Exercício 1a) Lista 3 para entregar

Calcule a integral de linha ao longo da curva indicada: $\int_\gamma (x-2y^2)\,ds$,

γ é o arco da parábola $y = x^2$ de $(-2,4)$ a $(1,1)$. Resp. 48.

Parametrizando a curva por $\gamma(t) = (t, t^2)$, $t \in [-2,1]$, obtemos a integral

$$\int_{-2}^{1} (t - 2t^4)\sqrt{1+4t^2}\,d\,t.$$

A primeira integral é imediata.

$$\int_{-2}^{1} t\sqrt{1+4t^2}\,d\,t = \frac{1}{12}(1+4t^2)^{3/2}|_{-2}^{1} = \frac{1}{12}(\sqrt{5}-\sqrt{17})$$

Usando a mudança de variáveis $t = \frac{1}{2}\tan\theta$, obtemos para a segunda integral

$$\int_{-2}^{1} 2t^4\sqrt{1+4t^2}\,d\,t = -\int_{\arctan(-4)}^{\arctan 2} \frac{1}{8}\tan^4\theta\,\sec^3(\theta)\,d\theta$$

Integrando por partes obtemos , para qualquer $n \in \mathbb{N}$

$$\int \tan^n\theta\, \sec^3\theta = \int \tan^n\theta\, \sec^2\theta\, \sec\theta\, d\theta$$

$$= \frac{1}{n+1}\tan^{(n+1)}\theta\, \sec\theta - \frac{1}{n+1}\int \tan^{(n+1)}\theta\, \sec\theta \tan\theta\, d\theta$$

$$= \frac{1}{n+1}\tan^{(n+1)}\theta\, \sec\theta - \frac{1}{n+1}\int \tan^{(n+2)}\theta\, \sec\theta\, d\theta$$

$$= \frac{1}{n+1}\tan^{(n+1)}\theta\, \sec\theta - \frac{1}{n+1}\int \tan^{(n)}\theta\, (\sec^2\theta - 1)\sec\theta\, d\theta$$

$$= \frac{1}{n+1}\tan^{(n+1)}\theta\, \sec\theta - \frac{1}{n+1}\int \tan^{(n)}\theta\, \sec^3\theta\, d\theta$$

$$+ \frac{1}{n+1}\int \tan^{(n)}\theta\, \sec\theta\, d\theta$$

Portanto, temos a lei de recorrência

$$\int \tan^n\theta\, \sec^3\theta = \; = \frac{1}{n+2}\tan^{(n+1)}\theta\, \sec\theta + \frac{1}{n+2}\int \tan^{(n)}\theta\, \sec\theta\, d\theta$$

Por outro lado, novamente integrando por partes obtemos, para qualquer $n \in \mathbb{N}$

$$\int \tan^n\theta\, \sec\theta = \int \tan^{n-1}\theta \tan\theta\, \sec\theta$$

$$= \tan^{n-1}\theta\, \sec\theta - \int (n-1)\tan^{n-2}\theta\, \sec^3\theta$$

$$= \tan^{n-1}\theta\, \sec\theta - (n-1)\int \tan^{n-2}\theta\, \sec\theta - (n-1)\int \tan^n\theta\, \sec\theta$$

Portanto, temos a seguinte lei de recorrência:

$$\int \tan^n \theta \, \sec \theta = \frac{1}{n} \tan^{n-1} \theta \, \sec \theta - \frac{n-1}{n} \int \tan^{n-2} \theta \, \sec \theta$$

Em particular

$$\begin{aligned}
\int \tan^4 \theta \, \sec^3 \theta &= \frac{1}{6} \tan^5 \theta \, \sec \theta + \frac{1}{6} \int \tan^4 \theta \, \sec \theta \, d\theta \\
&= \frac{1}{6} \tan^5 \theta \, \sec \theta + \frac{1}{6.4} \tan^3 \theta \, \sec \theta - \frac{3}{6.4} \int \tan^2 \theta \, \sec \theta \, d\theta \\
&= \frac{1}{6} \tan^5 \theta \, \sec \theta + \frac{1}{6.4} \tan^3 \theta \, \sec \theta - \frac{3}{6.4.2} \tan \theta \sec \theta + \frac{3}{6.4.2} \int \sec \\
&= \frac{1}{6} \tan^5 \theta \, \sec \theta + \frac{1}{6.4} \tan^3 \theta \, \sec \theta - \frac{3}{6.4.2} \tan \theta \sec \theta \\
&+ \frac{3}{6.4.2} \log(\sec \theta + \tan \theta) \, d\theta
\end{aligned}$$

Portanto

$$
\begin{aligned}
&-\int_{-2}^{1} 2t^4\sqrt{1+4t^2}\,d\,t = \int_{\arctan(-4)}^{\arctan 2} -\frac{1}{8}\tan^4\theta\ \sec^3(\theta)\\
&= \left[\left(-\frac{1}{6.8}2^5 - \frac{1}{8.6.4}2^3 + \frac{3}{8.6.4.2}2\right)\sqrt{5} - \frac{3}{8.6.4.2}\log(\sqrt{5}+2)\right]\\
&- \left[\left(\frac{1}{6.8}4^5 + \frac{1}{8.6.4}4^3 - \frac{3}{8.6.4.2}4\right)\sqrt{17} - \frac{3}{8.6.4.2}\log(\sqrt{17}-4)\right]\\
&= \left[\left(-\frac{2}{3} - \frac{1}{24} + \frac{3}{192}\right)\sqrt{5} - \frac{3}{384}\log(\sqrt{5}+2)\right]\\
&- \left[\left(\frac{64}{3} + \frac{1}{3} - \frac{3}{96}\right)\sqrt{17} - \frac{3}{384}\log(\sqrt{17}-4)\right].
\end{aligned}
$$

E, lembrando

$$\int_{-2}^{1} t\sqrt{1+4t^2}\,d\,t = \frac{1}{12}(5^{3/2} - 17^{3/2}).$$