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THE EXPONENTIAL MATRIX: AN EXPLICIT FORMULA BY AN ELEMENTARY METHOD

Abstract

We show an explicit formula, with a quite easy deduction, for the exponential matrix e^{tA} of a real and finite square matrix A (and for complex ones also). The elementary method developed avoids Jordan canonical form, eigenvectors, resolution of any linear system, matrix inversion, polynomial interpolation, complex integration, functional analysis, and generalized Fibonacci sequences. The basic tools are the Cayley-Hamilton theorem and the method of partial fraction decomposition. Two examples are given. We also show that such method applies to algebraic operators on infinite dimensional real Banach spaces.

1 Introduction.

In this article we give an explicit formula, with a quite easy deduction, for the exponential matrix e^{tA} of a square real (or complex) matrix A of order $n \times n$, where t is an arbitrary real number. The method developed in what follows requires neither Jordan canonical form (Gantmacher [3, pp. 149–152]), nor eigenvectors (Taylor [10, pp. 146-157]), nor resolution of linear systems of differential equations (Apostol [1, pp. 205–208], Kolodner [6]), nor matrix inversion (Kirchner [5]), nor polynomial interpolation methods (Apostol [1, pp. 209–213], Gantmacher [3]) , nor complex integration combined with functional analysis (Rudin [9, pp. 258–267]), nor generalized Fibonacci sequences (Bensaoud and Moline, [2]). The basic tools employed in this article are well-known

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results on power series and the method of partial fraction decomposition. Obviously, it is also necessary to know the roots of the characteristic polynomial.

There are many distinctive methods and well-known formulas for obtaining the exponential matrix and, as one should expect, some of these formulas (but maybe not their correspondent methods) easily imply each other. I have been unable to locate in the literature on the subject the method shown in this article.

As is well-known, the question of computing the exponential matrix e^{tA} arises from the problem of finding a real curve (a real solution) $x : \mathbb{R} \to \mathbb{R}^n$ to the real constant coefficients linear system of ordinary differential equations

$$\begin{cases} x'(t) = Ax(t) \\ x(0) = x_0, \end{cases}$$

where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

is a $n \times n$ real square matrix and x_0 is a fixed point in $\mathbb{R}^n$. As is also well-known, the unique solution is the curve $x(t) = e^{tA}x_0$. It turns out that this real problem is best dealt in $\mathbb{C}$ and then, at last, we arrive at a real solution.

Two of the best ways of finding e^{tA} are the method that employs Jordan canonical form and Putzer's method (see [1]). Some authors, short of employing Jordan canonical form, resort to the linear algebra primary decomposition theorem (see [10]). It is worth to point out that the method developed by Putzer requires solving another linear system of differential equations. An improvement of Putzer's method can be seen in Kolodner [6].

Among others strategies of computing the matrix e^{tA} we mention Kirchner [5]. In it Kirchner also finds an explicit formula for e^{tA}. However, his approach requires to compute the inverse of a matrix and this can be troublesome. On the contrary, the method provided in this article avoids matrix inversion.

The well-known polynomial interpolation methods, by Lagrange, Sylvester, and Hermite, to compute the exponential matrix (Apostol [1], Gantmacher [3]) have the disadvantage of requiring quite long justifications, besides either matrix inversion or resolution of linear systems.

The sophisticated *Symbolic Calculus* technique (e.g., Rudin [9]) employs the Cauchy integral formula for functions in one complex variable taking values in complex Banach spaces. Three comments are worthwhile regarding it. First, this technique requires a bit of functional analysis and complex integration theory, and thus it does not apply (as pointed out in [9, p. 248]) to algebraic operators defined on infinite dimensional Banach spaces over the real numbers (the elementary method in this article does apply to such operators). Second, the complex integral formula then developed is not explicit on how to use the partial fraction decomposition method in order to obtain the exponential matrix. Third, this technique is unnecessary to find the exponential of a matrix.

For those who are also interested on numerical analysis and computational algorithms to evaluate the matrix e^{tA}, we refer Moler and Van Loan [8]. In it, they focus specially the cases where A is a matrix of order $n\times n$ with $n \leq 100$.

2 The explicit formula for e^{tA}

Let z be in $\mathbb{C}$. We assume the following (proofs in Apostol [1], Lang [7]).

- **(Cayley-Hamilton Theorem).** Given A a $n \times n$ real matrix and $p(z) = \det(zI-A)$ its monic characteristic polynomial, we have $p(A) = 0$.
- **(Partial fraction decomposition).** Let f an q be everywhere convergent complex power series, and p and r be complex polynomials such that $f(z) = q(z)p(z) + r(z)$, where p is monic and degree(r) <degree(p) $= n$. If $\lambda_1, \ldots, \lambda_m$ are the distinct zeros of $p(z)$, with respective multiplicities $m_1, \ldots, m_m$, we write $p(z) = (z-\lambda_1)^{m_1} \cdots (z-\lambda_m)^{m_m}$. Then, there are n constants $C_{1,1}, \ldots, C_{1,m_1}, \ldots, C_{m,1}, \ldots, C_{m,m_m}$ such that

$$\frac{f(z)}{p(z)} = q(z)+\left[\frac{C_{1,1}}{z-\lambda_1} + \cdots + \frac{C_{1,m_1}}{(z-\lambda_1)^{m_1}}\right]+\cdots+\left[\frac{C_{m,1}}{z-\lambda_m} + \cdots + \frac{C_{m,m_m}}{(z-\lambda_m)^{m_m}}\right],$$

for all z outside $\{\lambda_1, \ldots, \lambda_m\}$. These constants are given by

$$C_{j,k} = \frac{g_j^{(m_j-k)}(\lambda_j)}{(m_j-k)!}, \text{ where } g_j(z) = \frac{f(z)(z-\lambda_j)^{m_j}}{p(z)}.$$

Remark 1. *A short proof of the decomposition follows by induction on degree(p).*

Theorem 2. *Let A be a real matrix of size $n\times n$ and characteristic polynomial $p(z) = (z-\lambda_1)^{m_1} \cdots (z-\lambda_m)^{m_m}$, with $\lambda_1, \ldots, \lambda_m$ the distinct zeros of p and $m_1, \ldots, m_m$ their respective algebraic multiplicities. For each $j = 1, \ldots, m$ and each $k = 1, \ldots, m_j$, let us consider the polynomial (a total of n polynomials)*

$$p_{j,k}(z) = (z-\lambda_j)^{m_j-k} \prod_{l\neq j} (z-\lambda_l)^{m_l} \quad \left[= \frac{p(z)}{(z-\lambda_j)^k}\right].$$

Giving $t \in \mathbb{R}$, we have (to simplify the notation, we omit the set where the indices take values)

$$e^{tA} = \sum C_{j,k} p_{j,k}(A), \ \ with\ C_{j,k} = \frac{1}{(m_j - k)!} \frac{d^{m_j - k}}{dz^{m_j - k}} \left\{ \frac{e^{tz}(z-\lambda_j)^{m_j}}{p(z)} \right\} \Big|_{z=\lambda_j}.$$

PROOF. Fixed $t \in \mathbb{R}$, the map $z \mapsto e^{tz}$ is given by a everywhere convergent power series. Dividing such power series by the polynomial $p(z)$ we find

$$e^{tz} = q(z)p(z) + r(z), \ \ \text{with} \ \begin{cases} q \text{ a everywhere convergent power series,} \\ r \text{ a polynomial with degree}(r) < \ \text{degree}(p). \end{cases}$$

Since A commutes with powers of A and the identity matrix I, we arrive at $e^{tA} = q(A)p(A) + r(A)$. The Cayley-Hamilton theorem yields $p(A) = 0$. Thus,

$$e^{tA} = r(A).$$

The partial fraction decomposition right above, and its notation, implies that

$$\frac{r(z)}{p(z)} = \sum \frac{C_{j,k}}{(z-\lambda_j)^k} \ \text{and} \ r(z) = \sum C_{j,k} p_{j,k}(z).$$

Hence, $e^{tA} = \sum C_{j,k} p_{j,k}(A)$. ☐

3 Examples.

Exemplo 3. - *Let us compute e^{tA} and e^{tB} for the real matrices*

$$(a)\ A = \begin{pmatrix} -1 & -3 & 3 \\ -6 & 2 & 6 \\ -3 & 3 & 5 \end{pmatrix}, \ (b)\ B = \begin{pmatrix} 5 & 2 & 2 \\ 1 & 1 & 2 \\ -1 & 4 & 3 \end{pmatrix}.$$

Solutions.

(a) The characteristic polynomial is $p_A(z) = (z-2)(z+4)(z-8)$. Following Theorem 2 and its notation we have $e^{tz} = q(z)p_A(z) + r(z)$ and

$$\frac{e^{tz}}{(z-2)(z+4)(z-8)} = q(z) + \frac{\alpha}{z-2} + \frac{\beta}{z+4} + \frac{\gamma}{z-8},$$

with $q(z)$ a convergent power series and $(\alpha, \beta, \gamma) = (-\frac{e^{2t}}{36}, \frac{e^{-4t}}{72}, \frac{e^{8t}}{72})$. Thus,

$$e^{tA} = -\frac{e^{2t}}{36}(A+4I)(A-8I) + \frac{e^{-4t}}{72}(A-2I)(A-8I) + \frac{e^{8t}}{72}(A-2I)(A+4I).$$

(b) The characteristic polynomial is $p_B(z) = (z+1)(z-5)^2$ and, as it is not difficult to see, the matrix B is non-diagonalizable. Following Theorem 2 and its notation we have $e^{tz} = q(z)p_B(z) + r(z)$ and

$$\frac{e^{tz}}{(z+1)(z-5)^2} = q(z) + \frac{\alpha}{z+1} + \frac{\beta}{(z-5)^2} + \frac{\gamma}{z-5},$$

with q a convergent power series and $(\alpha, \beta, \gamma) = (\frac{e^{-t}}{36}, \frac{e^{5t}}{6}, \frac{(6t-1)e^{5t}}{36})$. Thus,

$$e^{tB} = \frac{e^{-t}}{36}(A-5I)^2 + \frac{e^{5t}}{36}(A+I) + \frac{(6t-1)e^{5t}}{36}(A+I)(A-5I).$$

4 Exponential of Algebraic Operators.

Here we extend the method in section 3 to arbitrary dimensional Banach spaces (either real or complex).

- Complex matrices. Clearly, the method in section 3 is applicable to a complex square matrix of order $n \times n$.
- Complex Banach spaces. Given an infinite dimensional complex Banach space X and a continuous linear operator $T : X \to X$, we say that T is an *algebraic operator* if there exists a non null and monic complex polynomial $p_T(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, with $n \geq 1$, such that
$$p_T(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_1 T + a_0 I = 0,$$
where $I : X \to X$ is the identity operator.

  Examples of algebraic operators are: *nilpotent* operators (i.e., $T^m = 0$ for some m), *projections* (i.e., $T^2 = T$), *idempotent* operators (i.e., $T^m = T$ for some m), and *involution* operators (i.e., $T^m = I$ for some m). Furthermore, operators with *finite rank* (i.e., the image $T(X)$ has finite dimension) are also algebraic operators (see Kaplanski [4, pp. 40–41]).

  It is well-known that it is well defined the exponential operator

$$e^{tT} = \sum_{n=0}^{+\infty} \frac{(tT)^n}{n!} = I + tT + \frac{(tT)^2}{2!} + \frac{(tT)^3}{3!} + \cdots, \text{ for all real } t.$$

  Then, it is not difficult to see, analogously to what we have commented for the exponential of a complex matrix, we obtain a formula for e^{tT}.

- Real Banach spaces. The definition of an algebraic operator $T : X \to X$, with X a real Banach space, is analogous to the one right above. The non null monic polynomial p_T, such that $p_T(T) = 0$, has real coefficients and fixed a real number t it is not difficult to see that we have

$$e^{tz} = q_T(z)p_T(z) + r_T(z), \text{ for all } z,$$

with q_T a everywhere convergent (over the complex plane) power series with real coefficients and r_T a polynomial, with real coefficients and whose degree is smaller than that of p_T. Thus, we have $e^{tT} = r_T(T)$.

By employing the partial fraction decomposition we may write

$$\frac{r_T(z)}{p_T(z)} = \sum_{\substack{1\leq j\leq \mu \\ 1\leq k\leq \mu_j}} \frac{\alpha_{j,k}}{(z-z_j)^k} + \sum_{\substack{1\leq j\leq \mu \\ 1\leq k\leq \mu_j}} \frac{\beta_{j,k}}{(z-\overline{z_j})^k} + \sum_{\substack{1\leq l\leq \nu \\ 1\leq k\leq \nu_l}} \frac{\gamma_{l,k}}{(z-x_l)^k}, \quad (1)$$

where the polynomial p_T has complex roots $z_1, \overline{z_1}, \dots, z_\mu, \overline{z_\mu}$ and real roots $x_1, \dots, x_\nu$ (all the roots are distinct and the algebraic multiplicities of these are, respectively, $\mu_1, \mu_1, \dots, \mu_\mu, \mu_\mu, \nu_1, \nu_2, \dots \nu_\nu$), with degree$(p_T) = 2(\mu_1 + \cdots + \mu_\mu) + \nu_1 + \cdots + \nu_\nu = n$, and all the coefficients $\alpha_{j,k}, \beta_{j,k}$, and $\gamma_{l,k}$ are unique complex constants.

In what follows, to simplify the layout we omit the sets where the indices for the coefficients $\alpha_{j,k}$, $\beta_{j,k}$, and $\gamma_{l,k}$ take values.

Claim 1. All the constants $\gamma_{l,k}$ are real numbers. In fact, since the map $z \mapsto e^{tz}(z-x_l)^{\nu_l}$ may be developed as a power series with real coefficients and the polynomial p_T has real coefficients, it follows that

$$\gamma_{l,k} = \frac{1}{(\nu_l - k)!}\frac{d^{\nu_l-k}}{dz^{\nu_l-k}}\left\{\frac{e^{tz}(z-x_l)^{\nu_l}}{p_T(z)}\right\}\bigg|_{z=x_l} \in \mathbb{R}.$$

Claim 2. We have $\beta_{j,k} = \overline{\alpha_{j,k}}$ for all possible j and k. In order to verify this claim, we consider the functions

$$\varphi(z) = \frac{e^{tz}(z-z_j)^{\mu_j}}{p_T(z)} \text{ and } \psi(z) = \frac{e^{tz}(z-\overline{z_j})^{\mu_j}}{p_T(z)}.$$

The identity $\overline{\psi(\overline{z})} = \varphi(z)$ holds, since the power series e^{tz} and the polynomial p_T have real coefficients. This implies that $\varphi'(z) = \overline{\psi'(\overline{z})}$, $\varphi''(z) = \overline{\psi''(\overline{z})}$, $\varphi'''(z) = \overline{\psi'''(\overline{z})}$, etc. From these and Theorem 2 we have

$$\alpha_{j,k} = \frac{\varphi^{(\mu_j-k)}(z_j)}{(\mu_j-k)!} = \overline{\frac{\psi^{(\mu_j-k)}(\overline{z_j})}{(\mu_j-k)!}} = \overline{\beta_{j,k}}.$$

The proof of Claim 2 is complete.

Thus far, based on equation (1) we have seen that

$$\begin{aligned}\frac{r_T(z)}{p_T(z)} &= \sum\left[\frac{\alpha_{j,k}}{(z-z_j)^k}+\frac{\overline{\alpha_{j,k}}}{(z-\overline{z_j})^k}\right]+\sum\frac{\gamma_{l,k}}{(z-x_l)^k}\\ &= \sum\frac{\alpha_{j,k}(z-\overline{z_j})^k+\overline{\alpha_{j,k}}(z-z_j)^k}{(z-z_j)^k(z-\overline{z_j})^k}+\sum\frac{\gamma_{l,k}}{(z-x_l)^k}.\end{aligned}$$

Claim 3. The expansion of the map $u_{j,k}(z)=\alpha_{j,k}(z-\overline{z_j})^k+\overline{\alpha_{j,k}}(z-z_j)^k$ is a polynomial, in the variable z, with real coefficients. In fact, we have

$$\alpha_{j,k}(z-\overline{z_j})^k+\overline{\alpha_{j,k}}(z-z_j)^k=\sum_{k'=0}^{k'=k}\binom{k}{k'}(-1)^{k-k'}\left[\alpha_{j,k}\overline{z_j}^{k-k'}+\overline{\alpha_{j,k}}z_j^{k-k'}\right]z^{k'},$$

with the right-hand side a polynomial with real coefficients. Claim 3 is proven.

Therefore, we may write

$$r_T(z)=\sum u_{j,k}(z)\frac{p_T(z)}{(z-z_j)^k(z-\overline{z_j})^k}+\sum\gamma_{l,k}\frac{p_T(z)}{(z-x_l)^k}.$$

Eliminating singularities, with clear identifications we may write $r_T=\sum u_{jk}v_{jk}+\sum\gamma_{lk}w_{lk}$, where each u_{jk}, v_{jk} and w_{lk} is a polynomial with real coefficients (a total of n polynomials) and each γ_{lk} is a real number.

Summing up, and since $e^{tT}=r_T(T)$, these computations yield the formula

$$e^{tT}=\sum u_{j,k}(T)v_{j,k}(T)+\sum\gamma_{l,k}w_{l,k}(T).$$

The case for X a real Banach space is complete.

5 Some Final Remarks.

The author humbly hopes that this very short method may be quite useful, along the already well-known textbook methods, as a practical way of computing the exponential of matrices and algebraic operators. One cannot miss the opportunity to point out that we are right in the middle of a coronavirus pandemic crisis that turned "exponential growth" into a household expression.

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