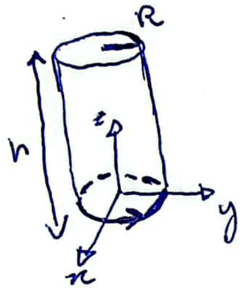
Gabarito - Primeira Prova

Questão 1



Considerando o campo $\vec{V}(\vec{r}) = 2\varphi\hat{\varphi} + 2z\hat{z}$

a) Calcular $\oint \vec{V}\cdot d\vec{l}$ na circunferência indicada:

$$\Rightarrow \oint \vec{V}\cdot d\vec{l} = \int_0^{2\pi}(2\varphi\hat{\varphi} + 2z\hat{z})\cdot R\,d\varphi\,\hat{\varphi} = \int_0^{2\pi} 2R\varphi\,d\varphi = R\varphi^2\Big|_0^{2\pi} = \underline{\underline{4\pi^2 R}}$$

b) Calcular o rotacional $\vec{\nabla}\times\vec{V}$:

$$\Rightarrow \vec{\nabla}\times\vec{V} = \left[\underbrace{\cancel{\frac{1}{r}\frac{\partial V_z}{\partial\phi}}}_{=0} - \underbrace{\cancel{\frac{\partial V_\phi}{\partial z}}}_{=0}\right]\hat{r} + \left[\underbrace{\cancel{\frac{\partial V_r}{\partial z}}}_{=0} - \underbrace{\cancel{\frac{\partial V_z}{\partial r}}}_{=0}\right]\hat{\varphi} + \frac{1}{r}\left[\frac{\partial(r V_\phi)}{\partial r} \,(= 2\phi) - \underbrace{\cancel{\frac{\partial V_r}{\partial\phi}}}_{=0}\right]\hat{z}$$

$$\Rightarrow \vec{\nabla}\times\vec{V} = \underline{\underline{\frac{2\varphi}{r}\cdot\hat{z}}}$$

c) Calcular o fluxo do rotacional através da lateral e do topo:

$$\Rightarrow \int_{\text{superfície}}(\vec{\nabla}\times\vec{V})\cdot d\vec{a} = \int_{\text{lateral}}(\vec{\nabla}\times\vec{V})\cdot d\vec{a} + \int_{\text{topo}}(\vec{\nabla}\times\vec{V})\cdot d\vec{a}$$

$$= \underbrace{\cancel{\int_0^h\int_0^{2\pi}\left(\frac{2\varphi}{r}\cdot\hat{z}\right)\cdot(r\,d\phi\,dz\,\hat{r})}}_{=0\ (\hat{z}\cdot\hat{r}=0)} + \int_0^R\int_0^{2\pi}\left(\frac{2\varphi}{r}\hat{z}\right)\cdot(r\,dr\,d\phi\,\hat{z})$$

$$= \int_0^R\int_0^{2\pi} 2\varphi\,dr\,d\phi = \underline{\underline{4\pi^2 R}}$$

d) O teorema de Stokes afirma que: $\int_S (\vec{\nabla}\times\vec{V})\cdot d\vec{a} = \oint \vec{V}\cdot d\vec{l}$

- Do item c) temos $\int_S (\vec{\nabla}\times\vec{V})\cdot d\vec{a} = 4\pi^2 R$
- Do item a) temos $\oint \vec{V}\cdot d\vec{l} = 4\pi^2 R$

Portanto os resultados são compatíveis com o teorema de Stokes.

Questão 2 (P1)

a) Da simetria do problema, temos que $\vec{E}$ tem direção de $\hat{r}$.

I: $0 \leq r < R$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \boxed{\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 E) = \frac{\rho}{\epsilon_0}}$$

II: $R < r < 2R$

$$\vec{\nabla} \cdot \vec{E} = \frac{-\rho}{\epsilon_0} \Rightarrow \boxed{\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 E) = \frac{-\rho}{\epsilon_0}}$$

III: $r > 2R$

$$\vec{\nabla} \cdot \vec{E} = \frac{0}{\epsilon_0} \Rightarrow \boxed{\frac{1}{r^2} \frac{\partial (r^2 E)}{\partial r} = 0}$$

b) $E(0) = 0$

$$E(R+\epsilon) = E(R-\epsilon)$$

$$E(2R+\epsilon) = E(2R-\epsilon)$$

c) I: $0 \leq r < R$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E\right) = \frac{\rho}{\epsilon_0}$$

$$\frac{\partial (r^2 E)}{\partial r} = \frac{\rho}{\epsilon_0} r^2$$

$$\int_0^r \frac{\partial (r'^2 E)}{\partial r'} dr' = \int_0^r \frac{\rho}{\epsilon_0} r'^2 dr'$$

$$r^2 E = \frac{r^3}{3}\frac{\rho}{\epsilon_0} \implies \boxed{\vec{E}(r) = \frac{\rho r}{3\epsilon_0}\hat{r}}, \quad 0 < r < R$$

Logo: $E(R) = \dfrac{\rho R}{3\epsilon_0}$

II: $R < r < 2R$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E\right) = -\frac{\rho}{\epsilon_0}$$

$$\frac{\partial (r^2 E)}{\partial r} = -\frac{r^2 \rho}{\epsilon_0}$$

$$\int_R^r \frac{\partial}{\partial r}(r^2 E)\, dr = \int_R^r -\frac{r^2 \rho}{\epsilon_0}\, dr$$

$$(r^2E)\Big|_R^r = \frac{\rho}{3\epsilon_0}(R^3 - r^3)$$

$$\vec{E}(r) = \left(\frac{\rho}{3\epsilon_0}\frac{(R^3-r^3)}{r^2} + \frac{R^2 \widetilde{E(R)}}{r^2}\right)\hat{r}$$

usarei o E(R) encontrado em I, bem como a condição de contorno $E(R+\epsilon) = E(R-\epsilon)$

$$\vec{E}(r) = \frac{\rho}{3\epsilon_0}\left[\frac{(R^3-r^3)}{r^2} + \frac{R^3}{r^2}\right]\hat{r}$$

$$\boxed{\vec{E}(r) = \frac{\rho}{3\epsilon_0}\left(\frac{2R^3 - r^3}{r^2}\right)\hat{r}}\quad , \quad R < r < 2R$$

Logo:

$$E(2R) = \frac{\rho}{3\epsilon_0}\left(\frac{2R^3 - 8R^3}{4R^2}\right) = \frac{\rho}{3\epsilon_0}\frac{(-6R^3)}{(4R^2)}$$

$$E(2R) = -\frac{\rho}{3\epsilon_0}\left(\frac{3R}{2}\right)$$

III: $r > 2R$

$$\frac{1}{r^2}\frac{\partial(r^2E)}{\partial r} = 0 \Rightarrow \int_{2R}^{r}\frac{\partial}{\partial r}(r^2E) = 0$$

$$r^2E(r) - 4R^2E(2R) = 0$$

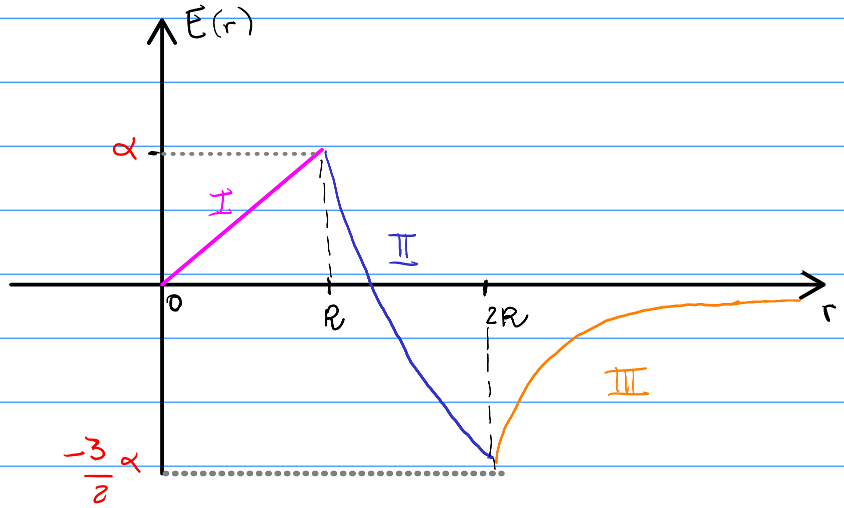
$$E(r) = \frac{4R^2\widetilde{E(2R)}}{r^2}\hat{r}$$

usarei o E(2R) que encontrei de II, bem como a condição de contorno $E(2R+\epsilon) = E(2R-\epsilon)$.

$$\vec{E}(r) = \frac{4R^2}{r^2}\left[\frac{-\rho}{3\epsilon_0}\left(\frac{3R}{2}\right)\right] = \frac{-\rho}{3\epsilon_0}\left(\frac{6R^3}{r^2}\right)\hat{r}$$

$$\boxed{\vec{E}(r) = \frac{-\rho}{3\epsilon_0}\left(\frac{6R^3}{r^2}\right)\hat{r} \quad ,} \quad r > 2R$$

d)



$\frac{\rho R}{3\epsilon_0} \equiv \alpha$

E(r) muda de sinal pois a contribuição da carga negativa para o campo elétrico aumenta na região II, enquanto a contribuição da carga positiva diminui, nessa região, conforme o raio aumenta.