Resolução Exercício 1 da Lista 3

L.D. Landau e E. M. Lifshitz - The classical theory of fields

§ 24. Lorentz transformation of the field

In this section we find the transformation formulas for fields, that is, formulas by means of which we can determine the field in one inertial system of reference, knowing the same field in another system.

The formulas for transformation of the potentials are obtained directly from the general formulas for transformation of four-vectors (6.1). Remembering that $A^i = (\phi, \mathbf{A})$, we get easily

$$\phi = \frac{\phi' + \frac{V}{c} A_x'}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad A_x = \frac{A_x' + \frac{V}{c}\phi'}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad A_y = A_y', \qquad A_z = A_z'. \tag{24.1}$$

The transformation formulas for an antisymmetric second-rank tensor (like F^{ik}) were found in problem 2 of § 6: the components F^{23} and F^{01} do not change, while the components F^{02}, F^{03}, and F^{12}, F^{13} transform like x^0 and x^1, respectively. Expressing the components of F^{ik} in terms of the components of the fields $\mathbf{E}$ and $\mathbf{H}$, according to (23.5), we then find the following formulas of transformation for the electric field:

$$E_x = E_x', \; E_y = \frac{E_y' + \frac{V}{c} H_z'}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad E_z = \frac{E_z' - \frac{V}{c} H_y'}{\sqrt{1-\frac{V^2}{c}}}, \tag{24.2}$$

and for the magnetic field:

$$H_x = H_x', \; H_y = \frac{H_y' - \frac{V}{c} E_z'}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad H_z = \frac{H_z' + \frac{V}{c} E_y'}{\sqrt{1-\frac{V^2}{c^2}}}. \tag{24.3}$$

Thus the electric and magnetic fields, like the majority of physical quantities, are relative; that is, their properties are different in different reference systems. In particular, the electric or the magnetic field can be equal to zero in one reference system and at the same time be present in another system.

§ 38. The field of a uniformly moving charge

We determine the field produced by a charge e, moving uniformly with velocity V. We call the laboratory frame the system K; the system of reference moving with the charge is the K' system. Let the charge be located at the origin of coordinates of the K' system. The system K' moves relative to K along the X axis; the axes Y and Z are parallel to Y' and Z'. At the time $t = 0$ the origins of the two systems coincide. The coordinates of the charge in the K system are consequently $x = Vt$, $y = z = 0$. In the K' system, we have a constant electric field with vector potential $\mathbf{A}' = 0$, and scalar potential equal to $\phi' = e/R'$, where $R'^2 = x'^2 + y'^2 + z'^2$. In the K system, according to (24.1) for $\mathbf{A}' = 0$,

$$\phi = \frac{\phi'}{\sqrt{1-\frac{V^2}{c^2}}} = \frac{e}{R'\sqrt{1-\frac{V^2}{c^2}}}. \tag{38.1}$$

We must now express R' in terms of the coordinates x, y, z, in the K system. According to the formulas for the Lorentz transformation

$$x' = \frac{x - Vt}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad y' = y, \qquad z' = z,$$

from which

$$R'^2 = \frac{(x-Vt)^2 + \left(1-\frac{V^2}{c^2}\right)(y^2+z^2)}{1-\frac{V^2}{c^2}}. \tag{38.2}$$

Substituting this in (38.1) we find

$$\phi = \frac{e}{R^*} \tag{38.3}$$

where we have introduced the notation

$$R^{*2} = (x-Vt)^2 + \left(1-\frac{V^2}{c^2}\right)(y^2+z^2). \tag{38.4}$$

The vector potential in the K system is equal to

$$\mathbf{A} = \phi\frac{\mathbf{V}}{c} = \frac{e\mathbf{V}}{cR^*}. \tag{38.5}$$

In the K' system the magnetic field $\mathbf{H}'$ is absent and the electric field is

$$\mathbf{E}' = \frac{e\mathbf{R}'}{R'^3}.$$

From formula (24.2), we find

$$E_x = E_x' = \frac{ex'}{R'^3}, \qquad E_y = \frac{E_y'}{\sqrt{1-\frac{V^2}{c^2}}} = \frac{ey'}{R'^3\sqrt{1-\frac{V^2}{c^2}}},$$

$$E_z = \frac{ez'}{R'^3\sqrt{1-\frac{V^2}{c^2}}}.$$

Substituting for R', x', y', z', their expressions in terms of x, y, z, we obtain

$$\mathbf{E} = \left(1-\frac{V^2}{c^2}\right)\frac{e\mathbf{R}}{R^{*3}}, \tag{38.6}$$

where $\mathbf{R}$ is the radius vector from the charge e to the field point with coordinates x, y, z (its components are $x - Vt$, y, z).

This expression for $\mathbf{E}$ can be written in another form by introducing the angle θ between the direction of motion and the radius vector $\mathbf{R}$. It is clear that $y^2 + z^2 = R^2 \sin^2\theta$, and therefore R^{*2} can be written in the form:

$$R^{*2} = R^2\left(1-\frac{V^2}{c^2}\sin^2\theta\right). \tag{38.7}$$

Then we have for $\mathbf{E}$,

$$\mathbf{E} = \frac{e\mathbf{R}}{R^3}\frac{1-\frac{V^2}{c^2}}{\left(1-\frac{V^2}{c^2}\sin^2\theta\right)^{3/2}}. \tag{38.8}$$

For a fixed distance R from the charge, the value of the field E increases as θ increases from 0 to $\pi/2$ (or as θ decreases from π to $\pi/2$). The field along the direction of motion $((\theta = 0, \pi)$ has the smallest value; it is equal to

$$E_{\parallel} = \frac{e}{R^2}\left(1-\frac{V^2}{c^2}\right).$$

The largest field is that perpendicular to the velocity ($\theta = \pi/2$), equal to

$$E_{\perp} = \frac{e}{R^2}\frac{1}{\sqrt{1-\frac{V^2}{c^2}}}.$$

We note that as the velocity increases, the field $E_{\parallel}$ decreases, while $E_{\perp}$ increases. We can describe this pictorially by saying that the electric field of a moving charge is "contracted" in the direction of motion. For velocities V close to the velocity of light, the denominator in formula (38.8) is close to zero in a narrow interval of values θ around the value $\theta = \pi/2$. The "width" of this interval is, in order of magnitude,

$$\Delta\theta \sim \sqrt{1-\frac{V^2}{c^2}}.$$

Thus the electric field of a rapidly moving charge at a given distance from it is large only in a narrow range of angles in the neighborhood of the equatorial plane, and the width of this interval decreases with increasing V like $\sqrt{1-(V^2/c^2)}$.

The magnetic field in the K system is

$$\mathbf{H} = \frac{1}{c}\mathbf{V}\times\mathbf{E} \tag{38.9}$$

[see (24.5)]. In particular, for $V \ll c$ the electric field is given approximately by the usual formula for the Coulomb law, $\mathbf{E} = e\mathbf{R}/R^3$, and the magnetic field is

$$\mathbf{H} = \frac{e}{c}\frac{\mathbf{V}\times\mathbf{R}}{R^3}, \tag{38.10}$$