Solução Exercício 2 da Lista 3

Denotando por $\frac{u}{c} = \tanh\alpha$ temos que a transformação de Lorentz de Σ para Σ' é

$$\begin{pmatrix} dx^{0\prime} \\ dx^{1\prime} \\ dx^{2\prime} \\ dx^{3\prime} \end{pmatrix} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix}$$

Denotando $\frac{u'}{c} = \tanh\alpha'$ temos que a transformação de Lorentz de Σ' para Σ'' é

$$\begin{pmatrix} dx^{0\prime\prime} \\ dx^{1\prime\prime} \\ dx^{2\prime\prime} \\ dx^{3\prime\prime} \end{pmatrix} = \begin{pmatrix} \cosh\alpha' & -\sinh\alpha' & 0 & 0 \\ -\sinh\alpha' & \cosh\alpha' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx^{0\prime} \\ dx^{1\prime} \\ dx^{2\prime} \\ dx^{3\prime} \end{pmatrix}$$

Logo a transformação de Lorentz de Σ para Σ'' é dada pela composição das duas.

$$\begin{pmatrix} dx^{0\prime\prime} \\ dx^{1\prime\prime} \\ dx^{2\prime\prime} \\ dx^{3\prime\prime} \end{pmatrix} = \begin{pmatrix} (\cosh\alpha\cosh\alpha' + \sinh\alpha\sinh\alpha') & -(\sinh\alpha\cosh\alpha' + \sinh\alpha'\cosh\alpha) & 0 & 0 \\ -(\sinh\alpha\cosh\alpha' + \sinh\alpha'\cosh\alpha) & (\cosh\alpha\cosh\alpha' + \sinh\alpha\sinh\alpha') & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix}$$

Om

$$\begin{pmatrix} dx^{0''} \\ dx^{1''} \\ dx^{2''} \\ dx^{3''} \end{pmatrix} = \begin{pmatrix} \cosh(\alpha+\alpha') & -\sinh(\alpha+\alpha') & 0 & 0 \\ -\sinh(\alpha+\alpha') & \cosh(\alpha+\alpha') & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx^{0} \\ dx^{1} \\ dx^{2} \\ dx^{3} \end{pmatrix}$$

Demostrando $\frac{u''}{c} = \tanh(\alpha+\alpha')$ temos

$$\frac{u''}{c} = \tan(\alpha+\alpha') = \frac{\sinh(\alpha+\alpha')}{\cosh(\alpha+\alpha')} =$$

$$= \frac{\sinh\alpha \cosh\alpha' + \sinh\alpha' \cosh\alpha}{\cosh\alpha \cosh\alpha' + \sinh\alpha' \sinh\alpha}$$

$$= \frac{\frac{\cosh\alpha'}{\sinh\alpha'} + \frac{\cosh\alpha}{\sinh\alpha}}{\frac{\cosh\alpha}{\sinh\alpha}\frac{\cosh\alpha'}{\sinh\alpha'} + 1}$$

$$= \frac{\tanh\alpha + \tanh\alpha'}{1 + \tanh\alpha \tan\alpha'}$$

$$= \frac{1}{c} \frac{u + u'}{1 + \frac{uu'}{c^2}}$$

Logo

$$\boxed{u'' = \frac{u + u'}{1 + \frac{uu'}{c^2}}}$$