Introduction to Causal Data Analysis and Modeling with Coincidence Analysis

Module 1.1

Methodological landscape and the essentials of Boolean algebra

Michael Baumgartner

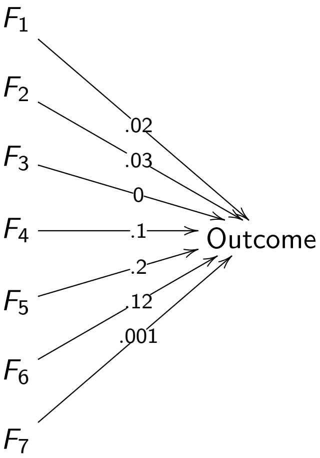
Prague University of Economics and Business

15 May 2023

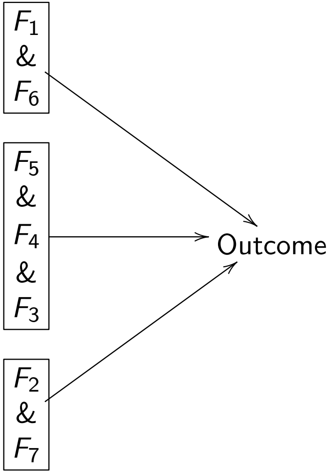
Overview

1. Conjunctivity and Disjunctivity
2. Methodological landscape
3. Foundations of Boolean algebra
4. Propositional logic
5. Boolean operations required by CNA

Two types of methods of causal learning

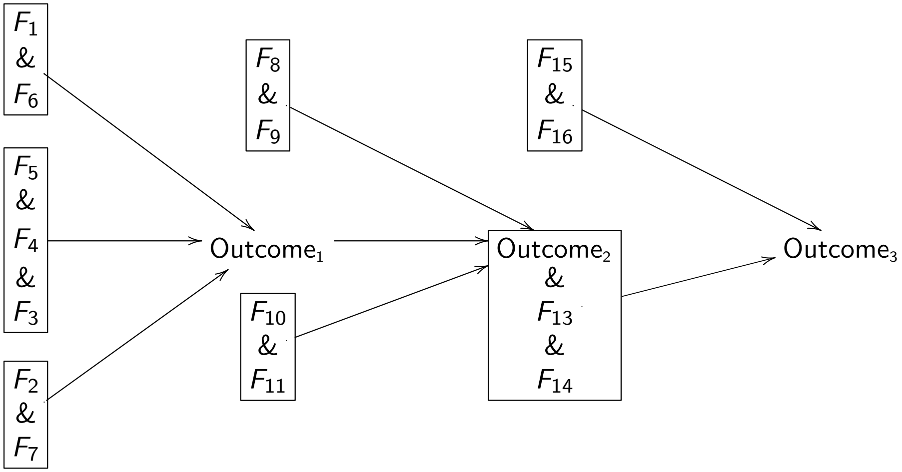


Most standard methods



Qualitative Comparative Analysis (QCA)
Logic Regression (LR)
Coincidence Analysis (CNA)

Coincidence Analysis (CNA)



Conjunctivity and Disjunctivity

Many disciplines investigate causal structures with one or both of the following features:

(a) Causes are arranged in complex bundles that only become operative when all of their components are properly co-instantiated, each of which in isolation is ineffective or leads to different outcomes.

(b) Outcomes can be brought about along alternative causal routes such that, when one route is suppressed, the outcome may still be produced via another one.

Conjunctivity and Disjunctivity

- A desired outcome in health facilities (e.g. high uptake of new treatments) is only obtained if certain implementation strategies are suitably combined (e.g. Yakovchenko et al. 2020), e.g.
  - S_1 AND S_2 AND S_{14} OR
  - S_4 AND S_{11} AND S_{21} OR
  - ...
- A variation in a phenotype only occurs if multiple single-nucleotide polymorphisms (SNPs) interact (e.g. Culverhouse et al. 2002), e.g.
  - SNP_1 AND SNP_4 AND SNP_7 OR
  - SNP_5 AND SNP_{41} AND SNP_{72} OR
  - ...

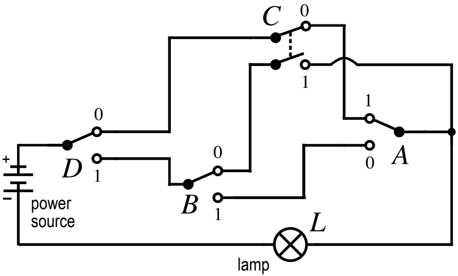
Conjunctivity and Disjunctivity

- **Various labels:** "component causation", "conjunctural causation", "interactions", "alternative causation", "equifinality"

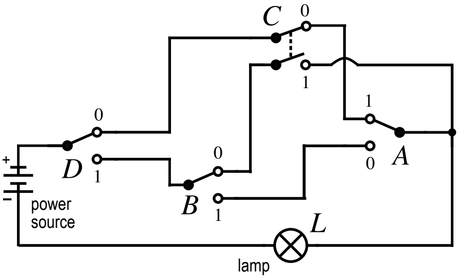
→ **conjunctivity:** cause_1 AND cause_2 AND cause_3

→ **disjunctivity:** path_1 OR path_2 OR path_3

The problem of conjunctivity and disjunctivity

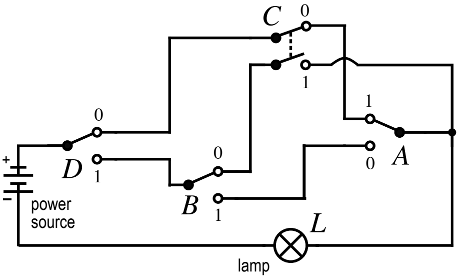


The problem of conjunctivity and disjunctivity



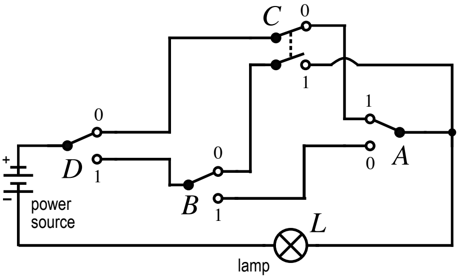
	A	B	D	C	L
c_1	0	1	1	1	1
c_2	1	0	1	1	1
c_3	0	0	1	1	1
c_4	0	1	1	0	1
c_5	1	1	0	0	1
c_6	1	0	0	0	1
c_7	1	1	1	1	0
c_8	1	1	0	1	0
c_9	0	1	0	1	0
c_{10}	1	0	0	1	0
c_{11}	0	0	0	1	0
c_{12}	1	1	1	0	0
c_{13}	1	0	1	0	0
c_{14}	0	0	1	0	0
c_{15}	0	1	0	0	0
c_{16}	0	0	0	0	0

The problem of conjunctivity and disjunctivity



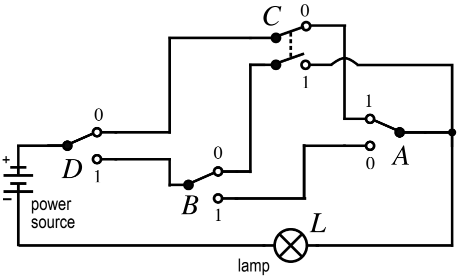
	A	B	D	C	L
c_1	0	1	1	1	1
c_2	1	0	1	1	1
c_3	0	0	1	1	1
c_4	0	1	1	0	1
c_5	1	1	0	0	1
c_6	1	0	0	0	1
c_7	1	1	1	1	0
c_8	1	1	0	1	0
c_9	0	1	0	1	0
c_{10}	1	0	0	1	0
c_{11}	0	0	0	1	0
c_{12}	1	1	1	0	0
c_{13}	1	0	1	0	0
c_{14}	0	0	1	0	0
c_{15}	0	1	0	0	0
c_{16}	0	0	0	0	0

The problem of conjunctivity and disjunctivity



	A	B	D	C	L
c_1	0	1	1	1	1
c_2	1	0	1	1	1
c_3	0	0	1	1	1
c_4	0	1	1	0	1
c_5	1	1	0	0	1
c_6	1	0	0	0	1
c_7	1	1	1	1	0
c_8	1	1	0	1	0
c_9	0	1	0	1	0
c_{10}	1	0	0	1	0
c_{11}	0	0	0	1	0
c_{12}	1	1	1	0	0
c_{13}	1	0	1	0	0
c_{14}	0	0	1	0	0
c_{15}	0	1	0	0	0
c_{16}	0	0	0	0	0

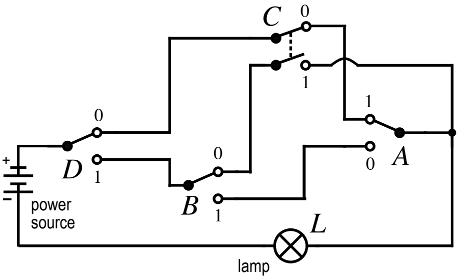
The problem of conjunctivity and disjunctivity



	A	B	C	D	L
A	1.00	0.00	0.00	0.00	0.00
B	0.00	1.00	0.00	0.00	0.00
C	0.00	0.00	1.00	0.00	0.00
D	0.00	0.00	0.00	1.00	0.26
L	0.00	0.00	0.00	0.26	1.00

	A	B	D	C	L
c_1	0	1	1	1	1
c_2	1	0	1	1	1
c_3	0	0	1	1	1
c_4	0	1	1	0	1
c_5	1	1	0	0	1
c_6	1	0	0	0	1
c_7	1	1	1	1	0
c_8	1	1	0	1	0
c_9	0	1	0	1	0
c_{10}	1	0	0	1	0
c_{11}	0	0	0	1	0
c_{12}	1	1	1	0	0
c_{13}	1	0	1	0	0
c_{14}	0	0	1	0	0
c_{15}	0	1	0	0	0
c_{16}	0	0	0	0	0

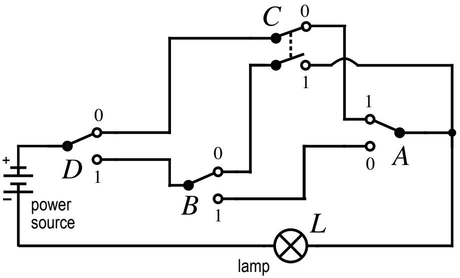
The problem of conjunctivity and disjunctivity



	A	a	Total
L	3	3	8
l	5	5	8
Total	8	8	16

	A	B	D	C	L
c_1	0	1	1	1	1
c_2	1	0	1	1	1
c_3	0	0	1	1	1
c_4	0	1	1	0	1
c_5	1	1	0	0	1
c_6	1	0	0	0	1
c_7	1	1	1	1	0
c_8	1	1	0	1	0
c_9	0	1	0	1	0
c_{10}	1	0	0	1	0
c_{11}	0	0	0	1	0
c_{12}	1	1	1	0	0
c_{13}	1	0	1	0	0
c_{14}	0	0	1	0	0
c_{15}	0	1	0	0	0
c_{16}	0	0	0	0	0

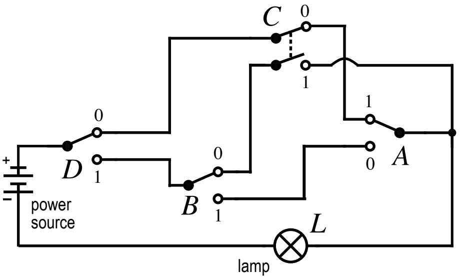
The problem of conjunctivity and disjunctivity



	A	a	Total
L	3	3	8
l	5	5	8
Total	8	8	16

	A	B	D	C	L
c_1	0	1	1	1	1
c_2	1	0	1	1	1
c_3	0	0	1	1	1
c_4	0	1	1	0	1
c_5	1	1	0	0	1
c_6	1	0	0	0	1
c_7	1	1	1	1	0
c_8	1	1	0	1	0
c_9	0	1	0	1	0
c_{10}	1	0	0	1	0
c_{11}	0	0	0	1	0
c_{12}	1	1	1	0	0
c_{13}	1	0	1	0	0
c_{14}	0	0	1	0	0
c_{15}	0	1	0	0	0
c_{16}	0	0	0	0	0

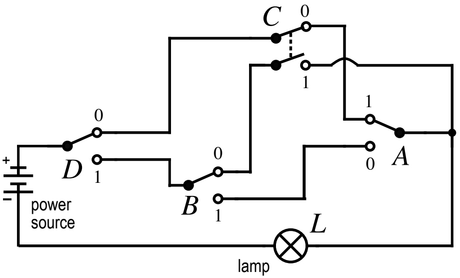
The problem of conjunctivity and disjunctivity



	A	a	Total
L	3	3	8
l	5	5	8
Total	8	8	16

	A	B	D	C	L
c_1	0	1	1	1	1
c_2	1	0	1	1	1
c_3	0	0	1	1	1
c_4	0	1	1	0	1
c_5	1	1	0	0	1
c_6	1	0	0	0	1
c_7	1	1	1	1	0
c_8	1	1	0	1	0
c_9	0	1	0	1	0
c_{10}	1	0	0	1	0
c_{11}	0	0	0	1	0
c_{12}	1	1	1	0	0
c_{13}	1	0	1	0	0
c_{14}	0	0	1	0	0
c_{15}	0	1	0	0	0
c_{16}	0	0	0	0	0

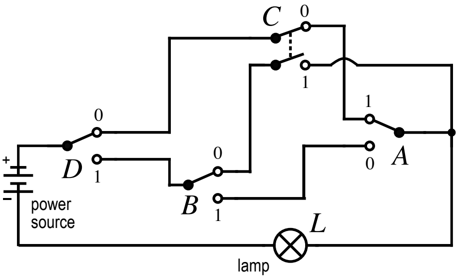
The problem of conjunctivity and disjunctivity



	A	a	Total
L	3	3	8
l	5	5	8
Total	8	8	16

	A	B	D	C	L
c_1	0	1	1	1	1
c_2	1	0	1	1	1
c_3	0	0	1	1	1
c_4	0	1	1	0	1
c_5	1	1	0	0	1
c_6	1	0	0	0	1
c_7	1	1	1	1	0
c_8	1	1	0	1	0
c_9	0	1	0	1	0
c_{10}	1	0	0	1	0
c_{11}	0	0	0	1	0
c_{12}	1	1	1	0	0
c_{13}	1	0	1	0	0
c_{14}	0	0	1	0	0
c_{15}	0	1	0	0	0
c_{16}	0	0	0	0	0

The problem of conjunctivity and disjunctivity



	A	a	Total
L	3	3	8
l	5	5	8
Total	8	8	16

	A	B	D	C	L
c_1	0	1	1	1	1
c_2	1	0	1	1	1
c_3	0	0	1	1	1
c_4	0	1	1	0	1
c_5	1	1	0	0	1
c_6	1	0	0	0	1
c_7	1	1	1	1	0
c_8	1	1	0	1	0
c_9	0	1	0	1	0
c_{10}	1	0	0	1	0
c_{11}	0	0	0	1	0
c_{12}	1	1	1	0	0
c_{13}	1	0	1	0	0
c_{14}	0	0	1	0	0
c_{15}	0	1	0	0	0
c_{16}	0	0	0	0	0

A problem: Conjunctivity and Disjunctivity

Problem

→ Causes and effects in causal structures featuring conjunctivity and disjunctivity may be **pairwise independent**.

- In order to discover such structures, causal factors must be embedded in complex Boolean functions that are fitted to data as a whole.
- **BUT:** The space of Boolean functions is vast: for n binary factors there exist 2^{2^n} possible Boolean functions. For 5 factors: 4'294'967'296

→ Methods for analyzing structures with conjunctivity and disjunctivity must find ways to efficiently navigate in that vast search space.

Different disciplines, different methods, same goal

- **In epidemiology:** sufficient component cause model (Rothman 1976)
- **In biostatistics:** logic regression (Ruczinski, Kooperberg, and LeBlanc 2003)
  - embedded in linear algebra; optimized for data with large sample sizes (1000 cases and more) and high noise levels, large numbers of exogenous factors, but no more than one endogenous factor; binary data only, heuristic model search.
- **In social science:** configurational comparative methods (CCMs) including **CNA** (Ragin 1987; Rihoux and Ragin 2009; Baumgartner and Ambühl 2020)
  - embedded in Boolean algebra; optimized for data with small to large sample sizes (20-2000 cases), low noise levels, low numbers of exogenous factors, but more than one endogenous factor; also multi-value and fuzzy-set data, exhaustive model search.

Different disciplines, different methods, same goal

- **In epidemiology:** sufficient component cause model (Rothman 1976)
- **In biostatistics:** logic regression (Ruczinski, Kooperberg, and LeBlanc 2003)
  - embedded in linear algebra; optimized for data with large sample sizes (1000 cases and more) and high noise levels, large numbers of exogenous factors, but no more than one endogenous factor; binary data only, heuristic model search.
- **In social science:** configurational comparative methods (CCMs) including **CNA** (Ragin 1987; Rihoux and Ragin 2009; Baumgartner and Ambühl 2020)
  - embedded in Boolean algebra; optimized for data with small to large sample sizes (20-2000 cases), low noise levels, low numbers of exogenous factors, but more than one endogenous factor; also multi-value and fuzzy-set data, exhaustive model search.

Different disciplines, different methods, same goal

- **In epidemiology:** sufficient component cause model (Rothman 1976)
- **In biostatistics:** logic regression (Ruczinski, Kooperberg, and LeBlanc 2003)
  - embedded in linear algebra; optimized for data with large sample sizes (1000 cases and more) and high noise levels, large numbers of exogenous factors, but no more than one endogenous factor; binary data only, heuristic model search.
- **In social science:** configurational comparative methods (CCMs) including **CNA** (Ragin 1987; Rihoux and Ragin 2009; Baumgartner and Ambühl 2020)
  - embedded in **Boolean algebra**; optimized for data with small to large sample sizes (20-2000 cases), low noise levels, low numbers of exogenous factors, but more than one endogenous factor; also multi-value and fuzzy-set data, exhaustive model search.

Boolean algebra

A Boolean algebra is a **mathematical object** which consists of a first set $\mathcal{I}$ comprising two distinguished identity elements typically denoted by "1" and "0", and a second set $\mathcal{O}$ consisting of two binary operations "$+$" and "$*$" as well as one unary operation "$-$". For any Boolean algebra, $\mathcal{O}$ is defined over $\mathcal{I}$ such that the following laws hold:

$$x + y = y + x \qquad x * y = y * x \tag{1}$$

$$(x + y) + z = x + (y + z) \qquad (x * y) * z = x * (y * z) \tag{2}$$

$$x + (y * z) = (x + y) * (x + z) \qquad x * (y + z) = (x * y) + (x * z) \tag{3}$$

$$x + 1 = 1 \qquad x * 0 = 0 \tag{4}$$

$$x + 0 = x \qquad x * 1 = x \tag{5}$$

$$x + x = x \qquad x * x = x \tag{6}$$

$$x + (-x) = 1 \qquad x * (-x) = 0 \tag{7}$$

$$(-x) * (-y) = -(x + y) \qquad (-x) + (-y) = -(x * y) \tag{8}$$

Interpretations of a Boolean algebra

- The main interpretations of the elements of a Boolean algebra are:
  - set theory
    - 1 universal set
    - 0 emtpy set
    - $+$ union $\cup$
    - $*$ intersection $\cap$
    - $-$ complement $\overline{x}$
  - propositional logic
    - 1 true
    - 0 false
    - $+$ disjunction $\vee$
    - $*$ conjunction $\wedge$
    - $-$ negation $\neg$
  - switching circuit theory
    - 1 closed switch
    - 0 open switch
    - $+$ switches in parallel
    - $*$ switches in series
    - $-$ change in switch transmittance

Propositional logic

Propositional logic is a **formal language** (L_{prop}) designed to render transparent dependence relations among sentences of other languages, typically natural languages, and to evaluate the validity of arguments in those other languages.

Propositional logic

Propositional logic is a **formal language** (L_{prop}) designed to render transparent dependence relations among sentences of other languages, typically natural languages, and to evaluate the validity of arguments in those other languages.

$$\begin{array}{r} \text{Socrates is human.} \\ \text{If } x \text{ is human, then } x \text{ is mortal.} \\ \hline \text{Socrates is mortal.} \end{array}$$

Propositional logic

Propositional logic is a **formal language** (L_{prop}) designed to render transparent dependence relations among sentences of other languages, typically natural languages, and to evaluate the validity of arguments in those other languages.

$$\frac{\text{Socrates is human.}\quad \text{If } x \text{ is human, then } x \text{ is mortal.}}{\text{Socrates is mortal.}}$$

$$\frac{\text{Socrates is mortal.}\quad \text{If } x \text{ is human, then } x \text{ is mortal.}}{\text{Socrates is human.}}$$

The syntax of L_{prop}

1. Sentence letters: ‘A’, ‘B’, ‘C’,..., ‘Z’, ‘A_0’,..., ‘Z_0’, ‘A_1’,..., ‘Z_1’, ‘A_2’,...
2. Sentential connectives/operators:
   - Negation: ‘$\neg$’ (’not’)
   - Conjunction: ‘$*$’ (’and’)
   - Disjunction: ‘$+$’ (’or’)
   - Material implication: ‘$\rightarrow$’ (‘if...then’)
   - Biconditional (equivalence): ‘$\leftrightarrow$’ (‘if and only if’, ‘iff’)
3. Auxiliary symbols: ‘(’, ‘)’

The syntax of L_{prop}

- In some textbooks, other signs are used for the sentential connectives of propositional logic:

	alternatives
$\neg$	$-$, $\sim$, 'a' for $\neg A$
$*$	&, ., $\wedge$, AB for $A{*}B$
$+$	$\vee$
$\rightarrow$	$\supset$
$\leftrightarrow$	$\equiv$

- In the CCM literature, you will find all sorts of mixtures of these signs. The only sign that is ubiquitously used is '$\rightarrow$'.

The syntax of L_{prop}

Definition – Well-formed formulas: The set of *well-formed formulas* (wffs) of L_{prop} is (recursively) defined as follows:

(1) Every sentence letter is a well-formed formula.

(2) If Φ is a wff, then $\neg\Phi$ is a wff.

(3) If Φ and Ψ are both wffs, then $(\Phi * \Psi)$ is a wff.

(4) If Φ and Ψ are both wffs, then $(\Phi + \Psi)$ is a wff.

(5) If Φ and Ψ are both wffs, then $(\Phi \rightarrow \Psi)$ is a wff.

(6) If Φ and Ψ are both wffs, then $(\Phi \leftrightarrow \Psi)$ is a wff.

(7) No expression that cannot be constructed by a finite number of successive applications of (1)–(6) is a wff.

The semantics of $\mathrm{L}_{\mathrm{prop}}$

The **semantics** of $\mathrm{L}_{\mathrm{prop}}$ is very simple:

- Sentence letters as 'P', 'Q', ... stand for true or false sentences, more specifically, for exactly one truth value from the set $\{1, 0\}$.
- '$\neg$' stands for this unary truth-function:

	$\neg$
1	0
0	1

- '$*$', '$+$', '$\to$', '$\leftrightarrow$' stand for the following binary truth-functions:

		$*$
1	1	1
1	0	0
0	1	0
0	0	0

		$+$
1	1	1
1	0	1
0	1	1
0	0	0

		$\to$
1	1	1
1	0	0
0	1	1
0	0	1

		$\leftrightarrow$
1	1	1
1	0	0
0	1	0
0	0	1

- Auxiliary symbols (parentheses) have no semantic values.

The semantics of L_{prop}

- The meaning of a complex well-formed formula Φ of L_{prop} is **recursively** constructed from the meanings of Φ's components.
- The semantics of complex wffs are standardly and most transparently represented in so-called **truth tables**:

Truth table for $P * \neg Q$:

P	Q	P	$*$	$\neg$	Q
1	1	1			1
1	0	1			0
0	1	0			1
0	0	0			0

The semantics of L_{prop}

- The meaning of a complex well-formed formula Φ of L_{prop} is **recursively** constructed from the meanings of Φ's components.
- The semantics of complex wffs are standardly and most transparently represented in so-called **truth tables**:

Truth table for $P * \neg Q$:

P	Q	P	$*$	$\neg$	Q
1	1	1			1
1	0	1			0
0	1	0			1
0	0	0			0

	$\neg$
1	0
0	1

The semantics of L_{prop}

- The meaning of a complex well-formed formula Φ of L_{prop} is **recursively** constructed from the meanings of Φ's components.
- The semantics of complex wffs are standardly and most transparently represented in so-called **truth tables**:

Truth table for $P * \neg Q$:

P	Q	P	$*$	$\neg$	Q
1	1	1		0	1
1	0	1			0
0	1	0			1
0	0	0			0

	$\neg$
1	0
0	1

The semantics of L_{prop}

- The meaning of a complex well-formed formula Φ of L_{prop} is **recursively** constructed from the meanings of Φ's components.
- The semantics of complex wffs are standardly and most transparently represented in so-called **truth tables**:

Truth table for $P * \neg Q$:

P	Q	P	$*$	$\neg$	Q
1	1	1		0	1
1	0	1		1	0
0	1	0			1
0	0	0			0

	$\neg$
1	0
0	1

The semantics of L_{prop}

- The meaning of a complex well-formed formula Φ of L_{prop} is **recursively** constructed from the meanings of Φ's components.
- The semantics of complex wffs are standardly and most transparently represented in so-called **truth tables**:

Truth table for $P * \neg Q$:

P	Q	P	$*$	$\neg$	Q
1	1	1		0	1
1	0	1		1	0
0	1	0		0	1
0	0	0		1	0

	$\neg$
1	0
0	1

The semantics of L_{prop}

- The meaning of a complex well-formed formula Φ of L_{prop} is **recursively** constructed from the meanings of Φ's components.
- The semantics of complex wffs are standardly and most transparently represented in so-called **truth tables**:

Truth table for $P * \neg Q$:

P	Q	P	$*$	$\neg$	Q
1	1	1		0	1
1	0	1		1	0
0	1	0		0	1
0	0	0		1	0

		$*$
1	1	1
1	0	0
0	1	0
0	0	0

The semantics of L_{prop}

- The meaning of a complex well-formed formula Φ of L_{prop} is **recursively** constructed from the meanings of Φ's components.
- The semantics of complex wffs are standardly and most transparently represented in so-called **truth tables**:

Truth table for $P * \neg Q$:

P	Q	P	$*$	$\neg$	Q
1	1	1	0	0	1
1	0	1		1	0
0	1	0		0	1
0	0	0		1	0

		$*$
1	1	1
1	0	0
0	1	0
0	0	0

The semantics of L_{prop}

- The meaning of a complex well-formed formula Φ of L_{prop} is **recursively** constructed from the meanings of Φ's components.
- The semantics of complex wffs are standardly and most transparently represented in so-called **truth tables**:

Truth table for $P * \neg Q$:

P	Q	P	$*$	$\neg$	Q
1	1	1	0	0	1
1	0	1	1	1	0
0	1	0		0	1
0	0	0		1	0

		$*$
1	1	1
1	0	0
0	1	0
0	0	0

The semantics of $\mathrm{L}_{\mathrm{prop}}$

- The meaning of a complex well-formed formula Φ of $\mathrm{L}_{\mathrm{prop}}$ is **recursively** constructed from the meanings of Φ's components.
- The semantics of complex wffs are standardly and most transparently represented in so-called **truth tables**:

Truth table for $P * \neg Q$:

P	Q	P	$*$	$\neg$	Q
1	1	1	0	0	1
1	0	1	1	1	0
0	1	0	0	0	1
0	0	0		1	0

		$*$
1	1	1
1	0	0
0	1	0
0	0	0

The semantics of L_{prop}

- The meaning of a complex well-formed formula Φ of L_{prop} is **recursively** constructed from the meanings of Φ's components.
- The semantics of complex wffs are standardly and most transparently represented in so-called **truth tables**:

Truth table for $P * \neg Q$:

P	Q	P	$*$	$\neg$	Q
1	1	1	0	0	1
1	0	1	1	1	0
0	1	0	0	0	1
0	0	0	0	1	0

		$*$
1	1	1
1	0	0
0	1	0
0	0	0

The semantics of L_{prop}

- The meaning of a complex well-formed formula Φ of L_{prop} is **recursively** constructed from the meanings of Φ's components.
- The semantics of complex wffs are standardly and most transparently represented in so-called **truth tables**:

Truth table for $P * \neg Q$:

P	Q	P	$*$	$\neg$	Q
1	1	1	0	0	1
1	0	1	1	1	0
0	1	0	0	0	1
0	0	0	0	1	0

		$*$
1	1	1
1	0	0
0	1	0
0	0	0

The semantics of L_{prop}

- The meaning of a complex well-formed formula Φ of L_{prop} is **recursively** constructed from the meanings of Φ's components.
- The semantics of complex wffs are standardly and most transparently represented in so-called **truth tables**:

Truth table for $P * \neg Q$:

P	Q	P	$*$	$\neg$	Q
1	1	1	0	0	1
1	0	1	1	1	0
0	1	0	0	0	1
0	0	0	0	1	0

		$*$
1	1	1
1	0	0
0	1	0
0	0	0

- ➙ Truth tables are to be filled out (evaluated) starting with the most binding connectives (inside parentheses) and gradually progressing to the least binding connective (outside of parentheses).
- ➙ The least binding connective (outside of parentheses) in a formula Φ is the **main operator** of Φ. The main operator ultimately determines the **truth conditions** (or the meaning) of Φ.

The semantics of L_{prop}

Truth table for $(P + Q) \to \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\to$	$\neg$	S
1	1	1	1		1			1
1	1	0	1		1			0
1	0	1	1		0			1
1	0	0	1		0			0
0	1	1	0		1			1
0	1	0	0		1			0
0	0	1	0		0			1
0	0	0	0		0			0

The semantics of L_{prop}

Truth table for $(P + Q) \to \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\to$	$\neg$	S
1	1	1	1		1			1
1	1	0	1		1			0
1	0	1	1		0			1
1	0	0	1		0			0
0	1	1	0		1			1
0	1	0	0		1			0
0	0	1	0		0			1
0	0	0	0		0			0

	$\neg$
1	0
0	1

The semantics of L_{prop}

Truth table for $(P + Q) \to \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\to$	$\neg$	S
1	1	1	1		1		0	1
1	1	0	1		1		1	0
1	0	1	1		0		0	1
1	0	0	1		0		1	0
0	1	1	0		1		0	1
0	1	0	0		1		1	0
0	0	1	0		0		0	1
0	0	0	0		0		1	0

	$\neg$
1	0
0	1

The semantics of L_{prop}

Truth table for $(P + Q) \to \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\to$	$\neg$	S
1	1	1	1		1		0	1
1	1	0	1		1		1	0
1	0	1	1		0		0	1
1	0	0	1		0		1	0
0	1	1	0		1		0	1
0	1	0	0		1		1	0
0	0	1	0		0		0	1
0	0	0	0		0		1	0

		$+$
1	1	1
1	0	1
0	1	1
0	0	0

The semantics of L_{prop}

Truth table for $(P + Q) \to \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\to$	$\neg$	S
1	1	1	1	1	1		0	1
1	1	0	1	1	1		1	0
1	0	1	1		0		0	1
1	0	0	1		0		1	0
0	1	1	0		1		0	1
0	1	0	0		1		1	0
0	0	1	0		0		0	1
0	0	0	0		0		1	0

		$+$
1	1	1
1	0	1
0	1	1
0	0	0

The semantics of L_{prop}

Truth table for $(P + Q) \to \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\to$	$\neg$	S
1	1	1	1	1	1		0	1
1	1	0	1	1	1		1	0
1	0	1	1	1	0		0	1
1	0	0	1		0		1	0
0	1	1	0		1		0	1
0	1	0	0		1		1	0
0	0	1	0		0		0	1
0	0	0	0		0		1	0

		$+$
1	1	1
1	0	1
0	1	1
0	0	0

The semantics of L_{prop}

Truth table for $(P + Q) \to \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\to$	$\neg$	S
1	1	1	1	1	1		0	1
1	1	0	1	1	1		1	0
1	0	1	1	1	0		0	1
1	0	0	1	1	0		1	0
0	1	1	0		1		0	1
0	1	0	0		1		1	0
0	0	1	0		0		0	1
0	0	0	0		0		1	0

		$+$
1	1	1
1	0	1
0	1	1
0	0	0

The semantics of L_{prop}

Truth table for $(P + Q) \to \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\to$	$\neg$	S
1	1	1	1	1	1		0	1
1	1	0	1	1	1		1	0
1	0	1	1	1	0		0	1
1	0	0	1	1	0		1	0
0	1	1	0	1	1		0	1
0	1	0	0	1	1		1	0
0	0	1	0		0		0	1
0	0	0	0		0		1	0

		$+$
1	1	1
1	0	1
0	1	1
0	0	0

The semantics of L_{prop}

Truth table for $(P + Q) \to \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\to$	$\neg$	S
1	1	1	1	1	1		0	1
1	1	0	1	1	1		1	0
1	0	1	1	1	0		0	1
1	0	0	1	1	0		1	0
0	1	1	0	1	1		0	1
0	1	0	0	1	1		1	0
0	0	1	0	0	0		0	1
0	0	0	0	0	0		1	0

		$+$
1	1	1
1	0	1
0	1	1
0	0	0

The semantics of L_{prop}

Truth table for $(P + Q) \rightarrow \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\rightarrow$	$\neg$	S
1	1	1	1	1	1		0	1
1	1	0	1	1	1		1	0
1	0	1	1	1	0		0	1
1	0	0	1	1	0		1	0
0	1	1	0	1	1		0	1
0	1	0	0	1	1		1	0
0	0	1	0	0	0		0	1
0	0	0	0	0	0		1	0

		$\rightarrow$
1	1	1
1	0	0
0	1	1
0	0	1

The semantics of L_{prop}

Truth table for $(P + Q) \to \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\to$	$\neg$	S
1	1	1	1	1	1	0	0	1
1	1	0	1	1	1		1	0
1	0	1	1	1	0		0	1
1	0	0	1	1	0		1	0
0	1	1	0	1	1		0	1
0	1	0	0	1	1		1	0
0	0	1	0	0	0		0	1
0	0	0	0	0	0		1	0

		$\to$
1	1	1
1	0	0
0	1	1
0	0	1

The semantics of L_{prop}

Truth table for $(P + Q) \to \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\to$	$\neg$	S
1	1	1	1	1	1	0	0	1
1	1	0	1	1	1	1	1	0
1	0	1	1	1	0		0	1
1	0	0	1	1	0		1	0
0	1	1	0	1	1		0	1
0	1	0	0	1	1		1	0
0	0	1	0	0	0		0	1
0	0	0	0	0	0		1	0

		$\to$
1	1	1
1	0	0
0	1	1
0	0	1

The semantics of L_{prop}

Truth table for $(P + Q) \to \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\to$	$\neg$	S
1	1	1	1	1	1	0	0	1
1	1	0	1	1	1	1	1	0
1	0	1	1	1	0	0	0	1
1	0	0	1	1	0	1	1	0
0	1	1	0	1	1		0	1
0	1	0	0	1	1		1	0
0	0	1	0	0	0		0	1
0	0	0	0	0	0		1	0

		$\to$
1	1	1
1	0	0
0	1	1
0	0	1

The semantics of L_{prop}

Truth table for $(P + Q) \to \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\to$	$\neg$	S
1	1	1	1	1	1	0	0	1
1	1	0	1	1	1	1	1	0
1	0	1	1	1	0	0	0	1
1	0	0	1	1	0	1	1	0
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	1	1	0
0	0	1	0	0	0		0	1
0	0	0	0	0	0		1	0

		$\to$
1	1	1
1	0	0
0	1	1
0	0	1

The semantics of L_{prop}

Truth table for $(P + Q) \rightarrow \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\rightarrow$	$\neg$	S
1	1	1	1	1	1	0	0	1
1	1	0	1	1	1	1	1	0
1	0	1	1	1	0	0	0	1
1	0	0	1	1	0	1	1	0
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	1	1	0
0	0	1	0	0	0	1	0	1
0	0	0	0	0	0		1	0

		$\rightarrow$
1	1	1
1	0	0
0	1	1
0	0	1

The semantics of L_{prop}

Truth table for $(P + Q) \to \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\to$	$\neg$	S
1	1	1	1	1	1	0	0	1
1	1	0	1	1	1	1	1	0
1	0	1	1	1	0	0	0	1
1	0	0	1	1	0	1	1	0
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	1	1	0
0	0	1	0	0	0	1	0	1
0	0	0	0	0	0	1	1	0

		$\to$
1	1	1
1	0	0
0	1	1
0	0	1

The semantics of L_{prop}

Truth table for $(P + Q) \to \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\to$	$\neg$	S
1	1	1	1	1	1	0	0	1
1	1	0	1	1	1	1	1	0
1	0	1	1	1	0	0	0	1
1	0	0	1	1	0	1	1	0
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	1	1	0
0	0	1	0	0	0	1	0	1
0	0	0	0	0	0	1	1	0

		$\to$
1	1	1
1	0	0
0	1	1
0	0	1

The semantics of L_{prop}

Truth table for $(P + Q) \to \neg S$:

P	Q	S	$(P$	$+$	$Q)$	$\to$	$\neg$	S
1	1	1	1	1	1	0	0	1
1	1	0	1	1	1	1	1	0
1	0	1	1	1	0	0	0	1
1	0	0	1	1	0	1	1	0
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	1	1	0
0	0	1	0	0	0	1	0	1
0	0	0	0	0	0	1	1	0

		$\to$
1	1	1
1	0	0
0	1	1
0	0	1

- The number of different sentence letters in a formula determines the size of the corresponding truth table: the truth table of a formula with n different sentence letters has 2^n rows.

The semantics of L_{prop}

Truth table for $P \rightarrow (Q + \neg Q)$:

P	Q	P	$\rightarrow$	$(Q$	$+$	$\neg$	$Q)$
1	1						
1	0						
0	1						
0	0						

The semantics of L_{prop}

Truth table for $P \to (Q + \neg Q)$:

P	Q	P	$\to$	$(Q$	$+$	$\neg$	$Q)$
1	1					0	
1	0						
0	1						
0	0						

The semantics of L_{prop}

Truth table for $P \to (Q + \neg Q)$:

P	Q	P	$\to$	$(Q$	$+$	$\neg$	$Q)$
1	1					0	
1	0					1	
0	1					0	
0	0					1	

The semantics of L_{prop}

Truth table for $P \rightarrow (Q + \neg Q)$:

P	Q	P	$\rightarrow$	$(Q$	$+$	$\neg$	$Q)$
1	1				1	0	
1	0					1	
0	1					0	
0	0					1	

The semantics of L_{prop}

Truth table for $P \to (Q + \neg Q)$:

P	Q	P	$\to$	$(Q$	$+$	$\neg$	$Q)$
1	1				1	0	
1	0				1	1	
0	1					0	
0	0					1	

The semantics of L_{prop}

Truth table for $P \rightarrow (Q + \neg Q)$:

P	Q	P	$\rightarrow$	$(Q$	$+$	$\neg$	$Q)$
1	1				1	0	
1	0				1	1	
0	1				1	0	
0	0				1	1	

The semantics of L_{prop}

Truth table for $P \rightarrow (Q + \neg Q)$:

P	Q	P	$\rightarrow$	$(Q$	$+$	$\neg$	$Q)$
1	1		1		1	0	
1	0		1		1	1	
0	1				1	0	
0	0				1	1	

The semantics of L_{prop}

Truth table for $P \rightarrow (Q + \neg Q)$:

P	Q	P	$\rightarrow$	$(Q$	$+$	$\neg$	$Q)$
1	1		1		1	0	
1	0		1		1	1	
0	1		1		1	0	
0	0		1		1	1	

The semantics of L_{prop}

Truth table for $P \rightarrow (Q + \neg Q)$:

P	Q	P	$\rightarrow$	$(Q$	$+$	$\neg$	$Q)$
1	1		1		1	0	
1	0		1		1	1	
0	1		1		1	0	
0	0		1		1	1	

The semantics of L_{prop}

Truth table for $P \rightarrow (Q + \neg Q)$:

P	Q	P	$\rightarrow$	$(Q$	$+$	$\neg$	$Q)$
1	1		1		1	0	
1	0		1		1	1	
0	1		1		1	0	
0	0		1		1	1	

➙ A formula Φ is a **tautology / logically true** iff the main operator of Φ only takes the truth value 1, i.e. iff every assignment of truth values satisfies Φ.

The semantics of L_{prop}

Truth table for $\neg((P \to Q) \leftrightarrow (\neg Q \to \neg P))$:

P	Q	$\neg$	$((P$	$\to$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\to$	$\neg$	$P))$
1	1										
1	0										
0	1										
0	0										

The semantics of L_{prop}

Truth table for $\neg((P \to Q) \leftrightarrow (\neg Q \to \neg P))$:

P	Q	$\neg$	$((P$	$\to$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\to$	$\neg$	$P))$
1	1						0				
1	0						1				
0	1						0				
0	0						1				

The semantics of L_{prop}

Truth table for $\neg((P \to Q) \leftrightarrow (\neg Q \to \neg P))$:

P	Q	$\neg$	$((P$	$\to$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\to$	$\neg$	$P))$
1	1						0			0	
1	0						1			0	
0	1						0			1	
0	0						1			1	

The semantics of L_{prop}

Truth table for $\neg((P \to Q) \leftrightarrow (\neg Q \to \neg P))$:

P	Q	$\neg$	$((P$	$\to$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\to$	$\neg$	$P))$
1	1			1			0			0	
1	0						1			0	
0	1						0			1	
0	0						1			1	

The semantics of $\mathrm{L}_{\mathrm{prop}}$

Truth table for $\neg((P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P))$:

P	Q	$\neg$	$((P$	$\rightarrow$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\rightarrow$	$\neg$	$P))$
1	1			1			0			0	
1	0			0			1			0	
0	1						0			1	
0	0						1			1	

The semantics of L_{prop}

Truth table for $\neg((P \to Q) \leftrightarrow (\neg Q \to \neg P))$:

P	Q	$\neg$	$((P$	$\to$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\to$	$\neg$	$P))$
1	1			1			0			0	
1	0			0			1			0	
0	1			1			0			1	
0	0						1			1	

The semantics of L_{prop}

Truth table for $\neg((P \to Q) \leftrightarrow (\neg Q \to \neg P))$:

P	Q	$\neg$	$((P$	$\to$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\to$	$\neg$	$P))$
1	1			1			0			0	
1	0			0			1			0	
0	1			1			0			1	
0	0			1			1			1	

The semantics of L_{prop}

Truth table for $\neg((P \to Q) \leftrightarrow (\neg Q \to \neg P))$:

P	Q	$\neg$	$((P$	$\to$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\to$	$\neg$	$P))$
1	1			1			0		1	0	
1	0			0			1			0	
0	1			1			0			1	
0	0			1			1			1	

The semantics of $\mathrm{L}_{\mathrm{prop}}$

Truth table for $\neg((P \to Q) \leftrightarrow (\neg Q \to \neg P))$:

P	Q	$\neg$	$((P$	$\to$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\to$	$\neg$	$P))$
1	1			1			0		1	0	
1	0			0			1		0	0	
0	1			1			0			1	
0	0			1			1			1	

The semantics of L_{prop}

Truth table for $\neg((P \to Q) \leftrightarrow (\neg Q \to \neg P))$:

P	Q	$\neg$	$((P$	$\to$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\to$	$\neg$	$P))$
1	1			1			0		1	0	
1	0			0			1		0	0	
0	1			1			0		1	1	
0	0			1			1			1	

The semantics of L_{prop}

Truth table for $\neg((P \to Q) \leftrightarrow (\neg Q \to \neg P))$:

P	Q	$\neg$	$((P$	$\to$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\to$	$\neg$	$P))$
1	1			1			0		1	0	
1	0			0			1		0	0	
0	1			1			0		1	1	
0	0			1			1		1	1	

The semantics of L_{prop}

Truth table for $\neg((P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P))$:

P	Q	$\neg$	$((P$	$\rightarrow$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\rightarrow$	$\neg$	$P))$
1	1			1			0		1	0	
1	0			0			1		0	0	
0	1			1			0		1	1	
0	0			1			1		1	1	

		$\leftrightarrow$
1	1	1
1	0	0
0	1	0
0	0	1

The semantics of L_{prop}

Truth table for $\neg((P \to Q) \leftrightarrow (\neg Q \to \neg P))$:

P	Q	$\neg$	$((P$	$\to$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\to$	$\neg$	$P))$
1	1			1		1	0		1	0	
1	0			0			1		0	0	
0	1			1			0		1	1	
0	0			1			1		1	1	

		$\leftrightarrow$
1	1	1
1	0	0
0	1	0
0	0	1

The semantics of L_{prop}

Truth table for $\neg((P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P))$:

P	Q	$\neg$	$((P$	$\rightarrow$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\rightarrow$	$\neg$	$P))$
1	1			1		1	0		1	0	
1	0			0		1	1		0	0	
0	1			1			0		1	1	
0	0			1			1		1	1	

		$\leftrightarrow$
1	1	1
1	0	0
0	1	0
0	0	1

The semantics of L_{prop}

Truth table for $\neg((P \to Q) \leftrightarrow (\neg Q \to \neg P))$:

P	Q	$\neg$	$((P$	$\to$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\to$	$\neg$	$P))$
1	1			1		1	0		1	0	
1	0			0		1	1		0	0	
0	1			1		1	0		1	1	
0	0			1			1		1	1	

		$\leftrightarrow$
1	1	1
1	0	0
0	1	0
0	0	1

The semantics of L_{prop}

Truth table for $\neg((P \to Q) \leftrightarrow (\neg Q \to \neg P))$:

P	Q	$\neg$	$((P$	$\to$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\to$	$\neg$	$P))$
1	1			1		1	0		1	0	
1	0			0		1	1		0	0	
0	1			1		1	0		1	1	
0	0			1		1	1		1	1	

		$\leftrightarrow$
1	1	1
1	0	0
0	1	0
0	0	1

The semantics of L_{prop}

Truth table for $\neg((P \to Q) \leftrightarrow (\neg Q \to \neg P))$:

P	Q	$\neg$	$((P$	$\to$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\to$	$\neg$	$P))$
1	1	0		1		1	0		1	0	
1	0	0		0		1	1		0	0	
0	1	0		1		1	0		1	1	
0	0	0		1		1	1		1	1	

The semantics of L_{prop}

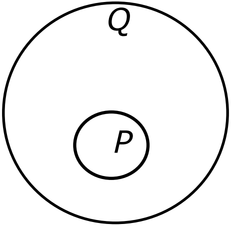
Truth table for $\neg((P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P))$:

P	Q	$\neg$	$((P$	$\rightarrow$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\rightarrow$	$\neg$	$P))$
1	1	0		1		1	0		1	0	
1	0	0		0		1	1		0	0	
0	1	0		1		1	0		1	1	
0	0	0		1		1	1		1	1	

- A formula Φ is a **contradiction / logically false** iff the main operator of Φ only takes the truth value 0, i.e. iff no assignment of truth values satisfies Φ.
- Two formulas Φ and Ψ are **equivalent** iff Φ and Ψ have the same distribution of truth values under their main operators, i.e. iff Φ and Ψ have the same truth conditions.

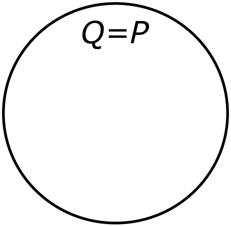
The implication operator $\rightarrow$

- Implication is the core operator for CNA, which searches for sufficient and necessary conditions of causally modeled outcomes.
- P is called a **sufficient condition** of Q if, and only if, $P \rightarrow Q$ is true, i.e. if P is a subset of Q.
- Q is called a **necessary condition** of P if, and only if, $P \rightarrow Q$ is true, i.e. if Q is superset of P.



The implication operator $\rightarrow$

- In $P \rightarrow Q$, P is called the **antecedent** and Q the **consequent**.
- That P is both sufficient and necessary for Q, i.e. $(P \rightarrow Q)*(Q \rightarrow P)$, is expressed by the biconditional (equivalence) $P \leftrightarrow Q$.



The implication operator $\rightarrow$

- The standard translation (into natural language) of '$P \rightarrow Q$' is 'If P, then Q'. However, the natural language usage of 'If…then' is very ambiguous. Compare:
  - If Peter correctly calculates the sum of two even numbers, then he gets an even number.
  - If it rains, then the street gets wet.
  - If Brian passes the final exam, then all humans grow wings.
  - If Brian passes the final exam, then he sells his logic book.

➙ If the 'If'-part of a natural language 'If…then' statement de facto is false, the statement as a whole is sometimes true, sometimes false, and sometimes undetermined.

- However, a formal language cannot allow for these kinds of ambiguities. Therefore, it has been conventionally determined that '$P \rightarrow Q$' is always to be understood in terms of an 'If…then' statement that is true when its 'If'-part is false, i.e. in terms of a so-called **material implication**.

The implication operator $\rightarrow$

- A material implication states exactly the same as a disjunction with a negated first disjunct or as a negated conjunction with a negated conjunct.

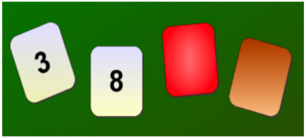
P	Q	$P \rightarrow Q$	$\neg P + Q$	$\neg(P * \neg Q)$
1	1	1	1	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

$\rightarrow$ The implication operator **has no causal connotation** whatsoever!

The implication operator $\rightarrow$

To test whether you correctly understand the implication operator, consider the **Wason Selection Task**:

- You are shown a set of four cards each of which has a number on one side and a coloured patch on the other side. The visible faces of the cards show 3, 8, red and brown:

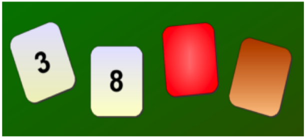


- **Question 1:** Which cards do you need to turn over in order to test the truth of the following implication (you should turn over as few cards as possible)?
    - If a card shows an even number, then its opposite face is red.

The implication operator $\to$

To test whether you correctly understand the implication operator, consider the **Wason Selection Task**:

- You are shown a set of four cards each of which has a number on one side and a coloured patch on the other side. The visible faces of the cards show 3, 8, red and brown:



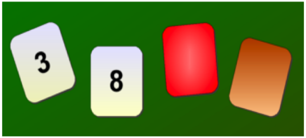
- **Question 1:** Which cards do you need to turn over in order to test the truth of the following implication (you should turn over as few cards as possible)?
  - If a card shows an even number, then its opposite face is red.

➔ You have to turn over the card showing the number 8 and the brown card.

The implication operator $\rightarrow$

To test whether you correctly understand the implication operator, consider the **Wason Selection Task**:

- You are shown a set of four cards each of which has a number on one side and a coloured patch on the other side. The visible faces of the cards show 3, 8, red and brown:

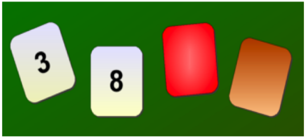


- **Question 2:** Which cards do you need to turn over in order to test the truth of the following implication (you should turn over as few cards as possible)?
  - If a card is not brown, its opposite face shows an even number.

The implication operator $\rightarrow$

To test whether you correctly understand the implication operator, consider the **Wason Selection Task**:

- You are shown a set of four cards each of which has a number on one side and a coloured patch on the other side. The visible faces of the cards show 3, 8, red and brown:



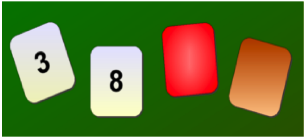
- **Question 2:** Which cards do you need to turn over in order to test the truth of the following implication (you should turn over as few cards as possible)?
    - If a card is not brown, its opposite face shows an even number.

➙ You have to turn over the card showing the number 3 and the red card.

The implication operator $\rightarrow$

To test whether you correctly understand the implication operator, consider the **Wason Selection Task**:

- You are shown a set of four cards each of which has a number on one side and a coloured patch on the other side. The visible faces of the cards show 3, 8, red and brown:

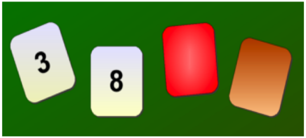


- **Question 3:** Which cards do you need to turn over in order to test the truth of the following implication (you should turn over as few cards as possible)?
  - If a card shows the number 5, its opposite face is brown.

The implication operator $\rightarrow$

To test whether you correctly understand the implication operator, consider the **Wason Selection Task**:

- You are shown a set of four cards each of which has a number on one side and a coloured patch on the other side. The visible faces of the cards show 3, 8, red and brown:



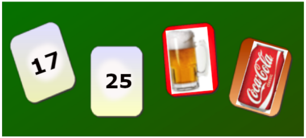
- **Question 3:** Which cards do you need to turn over in order to test the truth of the following implication (you should turn over as few cards as possible)?
  - If a card shows the number 5, its opposite face is brown.

➙ You have to turn over the red card.

The implication operator $\rightarrow$

To test whether you correctly understand the implication operator, consider the **Wason Selection Task**:

- You are shown a set of four cards each of which has an age on one side and a beverage on the other side. The visible faces of the cards show 17, 25, beer and Coke:

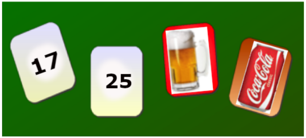


- **Question 4:** Which cards do you need to turn over in order to test the truth of the following implication (you should turn over as few cards as possible)?
  - If the beverage is alcoholic, the age must be 21 or more.

The implication operator $\rightarrow$

To test whether you correctly understand the implication operator, consider the **Wason Selection Task**:

- You are shown a set of four cards each of which has an age on one side and a beverage on the other side. The visible faces of the cards show 17, 25, beer and Coke:



- **Question 4:** Which cards do you need to turn over in order to test the truth of the following implication (you should turn over as few cards as possible)?
  - If the beverage is alcoholic, the age must be 21 or more.

➙ You have to turn over the 17 and the beer card.

Baumgartner, M. and M. Ambühl (2020).
Causal modeling with multi-value and fuzzy-set Coincidence Analysis.
Political Science Research and Methods 8, 526–542.

Culverhouse, R., B. K. Suarez, J. Lin, and T. Reich (2002).
A perspective on epistasis: Limits of models displaying no main effect.
The American Journal of Human Genetics 70(2), 461 – 471.

Ragin, C. C. (1987).
The Comparative Method.
Berkeley: University of California Press.

Rihoux, B. and C. C. Ragin (Eds.) (2009).
Configurational Comparative Methods. Qualitative Comparative Analysis (QCA) and Related Techniques.
Thousand Oaks: Sage.

Rothman, K. J. (1976).
Causes.
American Journal of Epidemiology 104(6), 587–592.

Ruczinski, I., C. Kooperberg, and M. LeBlanc (2003).
Logic regression.
Journal of Computational and Graphical Statistics 12(3), 475–511.

Yakovchenko, V., E. J. Miech, M. J. Chinman, M. Chartier, R. Gonzalez, J. E. Kirchner, T. R. Morgan, A. Park, B. J. Powell, E. K. Proctor, D. Ross, T. J. Waltz, and S. S. Rogal (2020).
Strategy configurations directly linked to higher hepatitis C virus treatment starts: An applied use of configurational comparative methods.
Medical Care 58(5).