Introduction to Causal Data Analysis and Modeling with Coincidence Analysis

Module 1.4

Top-down vs. Bottom-up Search / the CNA Algorithm

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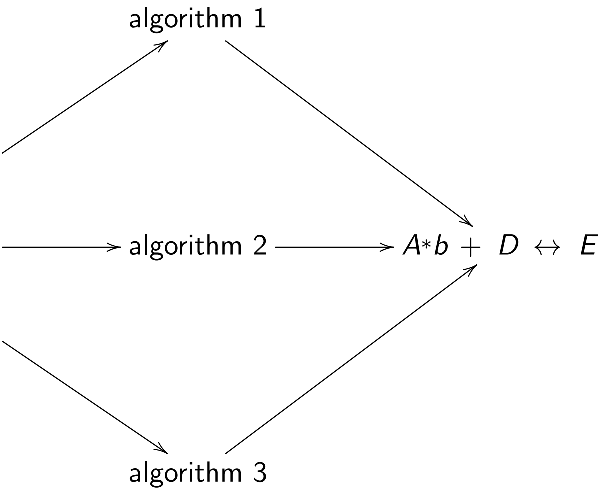
15 May 2022

Overview

1. Top-down search
2. Bottom-up search
3. The QCA algorithm
4. The CNA algorithm
5. Fragmentation
6. Model interpretation

Alternative algorithms

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0



Two search directions: Top-down

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Two search directions: Top-down

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A{*}B{*}D \rightarrow E$?

Two search directions: Top-down

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$A{*}B{*}D \rightarrow E \quad ?$$

$$A{*}B{*}d \rightarrow E \quad ?$$

Two search directions: Top-down

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$A{*}B{*}D \rightarrow E \quad ?$$

$$A{*}B{*}d \rightarrow E \quad ?$$

$$E \rightarrow A{*}B{*}D + A{*}b{*}D + A{*}b{*}d + a{*}B{*}D + a{*}b{*}D \quad ?$$

Two search directions: Top-down

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$A*B*D \rightarrow E \ ?$$

$$A*B*d \rightarrow E \ ?$$

$$E \rightarrow A*B*D + A*b*D + A*b*d + a*B*D + a*b*D \ ?$$

$$A*b + D \leftrightarrow E$$

Two search directions: Bottom-up

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Two search directions: Bottom-up

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A \rightarrow E$?

Two search directions: Bottom-up

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A \to E$?

$B \to E$?

Two search directions: Bottom-up

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A \rightarrow E$?

$B \rightarrow E$?

$A{*}B \rightarrow E$?

Two search directions: Bottom-up

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A \rightarrow E$?

$B \rightarrow E$?

$A{*}B \rightarrow E$?

$E \rightarrow A{*}b$?

Two search directions: Bottom-up

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A \rightarrow E$?

$B \rightarrow E$?

$A{*}B \rightarrow E$?

$E \rightarrow A{*}b$?

$E \rightarrow D$?

Two search directions: Bottom-up

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$A \rightarrow E \ ?$$
$$B \rightarrow E \ ?$$
$$A{*}B \rightarrow E \ ?$$
$$E \rightarrow A{*}b \ ?$$
$$E \rightarrow D \ ?$$

$$A{*}b + D \leftrightarrow E$$

The standard QCA algorithm: Quine-McCluskey optimization (QMC)

Standardly, QCA infers MINUS-formulas from data in two computational phases using QMC:

1. QCA identifies sufficient and necessary conditions in the data;
2. QCA eliminates redundancies from recovered sufficient and necessary conditions by means of *Quine-McCluskey optimization (QMC)*, which itself operates in two phases:
   i. QMC eliminates redundancies from sufficient conditions using Boolean distributive laws;
   ii. QMC eliminates redundancies from necessary conditions using so-called *PI charts*.

Phase 1

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Phase 1

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Phase 1

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$A*B*D \rightarrow E$$

Phase 1

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$A{*}B{*}D \to E$$

Phase 1

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$A*B*D \rightarrow E$$

Phase 1

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$A{*}B{*}D \rightarrow E$$

$$A{*}b{*}D \rightarrow E$$

Phase 1

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$A{*}B{*}D \rightarrow E$$
$$A{*}b{*}D \rightarrow E$$
$$A{*}b{*}d \rightarrow E$$
$$a{*}B{*}D \rightarrow E$$
$$a{*}b{*}D \rightarrow E$$

Phase 1

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$A{*}B{*}D \rightarrow E$$
$$A{*}b{*}D \rightarrow E$$
$$A{*}b{*}d \rightarrow E$$
$$a{*}B{*}D \rightarrow E$$
$$a{*}b{*}D \rightarrow E$$
$$E \rightarrow A{*}B{*}D \;+\; A{*}b{*}D \;+\; A{*}b{*}d \;+\; a{*}B{*}D \;+\; a{*}b{*}D$$

$$A{*}B{*}D + A{*}b{*}D + A{*}b{*}d + a{*}B{*}D + a{*}b{*}D \leftrightarrow E$$

Phase 2i

Sufficient conditions recovered in phase 1 are freed of redundancies to yield minimally sufficient conditions or so-called *prime implicants*.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$A{*}B{*}D \rightarrow E$$
$$A{*}b{*}D \rightarrow E$$
$$A{*}b{*}d \rightarrow E$$
$$a{*}B{*}D \rightarrow E$$
$$a{*}b{*}D \rightarrow E$$

Phase 2i

Sufficient conditions recovered in phase 1 are freed of redundancies to yield minimally sufficient conditions or so-called *prime implicants*.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A{*}B{*}D \rightarrow E$ $\qquad$ $A{*}D \rightarrow E$

$A{*}b{*}D \rightarrow E$

$A{*}b{*}d \rightarrow E$

$a{*}B{*}D \rightarrow E$

$a{*}b{*}D \rightarrow E$

Phase 2i

Sufficient conditions recovered in phase 1 are freed of redundancies to yield minimally sufficient conditions or so-called *prime implicants*.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A{*}B{*}D \rightarrow E$ $\quad$ $A{*}D \rightarrow E$

$A{*}b{*}D \rightarrow E$

$A{*}b{*}d \rightarrow E$ $\quad$ $B{*}D \rightarrow E$

$a{*}B{*}D \rightarrow E$

$a{*}b{*}D \rightarrow E$

Phase 2i

Sufficient conditions recovered in phase 1 are freed of redundancies to yield minimally sufficient conditions or so-called *prime implicants*.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A{*}B{*}D \to E$ $\quad$ $A{*}D \to E$

$A{*}b{*}D \to E$

$A{*}b{*}d \to E$ $\quad$ $B{*}D \to E$

$a{*}B{*}D \to E$ $\quad$ $A{*}b \to E$

$a{*}b{*}D \to E$

Phase 2i

Sufficient conditions recovered in phase 1 are freed of redundancies to yield minimally sufficient conditions or so-called *prime implicants*.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A{*}B{*}D \rightarrow E$ $\qquad A{*}D \rightarrow E$

$A{*}b{*}D \rightarrow E$ $\qquad b{*}D \rightarrow E$

$A{*}b{*}d \rightarrow E$ $\qquad B{*}D \rightarrow E$

$a{*}B{*}D \rightarrow E$ $\qquad A{*}b \rightarrow E$

$a{*}b{*}D \rightarrow E$

Phase 2i

Sufficient conditions recovered in phase 1 are freed of redundancies to yield minimally sufficient conditions or so-called *prime implicants*.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A{*}B{*}D \rightarrow E$ $A{*}D \rightarrow E$

$A{*}b{*}D \rightarrow E$ $b{*}D \rightarrow E$

$A{*}b{*}d \rightarrow E$ $B{*}D \rightarrow E$

$a{*}B{*}D \rightarrow E$ $A{*}b \rightarrow E$

$a{*}b{*}D \rightarrow E$ $a{*}D \rightarrow E$

Phase 2i

Sufficient conditions recovered in phase 1 are freed of redundancies to yield minimally sufficient conditions or so-called *prime implicants*.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A{*}B{*}D \rightarrow E$ $A{*}D \rightarrow E$

$A{*}b{*}D \rightarrow E$ $b{*}D \rightarrow E$

$A{*}b{*}d \rightarrow E$ $B{*}D \rightarrow E$ $D \rightarrow E$

$a{*}B{*}D \rightarrow E$ $A{*}b \rightarrow E$

$a{*}b{*}D \rightarrow E$ $a{*}D \rightarrow E$

Phase 2i

Sufficient conditions recovered in phase 1 are freed of redundancies to yield minimally sufficient conditions or so-called *prime implicants*.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A{*}B{*}D \rightarrow E$ $A{*}D \rightarrow E$

$A{*}b{*}D \rightarrow E$ $b{*}D \rightarrow E$

$A{*}b{*}d \rightarrow E$ $B{*}D \rightarrow E$

$a{*}B{*}D \rightarrow E$ $A{*}b \rightarrow E$

$a{*}b{*}D \rightarrow E$ $a{*}D \rightarrow E$

$D \rightarrow E$

Phase 2i

Sufficient conditions recovered in phase 1 are freed of redundancies to yield minimally sufficient conditions or so-called *prime implicants*.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A{*}B{*}D \to E$ $\quad$ $A{*}D \to E$

$A{*}b{*}D \to E$ $\quad$ $b{*}D \to E$

$A{*}b{*}d \to E$ $\quad$ $B{*}D \to E$ $\quad$ $D \to E$

$a{*}B{*}D \to E$ $\quad$ $A{*}b \to E$ $\quad$ $A{*}b \to E$

$a{*}b{*}D \to E$ $\quad$ $a{*}D \to E$

Phase 2i

Sufficient conditions recovered in phase 1 are freed of redundancies to yield minimally sufficient conditions or so-called *prime implicants*.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A{*}B{*}D \rightarrow E$ $\quad$ $A{*}D \rightarrow E$

$A{*}b{*}D \rightarrow E$ $\quad$ $b{*}D \rightarrow E$

$A{*}b{*}d \rightarrow E$ $\quad$ $B{*}D \rightarrow E$

$a{*}B{*}D \rightarrow E$ $\quad$ $A{*}b \rightarrow E$

$a{*}b{*}D \rightarrow E$ $\quad$ $a{*}D \rightarrow E$

$D \rightarrow E$

$A{*}b \rightarrow E$

$A{*}b + D \rightarrow E$

Phase 2ii

Necessary conditions recovered in phase 1 are freed of redundancies by means of *prime implicants (PI) charts*.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$E \rightarrow A{*}B{*}D + A{*}b{*}D + A{*}b{*}d + a{*}B{*}D + a{*}b{*}D$$

	$A{*}B{*}D$	$A{*}b{*}D$	$A{*}b{*}d$	$a{*}B{*}D$	$a{*}b{*}D$
$A{*}b$	–	×	×	–	–
D	×	×	–	×	×

➙ Overall, QCA recovers exactly the sought-after MINUS-formula of E:

$$A{*}b + D \leftrightarrow E$$

Phase 2ii

Necessary conditions recovered in phase 1 are freed of redundancies by means of *prime implicants (PI) charts*.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$E \rightarrow A{*}b + D$$

	$A{*}B{*}D$	$A{*}b{*}D$	$A{*}b{*}d$	$a{*}B{*}D$	$a{*}b{*}D$
$A{*}b$	–	×	×	–	–
D	×	×	–	×	×

➙ Overall, QCA recovers exactly the sought-after MINUS-formula of E:

$$A{*}b + D \leftrightarrow E$$

Phase 2ii

Necessary conditions recovered in phase 1 are freed of redundancies by means of *prime implicants (PI) charts*.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$E \rightarrow A{*}b \; + \; D$$

	$A{*}B{*}D$	$A{*}b{*}D$	$A{*}b{*}d$	$a{*}B{*}D$	$a{*}b{*}D$
$A{*}b$	–	×	×	–	–
D	×	×	–	×	×

➙ Overall, QCA recovers exactly the sought-after MINUS-formula of E:

$$A{*}b \; + \; D \; \leftrightarrow \; E$$

The CNA algorithm

CNA infers MINUS-formulas from data δ in four computational phases:

1. CNA identifies a set $\boldsymbol{O} = \{O_1, \ldots, O_n\}$ of candidate outcomes in δ.
2. CNA checks single exogenous factor values in δ for sufficiency for each O_i, then conjunctions of two, of three, etc. Whenever a conjunction is found to be sufficient it is a **minimally sufficient condition** (*msc*) and supersets are not tested anymore.
3. CNA checks single *msc* identified in phase (2) for necessity for each O_i, then disjunctions of two, of three, etc. Whenever a disjunction is found to be necessary it is an **atomic solution formula** (*asf*) and supersets are not tested anymore.
4. CNA conjunctively combines the *asf* identified in phase (3) to **complex solution formulas** (*csf*) and eliminates structural redundancies. The result is the set of all complex MINUS-formulas that fit δ.

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$D \rightarrow E$

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$D \rightarrow E$

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$D \rightarrow E$

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$D \rightarrow E$

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$D \rightarrow E$

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$D \rightarrow E$

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$D \rightarrow E$

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$

$$A{*}b \rightarrow E$$

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$
$$A{*}b \rightarrow E$$

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$

$$A{*}b \rightarrow E$$

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$

$$A{*}b \rightarrow E$$

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$

$$A{*}b \rightarrow E$$

Phase 2

from phase 1: $\boldsymbol{O} = \{E\}$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$
$$A{*}b \rightarrow E$$

$$msc(E) = D,\ A{*}b$$

Phase 3

from phase 2: $msc(E) = D, \; A{*}b$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Phase 3

from phase 2: $msc(E) = D, \ A*b$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Phase 3

from phase 2: $msc(E) = D, \; A{*}b$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Phase 3

from phase 2: $msc(E) = D,\ A{*}b$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Phase 3

from phase 2: $msc(E) = D, \; A{*}b$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Phase 3

from phase 2: $msc(E) = D, \; A{*}b$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Phase 3

from phase 2: $msc(E) = D,\ A*b$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$E \ \rightarrow \ A*b + D$$

Phase 3

from phase 2: $msc(E) = D,\ A{*}b$

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$\frac{E \;\rightarrow\; A{*}b + D}{asf(E) = A{*}b + D \;\leftrightarrow\; E}$$

➙ Overall, CNA recovers exactly the sought-after MINUS-formula of E:

$$A{*}b + D \;\leftrightarrow\; E$$

Fragmentation (limited diversity)

The top-down approach has problems with non-ideal data.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
e	0	1	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$A{*}B{*}D \rightarrow E$$
$$A{*}b{*}D \rightarrow E$$
$$A{*}b{*}d \rightarrow E$$
$$a{*}B{*}D \rightarrow E$$
$$a{*}b{*}D \rightarrow E$$

Fragmentation (limited diversity)

The top-down approach has problems with non-ideal data.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$A{*}B{*}D \rightarrow E$$

$$A{*}b{*}D \rightarrow E$$

$$A{*}b{*}d \rightarrow E$$

$$a{*}b{*}D \rightarrow E$$

Fragmentation (limited diversity)

The top-down approach has problems with non-ideal data.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A{*}B{*}D \rightarrow E$

$A{*}b{*}D \rightarrow E$

$A{*}b{*}d \rightarrow E$

$a{*}b{*}D \rightarrow E$

$A{*}D \rightarrow E$

Fragmentation (limited diversity)

The top-down approach has problems with non-ideal data.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A{*}B{*}D \rightarrow E$

$A{*}b{*}D \rightarrow E$

$A{*}b{*}d \rightarrow E$

$a{*}b{*}D \rightarrow E$

$A{*}D \rightarrow E$

$A{*}b \rightarrow E$

Fragmentation (limited diversity)

The top-down approach has problems with non-ideal data.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$A{*}B{*}D \rightarrow E$

$A{*}b{*}D \rightarrow E$

$A{*}b{*}d \rightarrow E$

$a{*}b{*}D \rightarrow E$

$A{*}D \rightarrow E$

$b{*}D \rightarrow E$

$A{*}b \rightarrow E$

Fragmentation (limited diversity)

The top-down approach has problems with non-ideal data.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$A{*}B{*}D \rightarrow E$$
$$A{*}b{*}D \rightarrow E$$
$$A{*}b{*}d \rightarrow E$$

$$a{*}b{*}D \rightarrow E$$

$$A{*}D \rightarrow E$$
$$b{*}D \rightarrow E$$
$$A{*}b \rightarrow E$$

$$E \rightarrow A{*}D + b{*}D + A{*}b$$

Fragmentation (limited diversity)

The top-down approach has problems with non-ideal data.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$A{*}B{*}D \rightarrow E$$
$$A{*}b{*}D \rightarrow E$$
$$A{*}b{*}d \rightarrow E$$

$$a{*}b{*}D \rightarrow E$$

$$A{*}D \rightarrow E$$
$$b{*}D \rightarrow E$$
$$A{*}b \rightarrow E$$

$$E \rightarrow A{*}D + b{*}D + A{*}b$$

$$A{*}D + b{*}D + A{*}b \leftrightarrow E$$

Fragmentation (limited diversity)

The top-down approach has problems with non-ideal data.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$A{*}B{*}D \rightarrow E$$
$$A{*}b{*}D \rightarrow E$$
$$A{*}b{*}d \rightarrow E$$

$$a{*}b{*}D \rightarrow E$$

$$A{*}D \rightarrow E$$
$$b{*}D \rightarrow E$$
$$A{*}b \rightarrow E$$

$$E \rightarrow A{*}D + b{*}D + A{*}b$$

$$A{*}D + b{*}D + A{*}b \leftrightarrow E$$

BUT: The true structure is $A{*}b + D \leftrightarrow E$.

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	*B*	*D*	*E*
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1

f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1

f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$D \rightarrow E$

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \to E$$

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$

$$A{*}b \rightarrow E$$

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$

$$A{*}b \rightarrow E$$

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$

$$A{*}b \rightarrow E$$

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$
$$A{*}b \rightarrow E$$

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$

$$A{*}b \rightarrow E$$

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$

$$A{*}b \rightarrow E$$

$$msc(E) = D,\ A{*}b$$

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$E \rightarrow D$?

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$E \rightarrow D$?

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$E \rightarrow D$?

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$E \rightarrow D$?

$E \rightarrow A*b$?

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$E \rightarrow D$?

$E \rightarrow A*b$?

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$E \rightarrow D\,?$$
$$E \rightarrow A{*}b\,?$$
$$E \rightarrow A{*}b + D\,?$$

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$E \rightarrow D\,?$

$E \rightarrow A{*}b\,?$

$E \rightarrow A{*}b + D$ ✓

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
d	1	0	0	1

f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$E \rightarrow D$?

$E \rightarrow A{*}b$?

$E \rightarrow A{*}b + D$ ✓

$$asf(E) = A{*}b + D \leftrightarrow E$$

Fragmentation (limited diversity)

The bottom-up approach has no such problems.

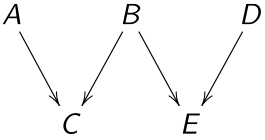
	A	B	D	E
a	1	1	1	1
b	1	1	0	0
c	1	0	1	1
f	0	1	0	0
g	0	0	1	1
h	0	0	0	0

$$D \rightarrow E$$
$$A{*}b \rightarrow E$$
$$E \rightarrow D$$

$$asf(E) = D \leftrightarrow E$$

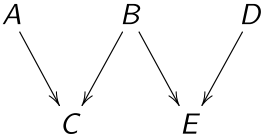
→ CNA recovers the true structure or proper parts thereof from fragmented data.

A multi-outcome example with CNA



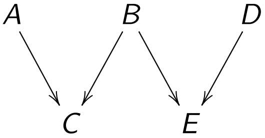
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

A multi-outcome example with CNA



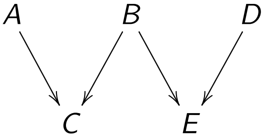
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

A multi-outcome example with CNA



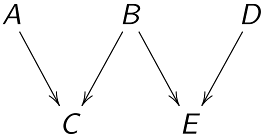
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

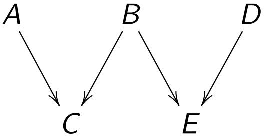
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$

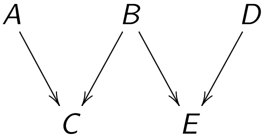
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$

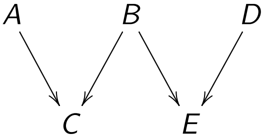
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$

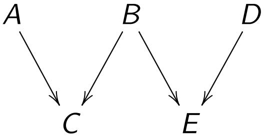
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$

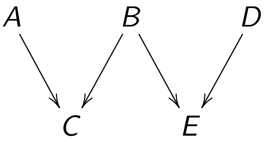
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E\,,\ D \rightarrow E$

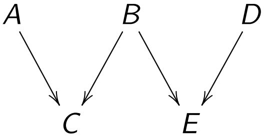
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E\,,\; D \rightarrow E$

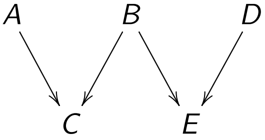
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E\,,\ D \rightarrow E$

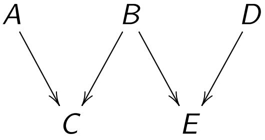
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$, $D \rightarrow E$

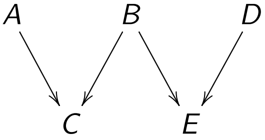
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$, $D \rightarrow E$

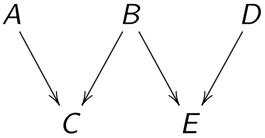
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E\,,\ D \rightarrow E$

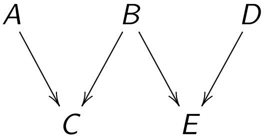
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$, $D \rightarrow E$

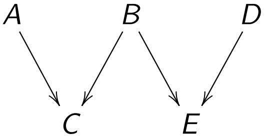
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \to E$, $D \to E$

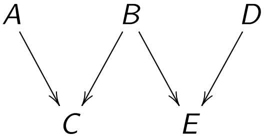
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$, $D \rightarrow E$

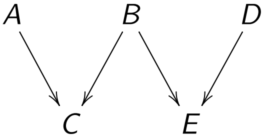
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \to E\,,\ D \to E$

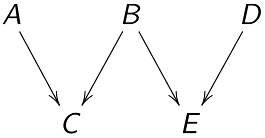
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E\,,\ D \rightarrow E$

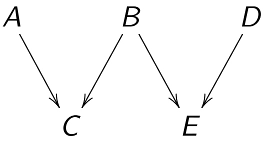
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$, $D \rightarrow E$

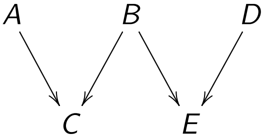
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$, $D \rightarrow E$, $a{*}C \rightarrow E$

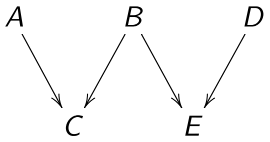
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$, $D \rightarrow E$, $a{*}C \rightarrow E$

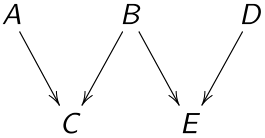
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E\,,\ D \rightarrow E\,,\ a{*}C \rightarrow E$

A multi-outcome example with CNA

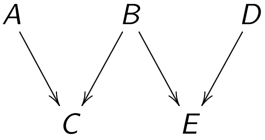


#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$, $D \rightarrow E$, $a*C \rightarrow E$

min_nec: $E \rightarrow B$

A multi-outcome example with CNA

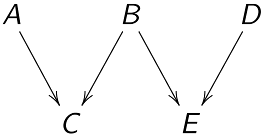


#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$, $D \rightarrow E$, $a{*}C \rightarrow E$

min_nec: ~~$E \rightarrow B$~~

A multi-outcome example with CNA

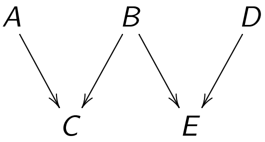


#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$, $D \rightarrow E$, $a{*}C \rightarrow E$

min_nec: ~~$E \rightarrow B$~~, $E \rightarrow D$

A multi-outcome example with CNA

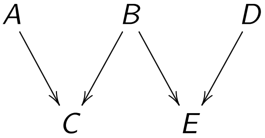


#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$, $D \rightarrow E$, $a{*}C \rightarrow E$

min_nec: ~~$E \rightarrow B$~~ , ~~$E \rightarrow D$~~

A multi-outcome example with CNA

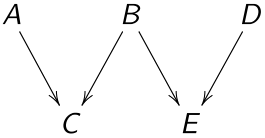


#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$, $D \rightarrow E$, $a{*}C \rightarrow E$

min_nec: ~~$E \rightarrow B$~~, ~~$E \rightarrow D$~~, $E \rightarrow a{*}C$

A multi-outcome example with CNA

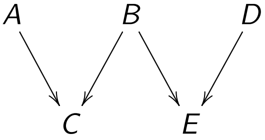


#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \to E$, $D \to E$, $a{*}C \to E$

min_nec: ~~$E \to B$~~ , ~~$E \to D$~~ , ~~$E \to a{*}C$~~

A multi-outcome example with CNA



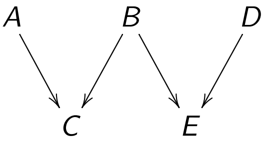
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \to E$, $D \to E$, $a{*}C \to E$

min_nec: ~~$E \to B$~~, ~~$E \to D$~~, ~~$E \to a{*}C$~~

$$E \to B + a{*}C$$

A multi-outcome example with CNA



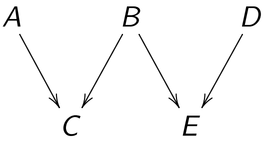
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \to E$, $D \to E$, $a{*}C \to E$

min_nec: ~~$E \to B$~~, ~~$E \to D$~~, ~~$E \to a{*}C$~~

~~$E \to B + a{*}C$~~

A multi-outcome example with CNA



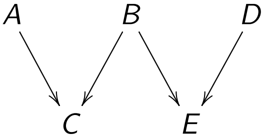
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$, $D \rightarrow E$, $a{*}C \rightarrow E$

min_nec: ~~$E \rightarrow B$~~ , ~~$E \rightarrow D$~~ , ~~$E \rightarrow a{*}C$~~

~~$E \rightarrow B + a{*}C$~~ , $E \rightarrow D + a{*}C$

A multi-outcome example with CNA



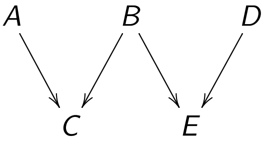
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$, $D \rightarrow E$, $a{*}C \rightarrow E$

min_nec: ~~$E \rightarrow B$~~, ~~$E \rightarrow D$~~, ~~$E \rightarrow a{*}C$~~

~~$E \rightarrow B + a{*}C$~~, ~~$E \rightarrow D + a{*}C$~~

A multi-outcome example with CNA



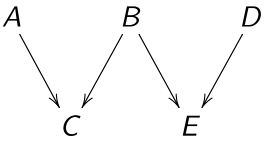
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$, $D \rightarrow E$, $a*C \rightarrow E$

min_nec: ~~$E \rightarrow B$~~, ~~$E \rightarrow D$~~, ~~$E \rightarrow a*C$~~

~~$E \rightarrow B + a*C$~~, ~~$E \rightarrow D + a*C$~~, $E \rightarrow B + D$

A multi-outcome example with CNA



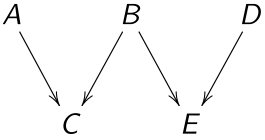
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $B \rightarrow E$, $D \rightarrow E$, $a{*}C \rightarrow E$

min_nec: ~~$E \rightarrow B$~~, ~~$E \rightarrow D$~~, ~~$E \rightarrow a{*}C$~~

~~$E \rightarrow B + a{*}C$~~, ~~$E \rightarrow D + a{*}C$~~, $E \rightarrow B + D$

A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

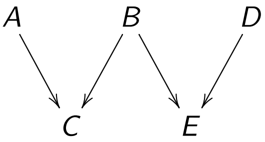
msc: $B \rightarrow E$, $D \rightarrow E$, $a{*}C \rightarrow E$

min_nec: ~~$E \rightarrow B$~~, ~~$E \rightarrow D$~~, ~~$E \rightarrow a{*}C$~~

~~$E \rightarrow B + a{*}C$~~, ~~$E \rightarrow D + a{*}C$~~, $E \rightarrow B + D$

atomic solution formula: $\boxed{B + D \leftrightarrow E}$

A multi-outcome example with CNA



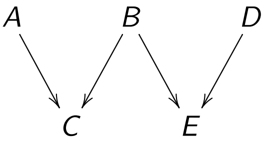
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc:

min_nec:

atomic solution formula:

A multi-outcome example with CNA



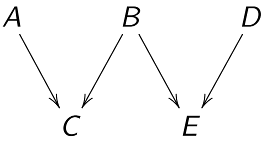
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc:

min_nec:

atomic solution formula:

A multi-outcome example with CNA



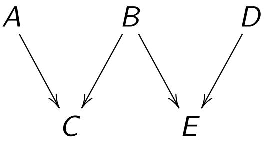
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc:

min_nec:

atomic solution formula:

A multi-outcome example with CNA



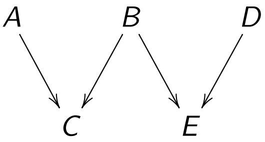
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc:

min_nec:

atomic solution formula:

A multi-outcome example with CNA



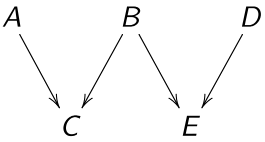
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc:

min_nec:

atomic solution formula:

A multi-outcome example with CNA



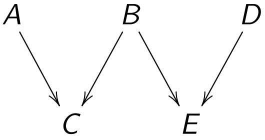
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc:

min_nec:

atomic solution formula:

A multi-outcome example with CNA



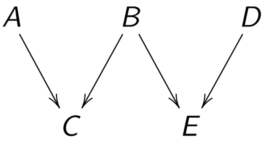
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $b{*}E \rightarrow D$

min_nec:

atomic solution formula:

A multi-outcome example with CNA



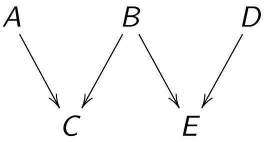
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $b{*}E \rightarrow D$

min_nec:

atomic solution formula:

A multi-outcome example with CNA



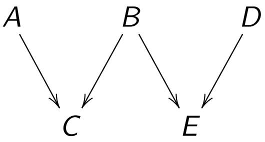
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $b{*}E \rightarrow D$, $c{*}E \rightarrow D$

min_nec:

atomic solution formula:

A multi-outcome example with CNA



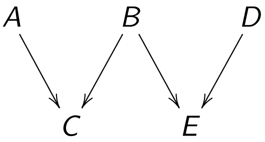
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $b{*}E \rightarrow D$, $c{*}E \rightarrow D$

min_nec: $D \rightarrow b{*}E$

atomic solution formula:

A multi-outcome example with CNA



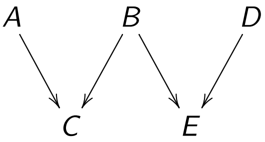
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $b{*}E \rightarrow D$, $c{*}E \rightarrow D$

min_nec: ~~$D \rightarrow b{*}E$~~

atomic solution formula:

A multi-outcome example with CNA



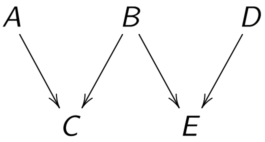
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $b{*}E \rightarrow D$, $c{*}E \rightarrow D$

min_nec: ~~$D \rightarrow b{*}E$~~ , $D \rightarrow c{*}E$

atomic solution formula:

A multi-outcome example with CNA



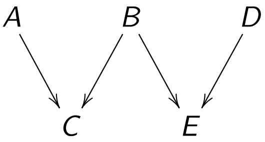
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $b{*}E \rightarrow D$, $c{*}E \rightarrow D$

min_nec: ~~$D \rightarrow b{*}E$~~ , ~~$D \rightarrow c{*}E$~~

atomic solution formula:

A multi-outcome example with CNA



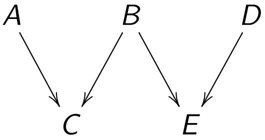
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $b{*}E \rightarrow D$, $c{*}E \rightarrow D$

min_nec: ~~$D \rightarrow b{*}E$~~ , ~~$D \rightarrow c{*}E$~~ , $D \rightarrow b{*}E + c{*}E$

atomic solution formula:

A multi-outcome example with CNA



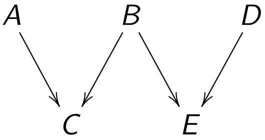
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $b{*}E \rightarrow D$, $c{*}E \rightarrow D$

min_nec: ~~$D \rightarrow b{*}E$~~ , ~~$D \rightarrow c{*}E$~~ , ~~$D \rightarrow b{*}E + c{*}E$~~

atomic solution formula:

A multi-outcome example with CNA



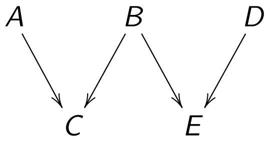
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $b{*}E \rightarrow D$, $c{*}E \rightarrow D$

min_nec: ~~$D \rightarrow b{*}E$~~ , ~~$D \rightarrow c{*}E$~~ , ~~$D \rightarrow b{*}E + c{*}E$~~

atomic solution formula: none

A multi-outcome example with CNA



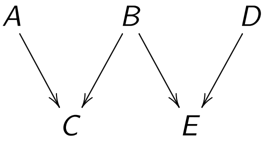
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

msc: $A \to C$, $B \to C$, $d{*}E \to C$

min_nec: $C \;\to\; A + C$

atomic solution formula: $\boxed{A + B \;\leftrightarrow\; C}$

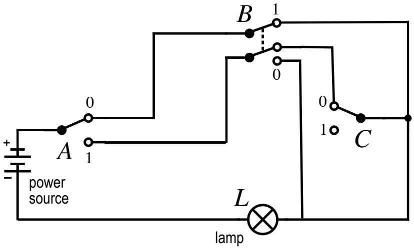
A multi-outcome example with CNA



#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

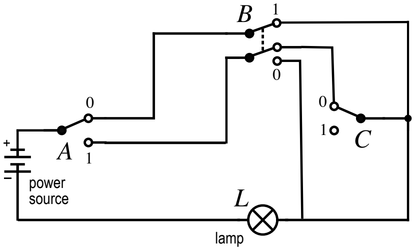
complex solution formula: $(A + B \leftrightarrow C) * (B + D \leftrightarrow E)$

Multiple models



#	A	B	C	L
c_1	0	1	1	1
c_2	1	0	1	1
c_3	1	1	0	1
c_4	0	1	0	1
c_5	1	0	0	1
c_6	1	1	1	0
c_7	0	0	1	0
c_8	0	0	0	0

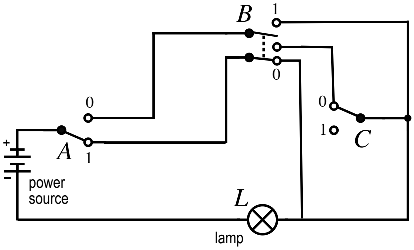
Multiple models



#	A	B	C	L
c_1	0	1	1	1
c_2	1	0	1	1
c_3	1	1	0	1
c_4	0	1	0	1
c_5	1	0	0	1
c_6	1	1	1	0
c_7	0	0	1	0
c_8	0	0	0	0

$msc(L)$: $a{*}B$

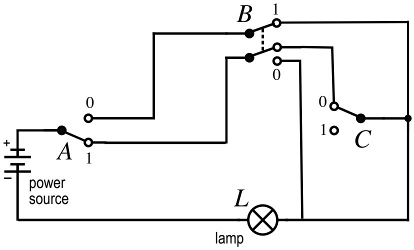
Multiple models



#	A	B	C	L
c_1	0	1	1	1
c_2	1	0	1	1
c_3	1	1	0	1
c_4	0	1	0	1
c_5	1	0	0	1
c_6	1	1	1	0
c_7	0	0	1	0
c_8	0	0	0	0

$msc(L)$: $A{*}b$, $a{*}B$

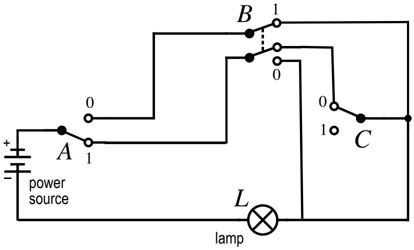
Multiple models



#	A	B	C	L
c_1	0	1	1	1
c_2	1	0	1	1
c_3	1	1	0	1
c_4	0	1	0	1
c_5	1	0	0	1
c_6	1	1	1	0
c_7	0	0	1	0
c_8	0	0	0	0

$msc(L)$: $A{*}b$, $a{*}B$, $A{*}c$

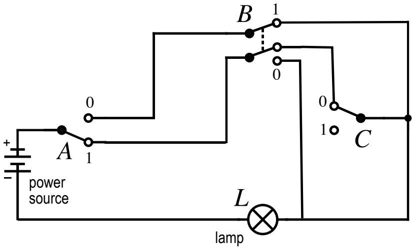
Multiple models



#	A	B	C	L
c_1	0	1	1	1
c_2	1	0	1	1
c_3	1	1	0	1
c_4	0	1	0	1
c_5	1	0	0	1
c_6	1	1	1	0
c_7	0	0	1	0
c_8	0	0	0	0

$msc(L)$: $A{*}b$, $a{*}B$, $A{*}c$, $B{*}c$

Multiple models



#	A	B	C	L
c_1	0	1	1	1
c_2	1	0	1	1
c_3	1	1	0	1
c_4	0	1	0	1
c_5	1	0	0	1
c_6	1	1	1	0
c_7	0	0	1	0
c_8	0	0	0	0

$$msc(L): \quad A{*}b, \quad a{*}B, \quad A{*}c, \quad B{*}c$$

$$asf(L): \quad A{*}b + a{*}B + A{*}c \leftrightarrow L$$

$$A{*}b + a{*}B + B{*}c \leftrightarrow L$$

Incomplete ground truth recovery

- As data δ analyzed by CNA typically are **fragmented**, models inferred from δ are **incomplete**.
- CNA models inferred from fragmented data make claims about causal relevancies, but not about causal irrelevancies. For instance, $A{*}B + a{*}b \leftrightarrow E$ inferred from δ entails
  - A, B, a, and b are causally relevant for E.
  - A and B are on one path and a and b on another.
  - (It does not follow that G is not relevant for E).

$\rightarrow$ A model $\boldsymbol{m_i}$ output by CNA is correct of a ground truth Δ if Ψ is **submodel** of Δ.

The submodel relation

Submodel relation

A model $\boldsymbol{m_i}$ is a submodel of another model $\boldsymbol{m_j}$ iff all causal relevance ascriptions as well as conjunctive and disjunctive groupings entailed by $\boldsymbol{m_i}$ are also entailed by $\boldsymbol{m_j}$.

(The submodel relation is reflexive.)

The submodel relation

#	m_i	m_j	submodel
1	$A{*}b{*}C + D \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	
2	$A{*}b{*}D + f \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	
3	$A + f \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	
4	$b + C \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	
5	$A{*}b + D + G \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	
6	$A{*}b{*}G + D \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	

The submodel relation

#	m_i	m_j	submodel
1	$A{*}b{*}C + D \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✓
2	$A{*}b{*}D + f \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	
3	$A + f \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	
4	$b + C \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	
5	$A{*}b + D + G \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	
6	$A{*}b{*}G + D \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	

The submodel relation

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1	$A{*}b{*}C + D \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✓
2	$A{*}b{*}D + f \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✗
3	$A + f \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	
4	$b + C \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	
5	$A{*}b + D + G \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	
6	$A{*}b{*}G + D \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	

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1	$A{*}b{*}C + D \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✓
2	$A{*}b{*}D + f \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✗
3	$A + f \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✓
4	$b + C \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	
5	$A{*}b + D + G \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	
6	$A{*}b{*}G + D \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	

The submodel relation

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2	$A{*}b{*}D + f \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✗
3	$A + f \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✓
4	$b + C \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✗
5	$A{*}b + D + G \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	
6	$A{*}b{*}G + D \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	

The submodel relation

#	m_i	m_j	submodel
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3	$A + f \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✓
4	$b + C \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✗
5	$A{*}b + D + G \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✗
6	$A{*}b{*}G + D \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	

The submodel relation

#	m_i	m_j	submodel
1	$A{*}b{*}C + D \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✓
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3	$A + f \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✓
4	$b + C \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✗
5	$A{*}b + D + G \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✗
6	$A{*}b{*}G + D \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✓

The submodel relation

#	m_i	m_j	submodel
1	$A{*}b{*}C + D \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✓
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3	$A + f \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✓
4	$b + C \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✗
5	$A{*}b + D + G \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✗
6	$A{*}b{*}G + D \leftrightarrow E$	$A{*}b{*}C{*}G + D{*}f \leftrightarrow E$	✓

➙ If m_j is the ground truth, models 1, 3, and 6 make only true causal claims, i.e. they are **correct**. The other models are incorrect.

Possible CNA outputs

1. **Empty:** There are no data-fitting models. CNA states that the evidence is insufficient to draw conclusions. That is not the same as an inference to causal irrelevance!

2. **One solution formula:** CNA states that the solution formula is a truthful (possibly incomplete) representation of the data-generating structure.

3. **Multiple solution formulas:** CNA states that at least one of the solution formulas is a truthful (possibly incomplete) representation of the data-generating structure but that there is not enough evidence to determine which one exactly.

Model interpretation

Suppose a CNA analysis of a data set over $\boldsymbol{F} = \{A, B, C, D, E, F\}$ yields (1) and (2). How to interpret that?

$$A{*}B + a{*}b + B{*}D \leftrightarrow E \quad (1)$$

$$A{*}B + a{*}b + a{*}D \leftrightarrow E \quad (2)$$

1. (1) OR (2) is a truthful (incomplete) representation of the data-generating structure.
2. $A{*}B$ and $a{*}b$ are parts of the same complex cause of E.
3. There is a further path to E, including either $B{*}D$ OR $a{*}D$.
4. There is no evidence in the data that F is a cause of E.

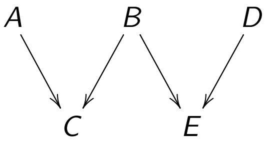
Benchmark criteria

Error-freeness A CNA output is error-free iff it is either empty or contains at least one model that only ascribes causal relevance relations that are present in the data-generating structure.

Correctness A CNA output is correct iff it contains at least one model that only ascribes causal relevance relations that are present in the data-generating structure.

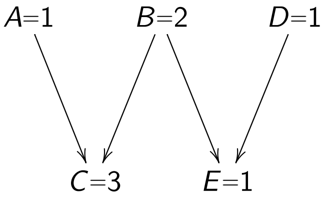
Completeness A CNA output is complete iff it contains at least one model that is identical to the data-generating structure.

Non-binary factors



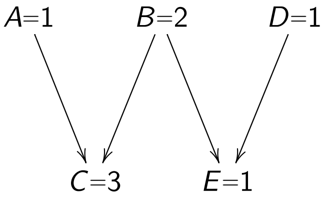
#	A	B	C	D	E
c_1	1	1	1	1	1
c_2	1	1	1	0	1
c_3	1	0	1	1	1
c_4	1	0	1	0	0
c_5	0	1	1	1	1
c_6	0	1	1	0	1
c_7	0	0	0	1	1
c_8	0	0	0	0	0

Non-binary factors



#	A	B	C	D	E
c_1	3	3	2	1	1
c_2	2	2	3	2	1
c_3	2	1	1	2	2
c_4	3	3	1	2	2
c_5	3	1	2	2	2
c_6	1	1	3	2	2
c_7	3	1	1	3	2
c_8	1	2	3	1	1
c_9	2	1	2	2	3
c_{10}	2	1	1	3	3
c_{11}	3	1	2	3	3
c_{12}	3	3	2	3	3

Non-binary factors



#	A	B	C	D	E
c_1	3	3	2	1	1
c_2	2	2	3	2	1
c_3	2	1	1	2	2
c_4	3	3	1	2	2
c_5	3	1	2	2	2
c_6	1	1	3	2	2
c_7	3	1	1	3	2
c_8	1	2	3	1	1
c_9	2	1	2	2	3
c_{10}	2	1	1	3	3
c_{11}	3	1	2	3	3
c_{12}	3	3	2	3	3

$$(A{=}1 + B{=}2 \leftrightarrow C{=}3) * (B{=}2 + D{=}1 \leftrightarrow E{=}1)$$

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