Resultados da distribuição normal e qui quadrado.

Dizemos que $Y \sim \chi^2_{(k)}$ se Y é v.a. contínua e sua f.d.p é dada por:

$$f_Y(y) = \frac{y^{k/2-1} \cdot e^{-y/2}}{2^{k/2} \cdot \Gamma(k/2)} \cdot \underset{(0,\infty)}{I}(y)$$

$\boxed{\text{Convenção: se } g(y) \text{ não estiver definida e } \underset{A}{I}(y)=0\text{, então considere que } g(y)\cdot \underset{A}{I}(y)=0}$

<u>Teorema:</u> $Y \sim \chi^2_{(k)} \Rightarrow M_Y(t) = \dfrac{1}{(1-2t)^{k/2}}, \quad t < 1/2$

<u>prova:</u>

$$M_Y(t) = E\left(e^{tY}\right) = \int_0^\infty \frac{e^{ty} \cdot y^{k/2-1} e^{-\frac{1}{2}y}}{2^{k/2} \cdot \Gamma(k/2)} \, dy, \quad t < 1/2$$

$$= \int_0^\infty \frac{y^{k/2-1} \cdot e^{-y(\frac{1}{2}-t)}}{2^{k/2}\, \Gamma(k/2)} \, dy, \quad t < 1/2$$

$u = y\left(\frac{1}{2}-t\right) \qquad du = \left(\frac{1}{2}-t\right) dy$

$y = \dfrac{u}{\frac{1}{2}-t}$

$$\Rightarrow M_Y(t) = \int_0^\infty \frac{u^{k/2-1} \cdot e^{-u}}{(1/2-t)^{k/2-\cancel{1}} \cdot 2^{k/2}\, \Gamma(k/2)} \frac{1}{\cancel{(1/2-t)}} \, du$$

$$= \frac{\cancel{2}^{k/2}}{(1-2t)^{k/2} \cdot \cancel{2}^{k/2}} \cdot \frac{1}{\Gamma(k/2)} \cdot \underbrace{\int_0^\infty u^{k/2-1} \cdot e^{-u} \, du}_{\Gamma(k/2)}$$

$$= \frac{1}{(1-2t)^{k/2}}, \quad t < 1/2. \qquad \square$$

<u>Teorema:</u> $X \sim N(0,1) \Rightarrow T = X^2 \sim \chi^2_{(1)}$.

<u>prova:</u>

Sejam $T_1 : R_+ \to \mathbb{R}_+$ e $T_2 : R_- \to R_+$ tais que

$$T_1(x) = x^2 \quad \text{e} \quad T_2(x) = x^2$$

$$\Rightarrow T_1^{-1}(t) = \sqrt{t} \qquad T_2^{-1}(t) = -\sqrt{t}$$

$$|J_1(t)| = \frac{1}{2\sqrt{t}} \qquad |J_2(t)| = \left|\frac{-1}{2\sqrt{t}}\right| = \frac{1}{2\sqrt{t}}$$

$$\Rightarrow T(x) = T_1(x) \cdot \underset{\mathbb{R}_+}{I}(x) + T_2(x) \cdot \underset{\mathbb{R}_-}{I}(x)$$

$$f_T(t) = f_X(T_1^{-1}(t)) \cdot |J_1(t)| \cdot \underset{T(\mathbb{R}_+)}{I}(t) + f_X(T_2^{-1}(t)) \cdot |J_2(t)| \cdot \underset{T(\mathbb{R}_-)}{I}(t)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{t})^2} \cdot \frac{1}{\cancel{2}\sqrt{t}} \cdot \underset{(0,\infty)}{I}(t) + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-\sqrt{t})^2} \frac{1}{\cancel{2}\sqrt{t}} \underset{(0,\infty)}{I}(t)$$

$$= \frac{1}{2^{1/2}\,\Gamma(1/2)} \cdot e^{-\frac{1}{2}t} \cdot \underset{\mathbb{R}_+}{I}(t) \Rightarrow T \sim \chi^2_{(1)}. \qquad \square$$

<u>Teorema:</u> $\underset{\sim}{X} = (X_1, \dots, X_n) \sim N_n(\underset{\sim}{0}, I_n)$

$$\Rightarrow \underset{\sim}{X}^T \underset{\sim}{X} = \sum X_i^2 \sim \chi^2_{(n)}$$

<u>prova:</u> Como $\underset{\sim}{X} \sim N_n(0, I_n)$, temos que

$$X_i \overset{iid}{\sim} N_1(0,1) \quad \forall\, i = 1, \dots, n$$

precisamos mostrar que $Y = \underset{\sim}{X}^T \underset{\sim}{X}$ tem f.g.m dada por $\dfrac{1}{(1-2t)^{n/2}}$.

$$M_Y(t) = E\left(e^{tY}\right) = E\left(e^{t\underset{\sim}{X}^T\underset{\sim}{X}}\right) = E\left(e^{t\sum X_i^2}\right) = E\left(\prod_{i=1}^n e^{tX_i^2}\right)$$

$$\overset{ind}{=} \prod_{i=1}^n E\left(e^{tX_i^2}\right) \overset{id}{=} E\left[e^{tX_1^2}\right]^n$$

Como $X_1 \sim N(0,1) \Rightarrow X_1^2 \sim \chi^2(1) \Rightarrow E\left(e^{tX_1^2}\right) = \dfrac{1}{(1-2t)^{1/2}}, t < 1/2$

$$\Rightarrow M_Y(t) = \frac{1}{(1-2t)^{n/2}}, \quad t < 1/2, \quad \text{logo } Y \sim \chi^2_{(n)}. \qquad \square$$

<u>Teorema:</u> $\underset{\sim}{X} \sim N_n(\underset{\sim}{\mu}, \Sigma)$ em que Σ é simétrica e positiva definida (autovalores positivos), então

$$\Sigma^{-1/2}(\underset{\sim}{X} - \underset{\sim}{\mu}) \sim N_n(\underset{\sim}{0}, I_n)$$

em que $\Sigma^a = \Gamma \Lambda^a \Gamma^T$, Γ é a matriz de autovetores de Σ e Λ é uma matriz diagonal de autovalores.

<u>prova:</u> $\underset{\sim}{Y} = \Sigma^{-1/2}(\underset{\sim}{X} - \underset{\sim}{\mu}) \Rightarrow \underset{\sim}{X} = \Sigma^{1/2}\underset{\sim}{Y} + \underset{\sim}{\mu}$

$$|J| = \left|\Sigma^{1/2}\right| = |\Sigma|^{1/2}$$

$$f_{\underset{\sim}{Y}}(y) = f_X\left(\Sigma^{1/2}\underset{\sim}{y} + \underset{\sim}{\mu}\right) \cdot |\Sigma|^{1/2} = \frac{1}{(2\pi)^{n/2}\,\cancel{|\Sigma|^{1/2}}} e^{-\frac{1}{2}(\Sigma^{1/2}y + \cancel{\mu} - \cancel{\mu})^T \Sigma^{-1} (\Sigma^{1/2}y + \cancel{\mu} - \cancel{\mu})} \cdot \cancel{|\Sigma|^{1/2}}$$

$$= \frac{1}{(2\pi)^{n/2}} \cdot e^{-\frac{1}{2} y^T \underbrace{\Sigma^{1/2} \cdot \Sigma^{-1} \cdot \Sigma^{1/2}}_{*} \cdot y}$$

$$* \; \Sigma^{1/2} \cdot \Sigma^{-1} \cdot \Sigma^{1/2} = \Gamma \Lambda^{1/2} \underbrace{\Gamma^T \cdot \Gamma}_{I} \Lambda^{-1} \underbrace{\Gamma^T \Gamma}_{I} \Lambda^{1/2} \cdot \Gamma^T = \Gamma \Lambda^0 \Gamma^T = I.$$

$$\Gamma^T \Gamma = I$$

$$\Rightarrow f_Y(y) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2} y^T y} \Rightarrow Y \sim N_n(\underset{\sim}{0}, I). \qquad \square$$

<u>Teorema:</u> Se $\underset{\sim}{X} \sim N_n(\underset{\sim}{0}, I_n)$ e A é uma matriz $(n \times n)$ idempotente $(A = A \cdot A)$ diferente de zero, então

$$\boxed{\underset{\sim}{X}^T A X \sim \chi^2_k}, \text{ em que}$$

$k = tr\{A\}$.

<u>prova:</u>

A é quadrada e simétrica, logo podemos escrever:

$$A = \Gamma \Lambda \Gamma^T, \text{ em que}$$

Γ é a matriz de autovetores de A tal que $\Gamma^T\Gamma = I$

Λ é uma matriz diagonal de autovalores de A

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{matrix} \leftarrow \text{max} \\ \\ \leftarrow \text{min} \end{matrix}$$

Como $A = A \cdot A$ temos que

$$\Gamma \Lambda \Gamma^T = \Gamma \Lambda \Gamma^T \Gamma \Lambda \Gamma^T$$
$$\Leftrightarrow \Gamma \Lambda \Gamma^T = \Gamma \Lambda^2 \Gamma^T$$

premultiplicando por Γ^T e pós multiplicando por Γ temos que

$$\Lambda = \Lambda^2 \Leftrightarrow \lambda_i = \lambda_i^2, \quad \forall i = 1, \dots, n$$

Como $\lambda_1, \dots, \lambda_n$ são valores reais, então

$$\lambda_i \in \{0, 1\}, \quad \forall i = 1, \dots, n$$

Obs: $\lambda_i = 1, \ \forall i = 1, \dots, n \Leftrightarrow A = I_n$

$\lambda_i = 0, \ \forall i = 1, \dots, n \Leftrightarrow A = 0_{n \times n}$

Considere que $\Lambda = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$ em que k é o posto de A (ou seja nº de autovalores $\neq$ de zero).

$$tr(A) = tr\{\Gamma \Lambda \Gamma^T\} = tr\{\Lambda \Gamma^T \Gamma\} = tr\{\Lambda\} = k$$

$= $ posto de A.

Observe que $X^T A X = X^T \Gamma \Lambda \Gamma^T X$ e defina

$Z = \Gamma^T X$. Então $Z \sim N_n(\mathbf{0}, \Gamma^T \Gamma)$

$\Rightarrow Z \sim N_n(\mathbf{0}, I_n)$ pa transformação de vetores

$\Rightarrow$

$$X^T A X = Z^T \Lambda Z.$$

Considere que $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ em Z_1 tem k componentes.

$$\Rightarrow Z^T \Lambda Z = \begin{pmatrix} Z_1^T & Z_2^T \end{pmatrix} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = Z_1^T Z_1$$

Como $Z \sim N_n(0, I_n) \Rightarrow Z_1 \sim N_k(\mathbf{0}, I_k)$

$$\Rightarrow Z_1^T Z \sim \chi^2_{(k)}, \quad k = tr\{A\}$$

$\Rightarrow X^T A X = Z_1^T Z_1$ temos que

$$X^T A X \sim \chi^2_{(k)}. \quad \blacksquare$$

Aplicação: $(X_1, \dots, X_n)$ a.a de $X \sim N(\mu, \sigma^2)$ em que $\theta = (\mu, \sigma^2) \in \Theta = \mathbb{R} \times \mathbb{R}_+$.

$$S_x^2 = \frac{1}{n} \sum (X_i - \bar{X})^2$$

$$\Rightarrow \frac{n S_x^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

Note que $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i = \frac{1}{n} \mathbf{1}^T \mathbf{X}$ $\qquad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

$$\boxed{\sum (X_i - \bar{X})^2} = \sum \left(X_i - \frac{1}{n} \mathbf{1}^T \mathbf{X}\right)\left(X_i - \frac{1}{n} \mathbf{1}^T \mathbf{X}\right)$$
$$= \left[\mathbf{X} - \mathbf{1}\left(\frac{1}{n} \mathbf{1}^T \mathbf{X}\right)\right]^T \left[\mathbf{X} - \mathbf{1}\left(\frac{1}{n} \mathbf{1}^T \mathbf{X}\right)\right]$$
$$= \mathbf{X}^T \underbrace{\left[I - \frac{1}{n} \mathbf{1}\mathbf{1}^T\right]}_{A} \underbrace{\left[I - \frac{1}{n} \mathbf{1}\mathbf{1}^T\right]}_{A} \mathbf{X}$$

$$A \cdot A = \left[I - \frac{1}{n} \mathbf{1}\mathbf{1}^T\right] \cdot \left[I - \frac{1}{n} \mathbf{1}\mathbf{1}^T\right]$$
$$= I - \frac{1}{n} \mathbf{1}\mathbf{1}^T - \frac{1}{n} \mathbf{1}\mathbf{1}^T + \frac{1}{n^2} \mathbf{1} \overset{= n}{\underset{\sum 1^2 = n}{\boxed{\mathbf{1}^T \cdot \mathbf{1}}}} \mathbf{1}^T$$
$$= I - \frac{2}{n} \mathbf{1}\mathbf{1}^T + \frac{1}{n} \mathbf{1}\mathbf{1}^T$$
$$= I - \frac{1}{n} \mathbf{1}\mathbf{1}^T = A$$

Note que

$$A \cdot \mu \mathbf{1} = \left[I - \frac{1}{n} \mathbf{1}\mathbf{1}^T\right] \mu \cdot \mathbf{1} = \mu \cdot \mathbf{1} - \mu \cdot \frac{1}{n} \mathbf{1} \overset{= n}{\boxed{\mathbf{1}^T \mathbf{1}}} = 0$$

$$\Rightarrow \sum (X_i - \bar{X})^2 = \mathbf{X}^T A \mathbf{X} = (\mathbf{X} - \mu \mathbf{1})^T A (\mathbf{X} - \mu \cdot \mathbf{1})$$

$$\Rightarrow \frac{1}{\sigma}(\mathbf{X} - \mu \cdot \mathbf{1}) \sim N_n(\mathbf{0}, I_n)$$

$$\Rightarrow \frac{\sum (X_i - \bar{X})^2}{\sigma^2} = \overbrace{\frac{1}{\sigma}(\mathbf{X} - \mu \mathbf{1})^T}^{\sim N_n(0, I_n)} A \overbrace{(\mathbf{X} - \mu \mathbf{1}) \frac{1}{\sigma}}^{\sim N_n(0, I_n)}$$

$$= \frac{n S_x^2}{\sigma} \sim \chi^2_{(tr\{A\})},$$

em que

$$tr\{A\} = tr\left\{I - \frac{1}{n} \mathbf{1} \mathbf{1}^T\right\} = n - \frac{1}{n} n = n - 1 \quad \blacksquare$$