3.1 INTRODUCTION

This chapter serves as an introduction to all the following chapters that deal with fluid motions. Fluid motions manifest themselves in many different ways. Some can be described very easily, while others require a thorough understanding of physical laws. In engineering applications, it is important to describe the fluid motions as simply as can be justified. This usually depends on the required accuracy. Often, accuracies of $\pm$ 10% are acceptable, although in some applications higher accuracies have to be achieved. The general equations of motion are very difficult to solve; consequently, it is the engineer's responsibility to know which simplifying assumptions can be made. This, of course, requires experience and, more importantly, an understanding of the physics involved.

KEY CONCEPT *Under certain conditions, viscous effects can be neglected.*

Some common assumptions used to simplify a flow situation are related to fluid properties. For example, under certain conditions, the viscosity can affect the flow significantly; in others, viscous effects can be neglected, greatly simplifying the equations without significantly altering the predictions. It is well known that the compressibility of a gas in motion should be taken into account if the velocities are very high. But compressibility effects do not have to be taken into account to predict wind forces on buildings or to predict any other physical quantity that is a direct effect of wind. Wind speeds are simply not high enough. Numerous examples could be cited. After our study of fluid motions, the appropriate assumptions used should become more obvious.

This chapter has three sections. In the first section we introduce the reader to some important general approaches used to analyze fluid mechanics problems. In the second section we give a brief overview of different types of flow, such as compressible and incompressible flows, and viscous and inviscid flows. Detailed discussions of each of these flow types follow in later chapters. The third section introduces the reader to the commonly used Bernoulli equation, an equation that establishes how pressures and velocities vary in a flow field. The use of this equation, however, requires many simplifying assumptions, and its application is, therefore, limited.

3.2 DESCRIPTION OF FLUID MOTION

The analysis of complex fluid flow problems is often aided by the visualization of flow patterns, which permit the development of a better intuitive understanding and help in formulating the mathematical problem. The flow in a washing machine is a good example. An easier, yet difficult problem is the flow in the vicinity of where a wing attaches to a fuselage, or where a bridge support interacts with the water at the bottom of a river. In Section 3.2.1 we discuss the description of physical quantities as a function of space and time coordinates. The second topic in this section introduces the different flow lines that are useful in our objective of describing a fluid flow. Finally, the mathematical description of motion is presented.

3.2.1 Lagrangian and Eulerian Descriptions of Motion

In the description of a flow field, it is convenient to think of individual particles each of which is considered to be a small mass of fluid, consisting of a large number of molecules, that occupies a small volume ΔV that moves with the flow. If the fluid is incompressible, the volume does not change in magnitude but may deform. If the fluid is compressible, as the volume deforms, it also changes its magnitude. In both cases the particles are considered to move through a flow field as an entity.

In the study of particle mechanics, where attention is focused on individual particles, motion is observed as a function of time. The position, velocity, and acceleration of each particle are listed as $\mathbf{s}(x_0, y_0, z_0, t)$, $\mathbf{V}(x_0, y_0, z_0, t)$, and $\mathbf{a}(x_0, y_0, z_0, t)$, and quantities of interest can be calculated. The point (x_0, y_0, z_0) locates the starting point—the name—of each particle. This is the **Lagrangian** description, named after Joseph L. Lagrange (1736–1813), of motion that is used in a course on dynamics. In the Lagrangian description many particles can be followed and their influence on one another noted. This becomes, however, a difficult task as the number of particles becomes extremely large in even the simplest fluid flow.

Lagrangian: *Description of motion where individual particles are observed as a function of time.*

An alternative to following each fluid particle separately is to identify points in space and then observe the velocity of particles passing each point; we can observe the rate of change of velocity as the particles pass each point, that is, $\partial \mathbf{V}/\partial x$, $\partial \mathbf{V}/\partial y$, and $\partial \mathbf{V}/\partial z$, and we can observe if the velocity is changing with time at each particular point, that is, $\partial \mathbf{V}/\partial t$. In this **Eulerian** description, named after Leonhard Euler (1707–1783), of motion, the flow properties, such as velocity, are functions of both space and time. In Cartesian coordinates the velocity is expressed as $\mathbf{V} = \mathbf{V}(x, y, z, t)$. The region of flow that is being considered is called a **flow field**.

Eulerian: *Description of motion where the flow properties are functions of both space and time.*

Flow field: *The region of interest in a flow.*

Eulerian vs. Lagrangian, 31–33

An example may clarify these two ways of describing motion. An engineering firm is hired to make recommendations that would improve the traffic flow in a large city. The engineering firm has two alternatives: Hire college students to travel in automobiles throughout the city recording the appropriate observations (the Lagrangian approach), or hire college students to stand at the intersections and record the required information (the Eulerian approach). A correct interpretation of each set of data would lead to the same set of recommendations, that is, the same solution. In this example it may not be obvious which approach would be preferred; in an introductory course in fluids, however, the Eulerian description is used exclusively since the physical laws using the Eulerian description are easier to apply to actual situations. Yet, there are examples where a Lagrangian description is needed, such as drifting buoys used to study ocean currents.

Steady flow: *Where flow quantities do not depend on time.*

If the quantities of interest do not depend on time, that is, $\mathbf{V} = \mathbf{V}(x, y, z)$, the flow is said to be a **steady flow**. Most of the flows of interest in this introductory textbook are steady flows. For a steady flow, all flow quantities at a particular point are independent of time, that is,

$$\frac{\partial \mathbf{V}}{\partial t} = 0 \qquad \frac{\partial p}{\partial t} = 0 \qquad \frac{\partial \rho}{\partial t} = 0 \tag{3.2.1}$$

to list a few. It is implied that x, y, and z are held fixed in the above. Note that the properties of a fluid particle do, in general, vary with time; the velocity and pressure vary with time as a particular fluid particle progresses along its path in a flow, even in a steady flow. In a steady flow, however, properties do not vary with time at a fixed point.

3.2.2 Pathlines, Streaklines, and Streamlines

Pathline: *The history of a particle's locations.*

Pathlines, 91

Three different lines help us in describing a flow field. A **pathline** is the locus of points traversed by a given particle as it travels in a field of flow; the pathline provides us with a "history" of the particle's locations. A photograph of a pathline would require a time exposure of an illuminated particle. A photograph showing pathlines of particles below a water surface with waves is given in Fig. 3.1.

Streakline: *An instantaneous line.*

Streamlines, 122

A **streakline** is defined as an instantaneous line whose points are occupied by all particles originating from some specified point in the flow field. Streaklines tell us where the particles are "right now." A photograph of a streakline would be a snapshot of the set of illuminated particles that passed a certain point. Figure 3.2 shows streaklines produced by the continuous release of a small-diameter stream of smoke as it moves around a cylinder.

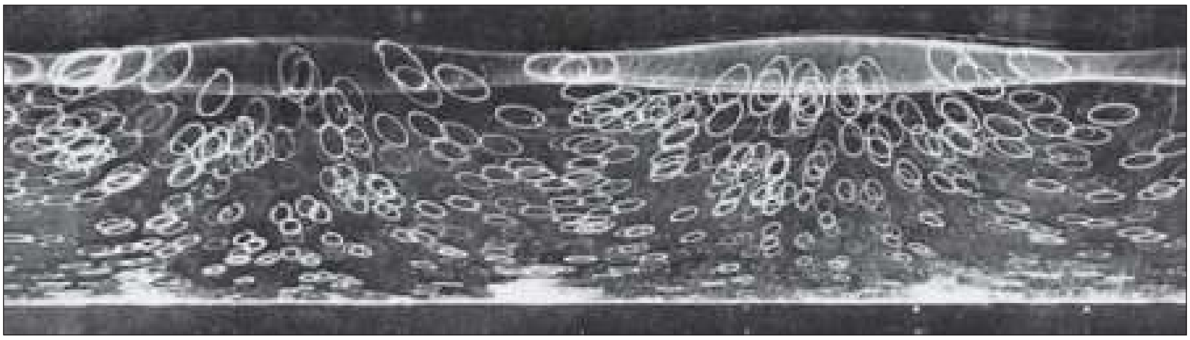

Fig. 3.1 Pathlines underneath a wave in a tank of water.
(Photograph by A. Wallet and F. Ruellan. Courtesy of M. C. Vasseur.)

Streaklines, 122

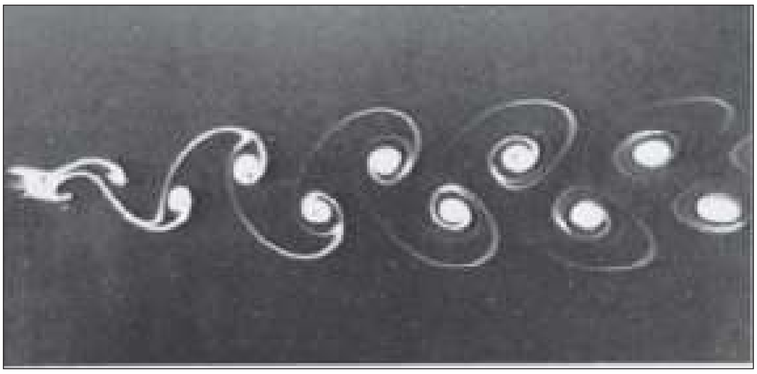

Fig. 3.2 Streaklines in the unsteady flow around a cylinder.
(Photography by Sadatoshi Taneda. From Album of Fluid Motion, 1982, The Parabolic Press, Stanford, California.)

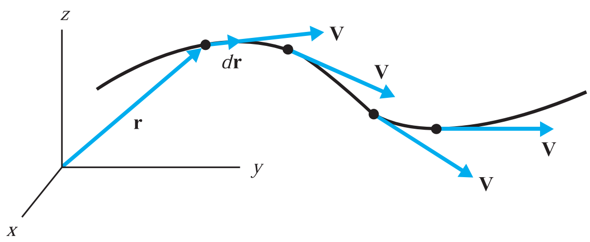

Fig. 3.3 Streamline in a flow field.

A **streamline** is a line in the flow possessing the following property: the velocity vector of each particle occupying a point on the streamline is tangent to the streamline. This is shown graphically in Fig. 3.3. An equation that expresses that the velocity vector is tangent to a streamline is

Streamline: *The velocity vector is tangent to the streamline.*

$$\mathbf{V} \times d\mathbf{r} = 0 \qquad (3.2.2)$$

since $\mathbf{V}$ and $d\mathbf{r}$ are in the same direction, as shown in the figure; recall that the cross product of two vectors in the same direction is zero. This equation will be used in future chapters as the mathematical expression of a streamline. A photograph of a streamline cannot be made directly. For a general unsteady flow the streamlines can be inferred from photographs of short pathlines of a large number of particles.

A **streamtube** is a tube whose walls are streamlines. Since the velocity is tangent to a streamline, no fluid can cross the walls of a streamtube. The streamtube is of particular interest in fluid mechanics. A pipe is a streamtube since its walls are streamlines; an open channel is a streamtube since no fluid crosses the walls of the channel. We often sketch a streamtube with a small cross section in the interior of a flow for demonstration purposes.

Streamtube: *A tube whose walls are streamlines.*

In a steady flow, pathlines, streaklines, and streamlines are all coincident. All particles passing a given point will continue to trace out the same path since the velocity in our Eulerian system does not change with time; hence the pathlines and streaklines coincide. In addition, the velocity vector of a particle at a given point will be tangent to the line that the particle is moving along; thus the line is also a streamline. Since the flows that we observe in laboratories are invariably steady flows, we call the lines that we observe streamlines even though they may actually be streaklines, or for the case of time exposures, pathlines.

KEY CONCEPT *In a steady flow, pathlines, streaklines, and streamlines are all coincident.*

3.2.3 Acceleration

The acceleration of a fluid particle is found by considering a particular particle shown in Fig. 3.4. Its velocity changes from $\mathbf{V}(t)$ at time t to $\mathbf{V}(t + dt)$ at time $t + dt$. The acceleration is, by definition,

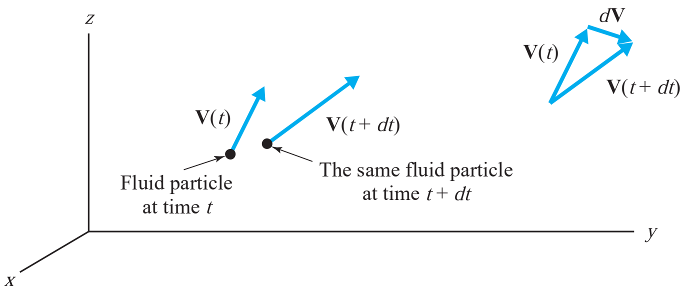

Fig. 3.4 Velocity of a fluid particle.

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} \tag{3.2.3}$$

where $d\mathbf{V}$ is shown in Fig. 3.4. The velocity vector $\mathbf{V}$ is given in component form as

$$\mathbf{V} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}} \tag{3.2.4}$$

where (u, v, w) are the velocity components in the x-, y-, and z- directions, respectively, and $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are the unit vectors. The quantity $d\mathbf{V}$ is, using the chain rule from calculus with $\mathbf{V} = \mathbf{V}(x, y, z, t)$,

$$d\mathbf{V} = \frac{\partial \mathbf{V}}{\partial x}dx + \frac{\partial \mathbf{V}}{\partial y}dy + \frac{\partial \mathbf{V}}{\partial z}dz + \frac{\partial \mathbf{V}}{\partial t}dt \tag{3.2.5}$$

This gives the acceleration using Eg. 3.2.3 as

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{V}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{V}}{\partial t} \tag{3.2.6}$$

Since we have followed a particular particle, as in Fig. 3.4, we recognize that

$$\frac{dx}{dt} = u \qquad \frac{dy}{dt} = v \qquad \frac{dz}{dt} = w \tag{3.2.7}$$

The acceleration is then expressed as

$$\mathbf{a} = u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} + \frac{\partial \mathbf{V}}{\partial t} \tag{3.2.8}$$

The scalar component equations of the above vector equation for Cartesian coordinates are written as

$$
\begin{aligned}
a_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\
a_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\
a_z &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}
\end{aligned} \tag{3.2.9}
$$

We often return to Eq. 3.2.3 and write Eq. 3.2.8 in a simplified form as

$$\mathbf{a} = \frac{D\mathbf{V}}{Dt} \tag{3.2.10}$$

where, in Cartesian coordinates,

$$\frac{D}{Dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t} \tag{3.2.11}$$

This derivative is called the **substantial derivative**, or **material derivative**. It is given a special name and special symbol (D/Dt instead of d/dt) because we followed a particular fluid particle, that is, we followed the substance (or material). It represents the relationship between a Lagrangian derivative in which a quantity depends on time t and an Eulerian derivative in which a quantity depends on position (x, y, z) and time t. The substantial derivative can be used with other dependent variables; for example, DT/Dt would represent the rate of change of the temperature of a fluid particle as we followed the particle along.

Substantial or material derivative: *The derivative D/Dt.*

The substantial derivative and acceleration components in cylindrical and spherical coordinates are presented in Table 3.1 on page 96.

The time-derivative term on the right side of Eqs. 3.2.8 and 3.2.9 for the acceleration is called the **local acceleration** and the remaining terms on the right side in each equation form the **convective acceleration**. Hence the acceleration of a fluid particle is the sum of the local acceleration and convective acceleration. In a pipe, local acceleration results if, for example, a valve is being opened or closed; and convective acceleration occurs in the vicinity of a change in the pipe geometry, such as a pipe contraction or an elbow. In both cases fluid particles change speed, but for very different reasons.

Local acceleration: *The time-derivative term $\partial \mathbf{V}/\partial t$ for acceleration.*

Convective acceleration: *All terms other than the local acceleration term.*

KEY CONCEPT *Convective acceleration occurs in the vicinity of a change in the geometry.*

We must note that the foregoing expressions for acceleration give the acceleration relative to an observer in the observer's reference frame only. In certain situations the observer's reference frame may be accelerating; then

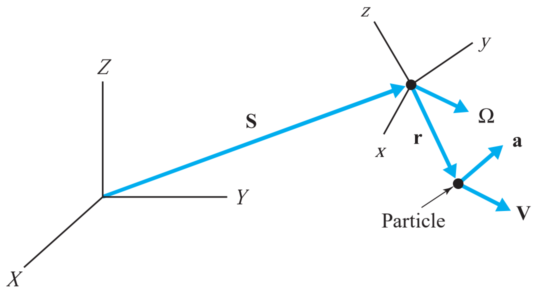

Fig. 3.5 Motion relative to a noninertial reference frame.

the acceleration of a particle relative to a fixed reference frame may be needed. It is given by

$$\mathbf{A} = \mathbf{a} + \underbrace{\frac{d^2\mathbf{S}}{dt^2}}_{\substack{\text{acceleration of}\\ \text{reference frame}}} + \underbrace{2\boldsymbol{\Omega} \times \mathbf{V}}_{\substack{\text{Coriolis}\\ \text{acceleration}}} + \underbrace{\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})}_{\substack{\text{normal}\\ \text{acceleration}}} + \underbrace{\frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r}}_{\substack{\text{angular}\\ \text{acceleration}}} \tag{3.2.12}$$

where **a** is given by Eq. 3.2.8, $d^2\mathbf{S}/dt^2$ is the acceleration of the observer's reference frame, **V** and **r** are the velocity and position vectors of the particle, respectively, in the observer's reference frame, and $\boldsymbol{\Omega}$ is the angular velocity of the observer's reference frame (see Fig. 3.5). Note that all vectors are written using the unit vectors of the XYZ-reference frame. For most engineering applications, reference frames attached to the earth yield $\mathbf{A} = \mathbf{a}$, since the other terms in Eq. 3.2.12 are often negligible with respect to **a**. We may decide, however, to attach the xyz-reference frame to an accelerating device (a rocket), or to a rotating device (a sprinkler arm); then certain terms of Eq. 3.2.12 must be included along with **a** of Eq. 3.2.8.

If the acceleration of all fluid particles is given by $\mathbf{A} = \mathbf{a}$ in a selected reference frame, it is an *inertial* reference frame. If $\mathbf{A} \neq \mathbf{a}$, it is a *noninertial* reference frame. A reference frame that moves with constant velocity without rotating is an inertial reference frame. When analyzing flow about, for example, an airfoil moving at a constant speed, we attach the reference frame to the airfoil so that a steady flow is observed in that reference frame.

3.2.4 Angular Velocity and Vorticity

A fluid flow may be thought of as the motion of a collection of fluid particles. As a particle travels along it may rotate or deform. The rotation and deformation of the fluid particles are of particular interest in our study of fluid mechanics. There are certain flows, or regions of a flow, in which the fluid particles do not rotate; such flows are of special importance, particularly in flows around objects, and are referred to as **irrotational flows**. Flow outside a thin boundary layer on airfoils, outside the separated flow region around autos and other moving vehicles, in the

Irrotational flows: *Flows where the fluid particles do not rotate.*

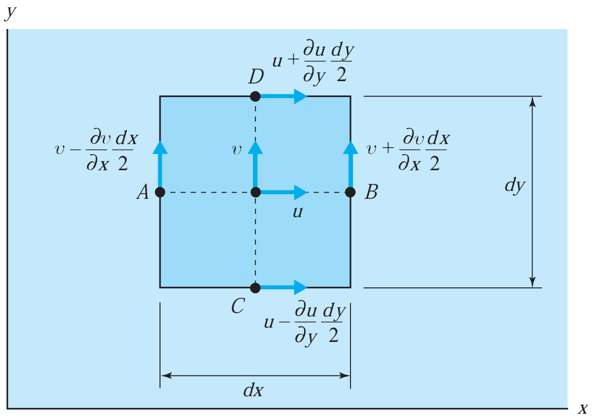

Fig. 3.6 Fluid particle occupying an infinitesimal parallelepiped at a particular instant.

Vorticity, 134

flow around submerged objects, and many other flows are examples of irrotational flows. Irrotational flows are extremely important.

Let us consider a small fluid particle that occupies an infinitesimal volume that has the xy-face as shown in Fig. 3.6. The **angular velocity** Ω_z about the z-axis is the average of the angular velocity of line segment AB and line segment CD. The two angular velocities, counterclockwise being positive, are

Angular velocity: *The average velocity of two perpendicular line segments of a fluid particle.*

$$\begin{aligned}\Omega_{AB} &= \frac{v_B - v_A}{dx} \\ &= \left[v + \frac{\partial v}{\partial x}\frac{dx}{2} - \left(v - \frac{\partial v}{\partial x}\frac{dx}{2}\right)\right] \Big/ dx = \frac{\partial v}{\partial x}\end{aligned} \tag{3.2.13}$$

$$\begin{aligned}\Omega_{CD} &= -\frac{u_D - u_C}{dy} \\ &= -\left[u + \frac{\partial u}{\partial y}\frac{dy}{2} - \left(u - \frac{\partial u}{\partial y}\frac{dy}{2}\right)\right] \Big/ dy = -\frac{\partial u}{\partial y}\end{aligned} \tag{3.2.14}$$

Consequently, the angular velocity Ω_z of the fluid particle is

$$\begin{aligned}\Omega_z &= \frac{1}{2}(\Omega_{AB} + \Omega_{CD}) \\ &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\end{aligned} \tag{3.2.15}$$

If we had considered the xz-face, we would have found the angular velocity about the y-axis to be

$$\Omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \tag{3.2.16}$$

Table 3.1 The Substantial Derivative, Acceleration, and Vorticity in Cartesian, Cylindrical, and Spherical Coordinates

Substantial Derivative

Cartesian

$$\frac{D}{Dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t}$$

Cylindrical

$$\frac{D}{Dt} = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_z\frac{\partial}{\partial z} + \frac{\partial}{\partial t}$$

Spherical

$$\frac{D}{Dt} = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial}{\partial \phi} + \frac{\partial}{\partial t}$$

Acceleration

Cartesian

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Cylindrical

$$a_r = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}$$

$$a_z = \frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}$$

Spherical

$$a_r = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi^2 + v_\theta^2}{r}$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial t} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\phi}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2\cot\theta}{r}$$

$$a_\phi = \frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi + v_\theta v_\phi\cot\theta}{r}$$

Vorticity

Cartesian

$$\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \qquad \omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \qquad \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Cylindrical

$$\omega_r = \frac{1}{r}\left(\frac{\partial v_z}{\partial \theta}\right) - \frac{\partial v_\theta}{\partial z} \qquad \omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \qquad \omega_z = \frac{1}{r}\left(\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)$$

Spherical

$$\omega_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(v_\phi \sin\theta) - \frac{\partial v_\theta}{\partial \phi}\right] \qquad \omega_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]$$

$$\omega_\theta = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]$$

and the yz-face would provide us with the angular velocity about the x-axis:

$$\Omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \tag{3.2.17}$$

These are the three components of the angular velocity vector. A cork placed in a water flow in a wide channel (the xy-plane) would rotate with an angular velocity about the z-axis, given by Eq. 3.2.15.

Vorticity: *Twice the angular velocity.*

It is common to define the **vorticity** $\boldsymbol{\omega}$ to be twice the angular velocity; its three components are then

$$\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \qquad \omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \qquad \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{3.2.18}$$

The vorticity components in cylindrical and spherical coordinates are included in Table 3.1 above.

An irrotational flow possesses no vorticity; the cork mentioned above would not rotate in an irrotational flow. We consider this special flow in Section 8.5.

The deformation of the particle of Fig. 3.6 is the rate of change of the angle that line segment AB makes with line segment CD. If AB is rotating with an angular velocity different from that of CD, the particle is deforming. The deformation is represented by the **rate-of-strain tensor**; its component ϵ_{xy} in the xy-plane is given by

Rate-of-strain tensor: *The rate at which deformation occurs.*

$$\epsilon_{xy} = \frac{1}{2}(\Omega_{AB} - \Omega_{CD}) = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{3.2.19}$$

For the xz-plane and the yz-plane we have

$$\epsilon_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad \epsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \tag{3.2.20}$$

Observe that $\epsilon_{xy}=\epsilon_{yx}$, $\epsilon_{xz}=\epsilon_{zx}$, and $\epsilon_{yz}=\epsilon_{zy}$. By observation, we see that the rate-of-strain tensor is symmetric.

The fluid particle could also deform by being stretched or compressed in a particular direction. For example, if point B of Fig. 3.6 is moving faster than point A, the particle would be stretching in the x-direction. This normal rate of strain is measured by

$$\epsilon_{xx} = \frac{u_B - u_A}{dx} = \left[u + \frac{\partial u}{\partial x}\frac{dx}{2} - \left(u - \frac{\partial u}{\partial x}\frac{dx}{2}\right)\right]\Big/dx = \frac{\partial u}{\partial x} \tag{3.2.21}$$

Similarly, in the y- and z-directions we would find that

$$\epsilon_{yy} = \frac{\partial v}{\partial y} \qquad \epsilon_{zz} = \frac{\partial w}{\partial z} \tag{3.2.22}$$

The symmetric rate-of-strain tensor can be displayed as

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{pmatrix} \tag{3.2.23}$$

where the subscripts i and j take on numerical values 1, 2, or 3. Then ϵ_{12} represents ϵ_{xy} in row 1 column 2.

We will see in Chapter 5 that the normal and shear stress components in a flow are related to the foregoing rate-of-strain components. In fact, in the one-dimensional flow of Fig. 1.6, the shear stress was related to $\partial u/\partial y$ with Eq. 1.5.5; note that $\partial u/\partial y$ is twice the rate-of-strain component given by Eq. 3.2.19 with $v = 0$.

Example 3.1

The velocity field is given by $\mathbf{V} = 2x\hat{\mathbf{i}} - yt\hat{\mathbf{j}}$ m/s, where x and y are in meters and t is in seconds. Find the equation of the streamline passing through (2, −1) and a unit vector normal to the streamline at (2, −1) at $t = 4$ s.

Solution

The velocity vector is tangent to a streamline so that $\mathbf{V} \times d\mathbf{r} = 0$ (the cross product of two parallel vectors is zero). For the given velocity vector we have, at $t = 4$ s,

$$(2x\hat{\mathbf{i}} - 4y\hat{\mathbf{j}}) \times (dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}) = (2x\,dy + 4y\,dx)\hat{\mathbf{k}} = 0$$

where we have used $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \mathbf{k}, \hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\mathbf{k}$, and $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = 0$. Consequently,

$$2x\,dy = -4y\,dx \qquad \text{or} \qquad \frac{dy}{y} = -2\frac{dx}{x}$$

Integrate both sides:

$$\ln y = -2\ln x + \ln C$$

where we used $\ln C$ for convenience. This is written as

$$\ln y = \ln x^{-2} + \ln C = \ln(Cx^{-2})$$

Hence

$$x^2 y = C$$

At (2, −1) $C = -4$, so that the streamline passing through (2, −1) has the equation

$$x^2 y = -4$$

A normal vector is perpendicular to the streamline, hence the velocity vector, so that using $\hat{\mathbf{n}} = n_x\hat{\mathbf{i}} + n_y\hat{\mathbf{j}}$ we have at (2, −1) and $t = 4$ s

$$\mathbf{V} \cdot \hat{\mathbf{n}} = (4\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) \cdot (n_x\hat{\mathbf{i}} + n_y\hat{\mathbf{j}}) = 0$$

Using $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = 1$ and $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0$, this becomes

$$4n_x + 4n_y = 0 \qquad \therefore n_x = -n_y$$

Then, because $\hat{\mathbf{n}}$ is a unit vector, $n_x^2 + n_y^2 = 1$ and we find that

$$n_x^2 = 1 - n_x^2 \qquad \therefore n_x = \frac{\sqrt{2}}{2}$$

The unit vector normal to the streamline is written as

$$\hat{\mathbf{n}} = \frac{\sqrt{2}}{2}(\hat{\mathbf{i}} - \hat{\mathbf{j}})$$

Example 3.2

A velocity field in a particular flow is given by $\mathbf{V} = 20y^2\hat{\mathbf{i}} - 20xy\hat{\mathbf{j}}$ m/s. Calculate the acceleration, the angular velocity, the vorticity vector, and any nonzero rate-of-strain components at the point $(1, -1, 2)$.

Solution

We could use Eq. 3.2.9 and find each component of the acceleration, or we could use Eq. 3.2.8 and find a vector expression. Using Eq. 3.2.8, we have

$$\begin{aligned}\mathbf{a} &= u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\overbrace{\frac{\partial \mathbf{V}}{\partial z}}^{0} + \overbrace{\frac{\partial \mathbf{V}}{\partial t}}^{0} \\ &= 20y^2(-\,20y\hat{\mathbf{j}}) - 20xy(40y\hat{\mathbf{i}} - 20x\hat{\mathbf{j}}) \\ &= -800xy^2\hat{\mathbf{i}} - 400(y^3 - x^2y)\hat{\mathbf{j}}\end{aligned}$$

where we have used $u = 20y^2$ and $v = -20xy$, as given by the velocity vector. All particles passing through the point $(1, -1, 2)$ have the acceleration

$$\mathbf{a} = -800\hat{\mathbf{i}} \text{ m/s}^2$$

The angular velocity has two zero components:

$$\Omega_x = \frac{1}{2}\left(\overbrace{\frac{\partial w}{\partial y}}^{0} - \overbrace{\frac{\partial v}{\partial z}}^{0}\right) = 0, \quad \Omega_y = \frac{1}{2}\left(\overbrace{\frac{\partial u}{\partial z}}^{0} - \overbrace{\frac{\partial w}{\partial x}}^{0}\right) = 0$$

The non-zero z-component is, at the point $(1, -1, 2)$,

$$\begin{aligned}\Omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\ &= \frac{1}{2}(-20y - 40y) = 30 \text{ rad/s}\end{aligned}$$

The vorticity vector is twice the angular velocity vector:

$$\boldsymbol{\omega} = 2\Omega_z\hat{\mathbf{k}} = 60\hat{\mathbf{k}} \text{ rad/s}$$

The nonzero rate-of-strain components are

$$\begin{aligned}\epsilon_{xy} &= \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ &= \frac{1}{2}(-20y + 40y) = -10 \text{ rad/s} \\ \epsilon_{yy} &= \frac{\partial v}{\partial y} \\ &= -20x = -20 \text{ rad/s}\end{aligned}$$

All other rate-of-strain components are zero.