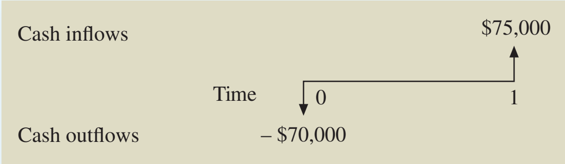
A Lending Example Consider a person who is concerned only about this year and the next. She has an income of $100,000 this year and expects to make the same amount next year. The interest rate is 10 percent. This individual is thinking about investing in a piece

[1]In this section, we are assuming a certain kind of financial market. In the language of economics, individuals who respond to rates and prices by acting as though they have no influence on them are called *price takers,* and this assumption is sometimes called the *price-taking assumption.* It is the condition of **perfectly competitive financial markets** (or, more simply, *perfect markets*). The following conditions are likely to lead to this:

1. Trading is costless. Access to the financial markets is free.
2. Information about borrowing and lending opportunities is available.
3. There are many traders, and no single trader can have a significant impact on market prices.

[2]Those familiar with consumer theory might be aware of the surprising case where raising the interest rate actually makes people borrow more or lowering the rate makes them lend more. The latter case might occur, for example, if the decline in the interest rate made the lenders have so little consumption next year that they have no choice but to lend out even more than they were lending before just to subsist. Nothing we do depends on excluding such cases, but it is much easier to ignore them, and the resulting analysis fits the real markets more closely.

Figure 4A.3
Cash Flows for Investment in Land



of land that costs $70,000. She is certain that next year the land will be worth $75,000, a sure $5,000 gain. Should she undertake the investment? This situation is described in Figure 4A.3 with the cash flow time chart.

A moment's thought should be all it takes to convince her that this is not an attractive business deal. By investing $70,000 in the land, she will have $75,000 available next year. Suppose instead that she puts the same $70,000 into a loan in the financial market. At the 10 percent rate of interest this $70,000 would grow to:

$$(1 + 0.1) \times \$70{,}000 = \$77{,}000$$

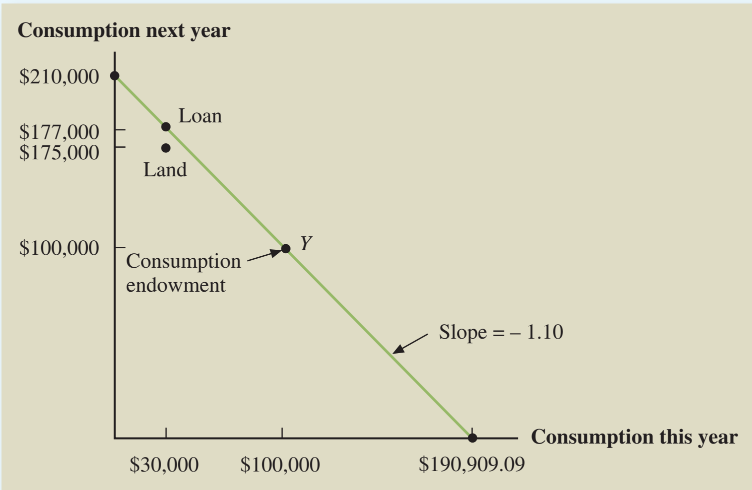
next year.

It would be foolish to buy the land when the same $70,000 investment in the financial market would beat it by $2,000 (that is, $77,000 from the loan minus $75,000 from the land investment). Figure 4A.4 illustrates this situation. Notice that the $70,000 loan gives no less income today and $2,000 more next year. This example illustrates some amazing features of the financial markets. It is remarkable to consider all of the information that we did *not* use when arriving at the decision not to invest in the land. We did not need to know how much income the person has this year or next year. We also did not need to know whether the person preferred more income this year or next.

We did not need to know any of these other facts, and more important, the person making the decision did not need to know them either. She needed only to be able to compare the investment with a relevant alternative available in the financial market. When this investment fell short of that standard—by $2,000 in the previous example—regardless of what the individual wanted to do, she knew that she should not buy the land.

A Borrowing Example Let us sweeten the deal a bit. Suppose that instead of being worth $75,000 next year, the land would be worth $80,000. What should our investor do now? This case is a bit more difficult. After all, even if the land seems like a good deal, this

Figure 4A.4
Consumption Opportunities with Borrowing and Lending



person's income this year is $100,000. Does she really want to make a $70,000 investment this year? Won't that leave only $30,000 for consumption?

The answers to these questions are yes, the individual should buy the land; yes, she does want to make a $70,000 investment this year; and, most surprising of all, even though her income is $100,000, making the $70,000 investment will not leave her with $30,000 to consume this year! Now let us see how finance lets us get around the basic laws of arithmetic.

The financial markets are the key to solving our problem. First, the financial markets can be used as a standard of comparison against which any investment project must be measured. Second, they can be used as a tool to actually help the individual undertake investments. These twin features of the financial markets enable us to make the right investment decision.

Suppose that the person borrows the $70,000 initial investment that is needed to purchase the land. Next year she must repay this loan. Because the interest rate is 10 percent, she will owe the financial market $77,000 next year. This is depicted in Figure 4A.5. Because the land will be worth $80,000 next year, she can sell it, pay off her debt of $77,000, and have $3,000 extra cash.

If she wishes, this person can now consume an extra $3,000 worth of goods and services next year. This possibility is illustrated in Figure 4A.6. In fact, even if she wants to do all of her consuming this year, she is still better off taking the investment. All she must do is take out a loan this year and repay it from the proceeds of the land next year and profit by $3,000.

Furthermore, instead of borrowing just the $70,000 that she needed to purchase the land, she could have borrowed $72,727.27. She could have used $70,000 to buy the land and consumed the remaining $2,727.27.

We will call $2,727.27 the net present value of the transaction. Notice that it is equal to $\$3{,}000 \times 1/1.1$. How did we figure out that this was the exact amount that she could borrow? It was easy: If $72,727.27 is the amount that she borrows, then, because the interest rate is 10 percent, she must repay:

$$\$72{,}727.27 \times (1 + 0.1) = \$80{,}000$$

Figure 4A.5
Cash Flows of Borrowing to Purchase the Land

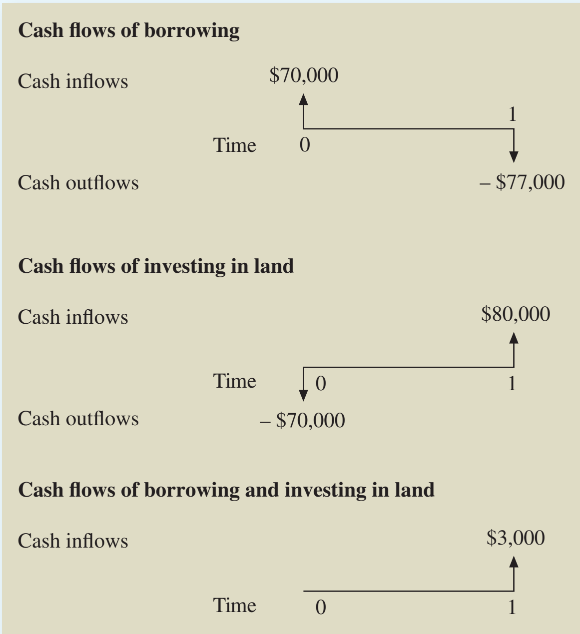
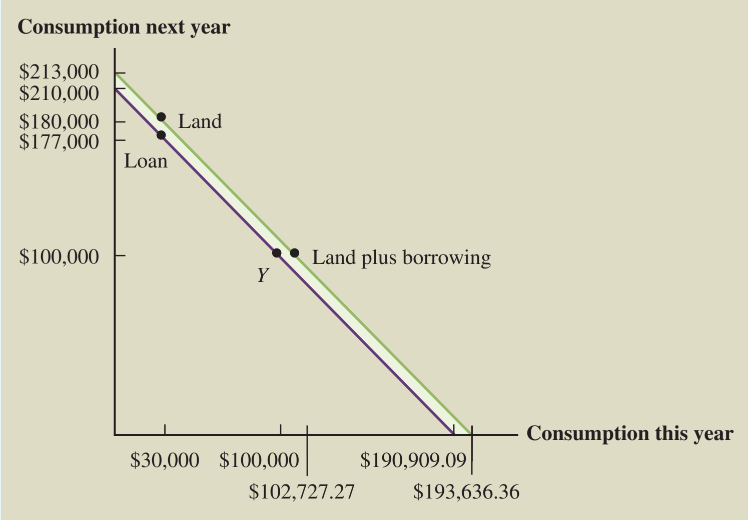


Figure 4A.6
Consumption Opportunities with Investment Opportunity and Borrowing and Lending



next year, and that is exactly what the land will be worth. The line through the investment position in Figure 4A.6 illustrates this borrowing possibility.

The amazing thing about both of these cases, one where the land is worth $75,000 next year and the other where it is worth $80,000 next year, is that we needed only to compare the investment with the financial markets to decide whether it was worth undertaking. This is one of the more important points in all of finance. It is true regardless of the consumption preferences of the individual. This is one of a number of **separation theorems** in finance. It states that the value of an investment to an individual is not dependent on consumption preferences. In our examples we showed that the person's decision to invest in land was not affected by consumption preferences. However, these preferences dictated whether she borrowed or lent.

4A.3 Illustrating the Investment Decision

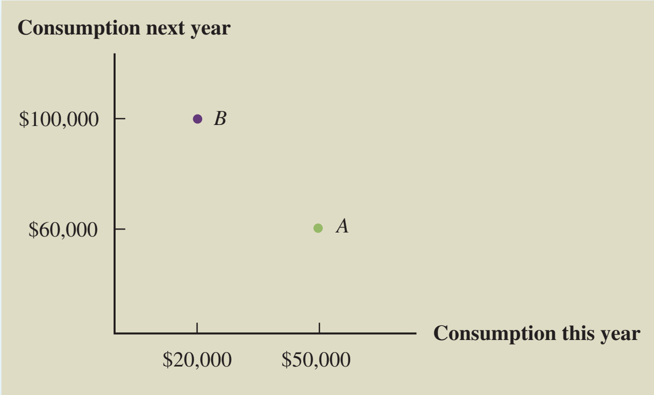
Figure 4A.1 describes the possibilities open to a person who has an income of $50,000 this year and $60,000 next year and faces a financial market in which the interest rate is 10 percent. But, at that moment, the person has no investment possibilities beyond the 10 percent borrowing and lending that is available in the financial market.

Suppose we give this person the chance to undertake an investment project that will require a $30,000 outlay of cash this year and that will return $40,000 to the investor next year. Refer to Figure 4A.1 and determine how you could include this new possibility in that figure and how you could use the figure to help you decide whether to undertake the investment.

Now look at Figure 4A.7. In Figure 4A.7 we have labeled the original point with $50,000 this year and $60,000 next year as point *A*. We have also added a new point *B*, with $20,000 available for consumption this year and $100,000 next year. The difference between point *A* and point *B* is that at point *A* the person is just where we started him off, and at point *B* the person has also decided to undertake the investment project. As a result of this decision the person at point *B* has:

$$\$50,000 - 30,000 = \$20,000$$

Figure 4A.7
Consumption Choices with Investment but No Financial Markets



left for consumption this year, and:

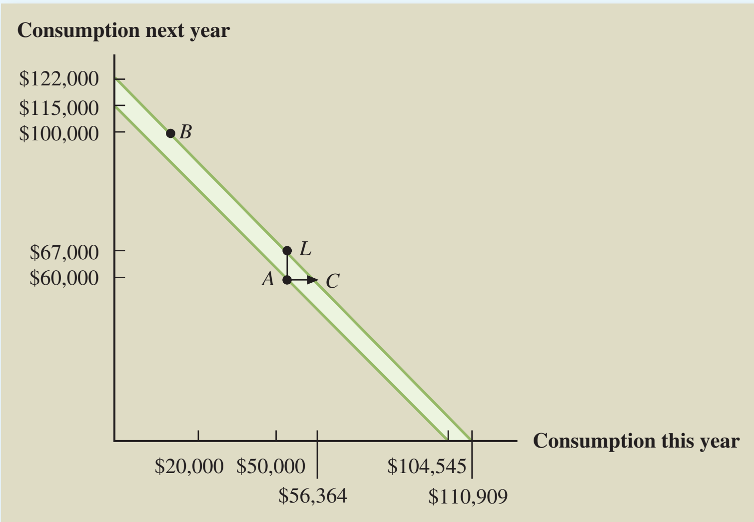
$$\$60{,}000 + 40{,}000 = \$100{,}000$$

available next year. These are the coordinates of point *B*.

We must use our knowledge of the individual's borrowing and lending opportunities to decide whether to accept or reject the investment. This is illustrated in Figure 4A.8. Figure 4A.8 is similar to Figure 4A.7, but in it we have drawn a line through point *A* that shows the possibilities open to the person if he stays at point *A* and does not make the investment. This line is exactly the same as the one in Figure 4A.1. We have also drawn a parallel line through point *B* that shows the new possibilities that are available to the person if he undertakes the investment. The two lines are parallel because the slope of each is determined by the same interest rate, 10 percent. It does not matter whether the person takes the investment and goes to point *B* or does not and stays at point *A*: in the financial market, each dollar of lending is a dollar less available for consumption this year and moves him to the left by a dollar along the *x*-axis. Because the interest rate is 10 percent, the $1 loan repays $1.10 and moves him up by $1.10 along the *y*-axis.

It is easy to see from Figure 4A.8 that the investment has made the person better off. The line through point *B* is higher than the line through point *A*. Thus, no matter what

Figure 4A.8
Consumption Choices with Investment Opportunities and Financial Markets



pattern of consumption this person wanted this year and next, he could have more in each year if he undertook the investment.

For example, suppose our individual wanted to consume everything this year. If he did not make the investment, the point where the line through point *A* intersected the *x*-axis would give the maximum amount of consumption he could enjoy this year. This point has $104,545 available this year. To recall how we found this figure, review the analysis of Figure 4A.1. But in Figure 4A.8 the line that goes through point *B* hits the *x*-axis at a higher point than the line that goes through point *A*. Along this line the person can have the $20,000 that is left after investing $30,000, plus all that he can borrow and repay with both next year's income and the proceeds from the investment. The total amount available to consume today is therefore:

$$= \$50{,}000 - 30{,}000 + (\$60{,}000 + 40{,}000)/(1 + 0.1)$$
$$= \$20{,}000 + (\$100{,}000/1.1)$$
$$= \$110{,}909$$

The additional consumption available this year from undertaking the investment and using the financial market is the difference on the *x*-axis between the points where these two lines intersect:

$$\$110{,}909 - 104{,}545 = \$6{,}364$$

This difference is an important measure of what the investment is worth to the person. It answers a variety of questions. For example, it reveals how much money we would need to give the investor this year to make him just as well off as he is with the investment.

Because the line through point *B* is parallel to the line through point *A* but has been moved over by $6,364, we know that if we were to add this amount to the investor's current income this year at point *A* and take away the investment, he would wind up on the line through point *B* and with the same possibilities. If we do this, the person will have $56,364 this year and $60,000 next year, which is the situation of the point on the line through point *B* that lies to the right of point *A* in Figure 4A.8. This is point *C*.

We could also ask a different question: How much money would we need to give the investor next year to make him just as well off as he is with the investment?

This is the same as asking how much higher the line through point *B* is than the line through point *A*. In other words, what is the difference in Figure 4A.8 between the point where the line through *A* intercepts the *y*-axis and the point where the line through *B* intercepts the *y*-axis?

The point where the line through *A* intercepts the *y*-axis shows the maximum amount the person could consume next year if all of his current income were lent out and the proceeds of the loan were consumed along with next year's income.

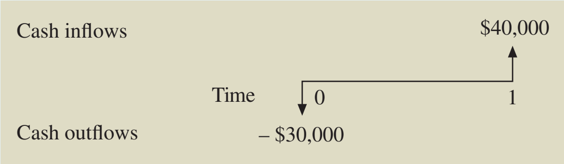
As we showed in our analysis of Figure 4A.1, this amount is $115,000. How does this compare with what the person can have next year if he makes the investment? By making the investment we saw that he would be at point *B*, where he has $20,000 left this year and would have $100,000 next year. By lending the $20,000 that is left this year and adding the proceeds of this loan to the $100,000, we find the line through *B* intercepts the *y*-axis at

$$(\$20{,}000 \times 1.1) + \$100{,}000 = \$122{,}000$$

The difference between this amount and $115,000 is

$$\$122{,}000 - 115{,}000 = \$7{,}000$$

Figure 4A.9
Cash Flows for the Investment Project



which is the answer to the question of how much we would need to give the person next year to make him as well off as he is with the investment.

There is a simple relationship between these two numbers. If we multiply \$6,364 by 1.1 we get \$7,000! Consider why this must be so. The \$6,364 is the amount of extra cash we must give the person this year to substitute for having the investment. In a financial market with a 10 percent rate of interest, however, \$1 this year is worth exactly the same as \$1.10 next year. Thus, \$6,364 this year is the same as \$6,364 × 1.1 next year. In other words, the person does not care whether he has the investment, \$6,364, this year or \$6,364 × 1.1 next year. But we already showed that the investor is equally willing to have the investment and to have \$7,000 next year. This must mean that:

$$\$6{,}364 \times 1.1 = \$7{,}000$$

You can also verify this relationship between these two variables by using Figure 4A.8. Because the lines through *A* and *B* each have the same slope of -1.1, the difference of \$7,000 between where they intersect on the *x*-axis must be in the ratio of 1.1 to 1.

Now we can show you how to evaluate the investment opportunity on a stand-alone basis. Here are the relevant facts: The individual must give up \$30,000 this year to get \$40,000 next year. These cash flows are illustrated in Figure 4A.9.

The investment rule that follows from the previous analysis is the net present value (NPV) rule. Here, we convert all consumption values to the present and add them up:

$$\begin{aligned}\text{Net present value} &= -\$30{,}000 + \$40{,}000 \times (1/1.1)\\ &= -\$30{,}000 + \$36{,}364\\ &= \$6{,}364\end{aligned}$$

The future amount, \$40,000, is called the *future value* (FV).

The net present value of an investment is a simple criterion for deciding whether to undertake an investment. NPV answers the question of how much cash an investor would need to have today as a substitute for making the investment. If the net present value is positive, the investment is worth taking on because doing so is essentially the same as receiving a cash payment equal to the net present value. If the net present value is negative, taking on the investment today is equivalent to giving up some cash today, and the investment should be rejected.

We use the term *net present value* to emphasize that we are already including the current cost of the investment in determining its value and not simply what it will return. For example, if the interest rate is 10 percent and an investment of \$30,000 today will produce a total cash return of \$40,000 in one year's time, the *present value* of the \$40,000 by itself is:

$$\$40{,}000/1.1 = \$36{,}364$$

but, the *net present value* of the investment is \$36,364 minus the original investment:

$$\text{Net present value} = \$36{,}364 - \$30{,}000 = \$6{,}364$$

The present value of a future cash flow is the value of that cash flow after considering the appropriate market interest rate. The net present value of an investment is the present value of the investment's future cash flows, minus the initial cost of the investment. We have just decided that our investment is a good opportunity. It has a positive net present value because it is worth more than it costs.

In general, our results can be stated in terms of the net present value rule:

> An investment is worth making if it has a positive NPV. If an investment's NPV is negative, it should be rejected.