Financial Networks and Contagion*

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Abstract

We model contagions and cascades of failures among organizations linked through a network of financial interdependencies. We identify how the network propagates discontinuous changes in asset values triggered by failures (e.g., bankruptcies, defaults, and other insolvencies) and use that to study the consequences of integration (each organization becoming more dependent on its counterparties) and diversification (each organization interacting with a larger number of counterparties). Integration and diversification have different, nonmonotonic effects on the extent of cascades. Initial increases in diversification connect the network which permits cascades to propagate further, but eventually, more diversification makes contagion between any pair of organizations less likely as they become less dependent on each other. Integration also faces tradeoffs: increased dependence on other organizations versus less sensitivity to own investments. Finally, we illustrate some aspects of the model with data on European debt cross-holdings.

Keywords: financial networks, networks, contagion, cascades, financial crises, bankruptcy, diversification, integration, globalization

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1 Introduction

Globalization brings with it increased financial interdependencies among many kinds of organizations – governments, central banks, investment banks, firms, etc. – that hold each other's shares, debts and other obligations. Such interdependencies can lead to cascading defaults and failures, which are often avoided through massive bailouts of institutions deemed "too big to fail." Recent examples include the U.S. government's interventions in A.I.G., Fannie Mae, Freddie Mac, and General Motors; and the European Commission's interventions in Greece and Spain. Although such bailouts circumvent the widespread failures that were more prevalent in the nineteenth and early twentieth centuries, they emphasize the need to study the risks created by a network of interdependencies. Understanding these risks is crucial to designing incentives and regulatory responses that defuse cascades before they are imminent.

In this paper we develop a general model that produces new insights regarding financial contagions and cascades of failures among organizations linked through a network of financial interdependencies. Organizations' values depend on each other – e.g., through cross-holdings of shares, debt or other liabilities. If an organization's value becomes sufficiently low, it hits a failure threshold at which it discontinuously loses further value; this imposes losses on its counterparties, and these losses then propagate to others, even those who did not interact directly with the organization initially failing. At each stage, other organizations may hit failure thresholds and also discontinuously lose value. Relatively small and even organization-specific shocks can be greatly amplified in this way.[1]

In our model, organizations hold primitive assets (any factors of production or other investments) as well as shares in each other.[2] The basic network we start with describes which organizations directly hold which others. Cross-holdings lead to a well-known problem of inflating book values[3], and so we begin our analysis by deriving a formula for a non-inflated "market value" that any organization delivers to final investors outside the system of cross-holdings. This formula shows how each organization's market value depends on the values of the primitive assets and on any failure costs that have hit the economy. We can therefore track how asset values and failure costs propagate through the network of interdependencies. An implication of failures being complementary is that cascades occur in "waves" of dependencies. Although in practice these might occur all at once, it can be useful to distinguish the sequence of dependencies in order to figure out how they might be avoided. Some initial failures are enough to cause a second wave of organizations to fail.

[1]The discontinuities incurred when an organization fails can include the cost of liquidating assets, the (temporary) misallocation of productive resources, as well as direct legal and administrative costs. Given that efficient investment or production can involve a variety of synergies and complementarities, any interruption in the ability to invest or pay for and acquire some factors of production can lead to discontinuously inefficient uses of other factors, or of investments. See Section 2.3 for more details.

[2]We model cross-holdings as direct (linear) claims on values of organizations for simplicity, but the model extends to all sorts of debt and other contracts as discussed in Section A.2 in the Online Appendix.

[3]See Brioschi, Buzzacchi, and Colombo (1989) and Fedenia, Hodder, and Triantis (1994).

Once these organizations fail, a third wave of failures may occur, and so on. A variation on a standard algorithm[4] then allows us compute the extent of these cascades by using the formula discussed above to propagate the failure costs at each stage and determine which organizations fail in the next wave. Policymakers can use this algorithm in conjunction with the market value formula to run counterfactual scenarios and identify which organizations might be involved in a cascade under various initial scenarios.

With this methodology in hand, our main results show how the probability of cascades and their extent depend on two key aspects of cross-holdings: integration and diversification. Integration refers to the *level* of exposure of organizations to each other: how much of an organization is privately held by final investors, and how much is cross-held by other organizations. Diversification refers to *how spread out* cross-holdings are: is a typical organization held by many others, or by just a few? Integration and diversification have different, nonmonotonic effects on the extent of cascades.

If there is no integration then clearly there cannot be any contagion. As integration increases, the exposure of organizations to each other increases and so contagions become possible. Thus, on a basic level increasing integration leads to increased exposure which tends to increase the probability and extent of contagions. The countervailing effect here is that an organization's dependence on its own primitive assets decreases as it becomes integrated. Thus, although integration can increase the likelihood of a cascade once an initial failure occurs, it can also decrease the likelihood of that first failure.

With regard to diversification, there are also tradeoffs, but on different dimensions. Here the overall exposure of organizations is held fixed but the number of organizations cross-held is varied. With low levels of diversification, organizations can be very sensitive to particular others, but the network of interdependencies is disconnected and overall cascades are limited in extent. As diversification increases, a "sweet spot" is hit where organizations have enough of their cross-holdings concentrated in particular other organizations so that a cascade can occur, and yet the network of cross-holdings is connected enough for the contagion to be far-reaching. Finally, as diversification is further increased, organizations' portfolios are sufficiently diversified so that they become insensitive to any particular organization's failure.

Putting these results together, an economy is most susceptible to widespread financial cascades when two conditions hold. The first is that integration is intermediate: each organization holds enough of its own assets that the idiosyncratic devaluation of those assets can spark a first failure, and holds enough of other organizations for failures to propagate. The second condition is that organizations are partly diversified: the network is connected enough for cascades to spread widely, but nodes don't have so many connections that they are well-insured against the failure of any counterparty. Our analysis of these tradeoffs includes both analytical results on a class of networks for which the dynamics of cascades are tractable, as well as simulation results on other random cross-holding networks.

[4]This sort of algorithm is the obvious one for finding extreme points of a lattice, and so is standard in a variety of equilibrium settings. Ours is a variation on one from Eisenberg and Noe (2001).

In the simulations, we examine several important specific network structures. One is a network with a clique of large "core" organizations surrounded by many smaller "peripheral" organizations, each of which is linked to a core organization. This emulates the network of interbank loans. There we see a further nonmonotonicity in integration: if core organizations have low levels of integration then the failure of some peripheral organization is contained, with only one core organization failing; if core organizations have middle levels of integration then widespread contagions occur; if core organizations are highly integrated then they become less exposed to any particular peripheral organization and more resistant to peripheral failures. A second model is one with concentrations of cross-holdings within sectors or other groups. As cross-holdings become more sector-specific, particular sectors become more susceptible to cascades, but widespread cascades become less likely. The level of segregation at which this change happens depends on diversification. With lower diversification, cascades disappear at *lower* rates of segregation – it takes less segregation to fragment the network and prevent cascades.

We also consider what a regulator or government might do to mitigate the possibility of cascades of failures. Preventing a first failure prevents the potential ensuing cascade of failures and it might be hoped that a clever reallocation of cross-holdings could achieve this. Unfortunately, we show that any fair exchange of cross-holdings or assets involving the organization most at risk of failing makes that organization more likely to fail at some asset prices close to the current asset prices. Making the system unambiguously less susceptible to a first failure *necessitates* "bailing out" the organization most at risk of failing.

Finally, we illustrate the model in the context of cross-holdings of European debt.

While there is a growing literature on networks of interdependencies in financial markets[5] our methodology and results are different from any that we are aware of, especially the results on nonmonotonicities in cascades due to integration and diversification.

An independent study by Acemoglu, Ozdaglar and Tahbaz-Salehi (2012b), as well as related earlier studies of Gouriéroux, Héam and Monfort (2012) and Gai and Kapadia (2010), are the closest to ours.[6] They each examine how shocks propagate through a network based on debt holdings or interbank lending, where shocks lead an organization to pay only a

[5]For example, see Rochet and Tirole (1996), Kiyotaki and Moore (1997), Allen and Gale (2000), Eisenberg and Noe (2001), Upper and Worms (2004), Cifuentes, Ferrucci and Shin (2005), Leitner (2005), Allen and Babus (2009), Lorenza, Battiston, Schweitzer (2009), Gai and Kapadia (2010), Wagner (2010), Billio et al. (2012), Demange (2011), Diebold and Yilmaz (2011), Dette, Pauls, and Rockmore (2011), Gai, Haldane, and Kapadia (2011), Greenwood, Landier, and Thesmar (2012), Ibragimov, Jaffee and Walden (2011), Upper (2011), Acemoglu et al. (2012a), Allen, Babus and Carletti (2012), Cohen-Cole, Patacchini and Zenou (2012), Gouriéroux, Héam and Monfort (2012), Alvarez and Barlevy (2013), Glasserman and Young (2013) and Gofman (2013).

[6]Cabrales, Gottardi, and Vega-Redondo (2013) study the tradeoff between the risk-sharing enabled by greater interconnection and the greater exposure to cascades resulting from larger components in the financial network. Their focus is also on some benchmark networks (minimally connected and complete ones) and they examine which ones are best for different distributions of shocks. Again, our work is complementary not only in terms of distinguishing diversification and integration but also analyzing comparative statics for intermediate network structures and finding nonmonotonicites there.

portion of its debts. They are also interested in how shocks propagate as a function of network architecture. However, beyond the basic motivation and focus on the network propagation of shocks, the studies are quite different and complementary. The main results of Acemoglu, Ozdaglar and Tahbaz-Salehi (2012b) characterize the best and worst networks from a social planner's perspective. For moderate shocks a perfectly diversified pattern of holdings is optimal, while for very large shocks perfectly diversified holdings become the worst possible.[7] Our focus is on the complementary question of what happens for intermediate shocks and for a variety of networks. To this end, we consider a class of random networks and ask how the consequences of a given moderate shock depend on diversification and integration. The results highlight that *intermediate* levels of diversification and integration can be the most problematic.

Gai and Kapadia (2010) made two observations. First: rare, large shocks may have extreme consequences when they occur – a point elaborated upon in the subsequent literature discussed above. Second, a shock of a given magnitude may have very different consequences depending on where in the network it hits and on the average connectivity of the network. Gai and Kapadia develop these points in a standard model of epidemics in which the network is characterized by its degree distribution. An innovation of our model is to go beyond the degree distribution of a network and calculate equilibrium (fixed-point) values and interdependencies for organizations. Doing so allows us to distinguish an important dimension of *financial* networks: integration, which can be varied independently of diversification. Building on that, we show how diversification and integration each affect the ingredients of financial cascades – and the final outcomes – in *different* and non-monotonic ways. In doing so, we recover, as a special case, Gai and Kapadia's observation that cascades can be non-monotonic in connectivity.[8] But we also gain key new results on when and how the "danger zone" of intermediate diversification can be blunted by changing the level of integration in the system. Finally, we study how the integration of a financial network interacts with a core-periphery structure and with segregation, and other correlation structures.

2 The Model and Determining Organizations' Values with Cross-Holdings

2.1 Primitive Assets, Organizations, and Cross-Holdings

There are n organizations (e.g., countries, banks, or firms) making up a set $N = \{1, \dots, n\}$.

The values of organizations are ultimately based on the values of primitive assets or factors of production – from now on simply *assets* – $M = \{1, \dots m\}$. For concreteness, a

[7]Shaffer (1994) also identifies a trade-off between risk sharing and systemic failures. While diversified portfolios reduce risk, they also result in organizations holding similar portfolios and a system susceptible to simultaneous failures. See also Ibragimov, Jaffee and Walden (2011) and Allen, Babus and Carletti (2012).

[8]In different settings, Cifuentes, Ferrucci and Shin (2005) and Gofman (2013) also find that cascades can be non-monotonic in connectivity.

primitive asset may be thought of as a project that generates a net flow of cash over time.[9] The present value (or market price) of asset k is denoted p_k. Let $D_{ik} \geq 0$ be the share of the value of asset k held by (i.e., flowing directly into) organization i and let $\mathbf{D}$ denote the matrix whose (i,k)-th entry is equal to D_{ik}. (Analogous notation is used for all matrices.)

An organization can also hold shares of other organizations. For any $i, j \in N$ the number $C_{ij} \geq 0$ is the fraction of organization j owned by organization i, where $C_{ii} = 0$ for each i.[10] The matrix $\mathbf{C}$ can be thought of as a network in which there is a directed link from j to i if value flows in that direction – i.e., if i owns a positive share of j, so that $C_{ij} > 0$.[11]

After all these cross-holding shares are accounted for, there remains a share $\widehat{C}_{ii} := 1 - \sum_{j\in N} C_{ji}$ of organization i not owned by any organization in the system – a share assumed to be positive.[12] This is the part that is owned by *outside* shareholders of i, external to the system of cross-holdings. The off-diagonal entries of the matrix $\widehat{\mathbf{C}}$ are defined to be 0.

Cross-holdings are modeled as linear dependencies in this paper, and we now briefly discuss the interpretation of this. We view the functional form as an approximation of debt contracts around and below organizations' failure thresholds – the region of organizations' values that are important whenever one's failure causes another to fail. In this region, under most bankruptcy procedures[13] there is linear rationing in how much of the debt is paid back. Some organizations may be far from their failure thresholds, and for those, others' changes in value have a smaller effect on the risk of failure. The linear model can incorporate both of these effects through the slope parameters in the cross-holdings matrix; this is discussed in detail in Section 2.5, as well as Section A.2 of the Online Appendix. Of course, this is a crude approximation, but allows a tractable analysis of cross-dependencies, and provides basic insights that should still be useful when nonlinearities are addressed in detail. More generally, cross-holdings can involve all sorts of contracts; any liability in the form of some payment that is due could be included.[14] Directly modeling other sorts of contracting

[9]The primitive assets could be more general factors: prices of inputs, values of outputs, the quality of organizational know-how, investments in human capital, etc. To keep the exposition simple, we model these as abstract investments and assume that net positions are nonnegative in all assets.

[10]It is possible to instead allow $C_{ii} > 0$, which leads to some straightforward adjustments in the derivations that follow; but one needs to be careful in interpreting what it means for an organization to have cross-holdings in itself – which effectively translates into a form of private ownership.

[11] Some definitions: a *path* from i_1 to i_ℓ in a matrix $\mathbf{M}$ is a sequence of distinct nodes $i_1, i_1, \ldots, i_\ell$ such that $M_{i_{r+1}i_r} > 0$ for each $r \in \{1, 2, \ldots, \ell-1\}$. A *cycle* is a sequence of (not necessarily distinct) nodes $i_1, i_1, \ldots, i_\ell$ such that $M_{i_{r+1}i_r} > 0$ for each $r \in \{1, 2, \ldots, \ell-1\}$ and $M_{i_1 i_r} > 0$.

[12]This assumption ensures that organization's market values (discussed below) are well-defined. It is slightly stronger than necessary. It would suffice to assume that, for every organization i, there is some j such that $\widehat{C}_{jj} > 0$ and there is a path from j to i. An organization with $\widehat{C}_{ii} = 0$ would essentially be a holding company, and the important aspect is to have an economy where there are at least some organizations that are not holding companies and some outside shareholders that no organizations have claims on.

[13]A richer model would include priority classes, but the basic issues that we address in the simplified model should still appear in such a richer model.

[14]In essence, our modeling is a reduced form that aggregates all effects into a linear dependence of each organization on others, allowing for a discontinuous loss at a critical organization value. In cases where organizations can short sell other organizations, or hold options or other derivatives that appreciate in value when another organization falls in value, some of our lattice results (discussed in Sections 2.6 and 3.2.3)

between organizations would complicate the analysis and so we focus on this formulation for now to illustrate the basic issues. Section A.2 in the Online Appendix discusses extending the model to more general liabilities.

2.2 Values of Organizations: Accounting and Adjusting for Cross-Holdings

In a setting with cross-holdings, there are subtleties in determining the "fair market" value of an organization, and the real economic costs of organizations' failures. Doing the accounting correctly is essential to analyzing cascades of failure. The basic framework for the accounting was developed by Brioschi, Buzzacchi, and Colombo (1989) and Fedenia, Hodder, and Triantis (1994). In this section, we briefly review the accounting and the key valuation equations in the absence of failure costs. In ensuing sections, we incorporate failures and associated discontinuities.

The equity value V_i of an organization i is the total value of its shares – those held by other organizations as well as those held by outside shareholders. This is equal to the value of organization i's primitive assets plus the value of its claims on other organizations:

$$V_i = \sum_k D_{ik} p_k + \sum_j C_{ij} V_j. \tag{1}$$

Equation (1) can be written in matrix notation as

$$\mathbf{V} = \mathbf{Dp} + \mathbf{CV}$$

and solved to yield[15]

$$\mathbf{V} = (\mathbf{I} - \mathbf{C})^{-1} \mathbf{Dp}. \tag{2}$$

Adding up equation (1) across organizations (and recalling that each column of $\mathbf{D}$ adds up to 1) shows that the sum of the V_i exceeds the total value of primitive assets held by the organizations. Essentially, each dollar of net primitive assets directly held by organization i contributes a dollar to the equity value of organization i, but then is also counted partially on the books of all the organizations that have an equity stake in i.[16]

As argued by both Brioschi, Buzzacchi, and Colombo (1989) and Fedenia, Hodder, and Triantis (1994), the ultimate (non-inflated) value of an organization to the economy – what we call the "market" value – is well-captured by the equity value of that organization that is held by its *outside* investors. This value captures the flow of real assets that accrues to

would no longer hold. That is an interesting topic for further research.

[15]Under the assumption that each column of $\mathbf{C}$ sums to less than 1 (which holds by our assumption of nonzero outside holdings in each organization), the inverse $(\mathbf{I} - \mathbf{C})^{-1}$ is well-defined and nonnegative (Meyer, 2000, Section 7.10).

[16]This initially counterintuitive feature is discussed in detail by French and Poterba (1991) and Fedenia, Hodder, and Triantis (1994).

final investors of that organization. The market value, which we denote by v_i, is equal to $\widehat{C}_{ii}V_i$, and therefore:[17]

$$\mathbf{v} = \widehat{\mathbf{C}}\mathbf{V} = \widehat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-1}\mathbf{D}\mathbf{p} = \mathbf{A}\mathbf{D}\mathbf{p}. \tag{3}$$

We refer to $\mathbf{A} = \widehat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-1}$ as the *dependency* matrix. It is reminiscent of Leontief's input-output analysis. Equation 3 shows that value of an organization can be represented as a sum of the value of its ultimate claims on primitive assets, with organization i owning a share A_{ij} of j's direct holdings of primitive assets. This is the portfolio of underlying assets an outside investor would hold to replicate the returns generated by holding organization i. To see this, suppose each organization fully owns exactly one proprietary asset, so that $m = n$ and $\mathbf{D} = \mathbf{I}$. In this case, A_{ij} describes the dependence of i's value on j's proprietary asset. It is reassuring that $\mathbf{A}$ is column stochastic so that indeed the total values of all organizations add up to the total values of all underlying assets – for all $j \in N$, we have[18]

$$\sum_{i \in N} A_{ij} = 1.$$

2.3 Discontinuities in Values and Failure Costs

An important part of our model is that organizations can lose productive value in discontinuous ways if their values fall below certain critical thresholds. These discontinuities can lead to cascading failures and also the presence of multiple equilibria.

There are many sources of such discontinuities. For example, if an airline can no longer pay for fuel, then its planes may be forced to sit idle (as happened with Spanair in February of 2012) which leads to a discontinuous drop in revenue in response to lost new bookings, and so forth. If a country or firm's debt rating is downgraded, it often experiences a discontinuous jump in its cost of capital. Dropping below a critical value might also involve bankruptcy proceedings and legal costs. Broadly, many of these discontinuities stem from an illiquidity

[17] A way to double check this equation is to derive the market value of an organization from the book value of its underlying assets and cross-holdings less the part of its book value promised to other organizations in cross-holdings:

$$v_i = \sum_j C_{ij}V_j - \sum_j C_{ji}V_i + \sum_k D_{ik}p_k$$

or

$$\mathbf{v} = \mathbf{C}\mathbf{V} - (\mathbf{I} - \widehat{\mathbf{C}})\mathbf{V} + \mathbf{D}\mathbf{p} = (\mathbf{C} - (\mathbf{I} - \widehat{\mathbf{C}}))\mathbf{V} + \mathbf{D}\mathbf{p}.$$

Substituting for the book value $\mathbf{V}$ from (2), this becomes

$$\mathbf{v} = (\mathbf{C} - \mathbf{I} + \widehat{\mathbf{C}})(\mathbf{I} - \mathbf{C})^{-1}\mathbf{D}\mathbf{p} + \mathbf{D}\mathbf{p} = (\mathbf{C} - \mathbf{I} + \widehat{\mathbf{C}} + (\mathbf{I} - \mathbf{C}))(\mathbf{I} - \mathbf{C})^{-1}\mathbf{D}\mathbf{p} = \mathbf{A}\mathbf{D}\mathbf{p}.$$

[18] This can be seen by defining an augmented system in which there is a node corresponding to each organization's external investor and noting that, under our assumptions, the added nodes are the only absorbing states of the Markov chain corresponding to the system of asset flows. Column j of $\mathbf{A}$ describes how the proprietary assets entering at node j are shared out among the external absorbing nodes. Since all the flow must end up at some external absorbing node, $\mathbf{A}$ must be column-stochastic.

which then leads to an inefficient use of assets. Indeed, given that efficient production can involve a variety of synergies and complementarities, any interruption in the ability to pay for and acquire some factors of production can lead to discontinuously inefficient uses of other factors, or of investments. One detailed and simple microfoundation is laid out in Section 2.5 below.

If the value v_i of a organization i falls below some threshold level $\underline{v}_i$, then i is said to *fail* and incurs failure costs $\beta_i(\mathbf{p})$.[19] These failure costs are subtracted from a failing organization's cash flow. They can represent the diversion of cash flow towards dealing with the failure or a reduction in the returns generated by proprietary assets. Either way this introduces critical non-linearities – indeed, discontinuities – into the system.

We base failure costs on the (market) value of an organization, v_i, and not the book value, V_i. This captures the idea that failure occurs when an organization has difficulties or disruptions in operating, and the artificial inflation in book values that accompanies cross-holdings is irrelevant in avoiding a failure threshold.[20] Nonetheless, the model could instead make failures dependent upon the book values V_i, in cases where cash flows relate to book values. Nothing qualitative would change in what follows, as the critical ingredients of thresholds of discontinuities and cascades that depend on cross-holdings would still all be present, just with different trigger points.

Let us say a few words about the relative sizes of these discontinuities. Recent work has estimated the cost of default to average 21.7 percent of the market value of an organization's assets, (with substantial variation – see Davydenko, Strebulaev, and Zhao (2012), as well as James (1991)).[21] It might be hoped that organizations will reduce the scope for cascades of failures by minimizing their failure costs and reducing the threshold values at which they fail. In fact, as we show in the Online Appendix (Section A.3), financial networks can create moral hazard and favor the opposite outcome. As discussed in Leitner Leitner (2005), counterparties have incentives to bail out a failing organization[22] to avoid (indirectly) incurring failure costs. To improve its bargaining position in negotiating for such aid, an organization may then want to increase its failure costs and make its failure more likely. Nevertheless, although default costs can be large both absolutely and relative to the value of an organization's assets (e.g., the size of the recent Greek write-down in debt, or the fire-sale of Lehman Brothers' assets), it can also be that smaller effects snowball. Given that a major recession in an economy is only a matter of a change of a few percentage points in its growth rate, when contagions are far-reaching, the particular drops in value of any single organization need not be very large in order to have a large effect on the economy. We develop this observation further in Section 3.1.

[19]The argument $\mathbf{p}$ reflects that these costs can depend on the values of underlying assets, as would be the case when these are liquidated for a fraction of their former value. See Section 2.5 for more detail.

[20]For example, if the failure threshold were based on book values, then two organizations about to fail would be able to avoid failure by exchanging cross-holdings and inflating their book values.

[21]Capping the failure costs is not important for our model, but they could easily be capped at $\underline{v}_i$ or $(Dp)_i$ or some other natural level.

[22]For example, in the form a debt write-down.

2.4 Including Failure Costs in Market Values

The valuations in (2) and (3) have analogs when we include discontinuities in value due to failures. The discontinuous drop imposes cost directly on an organization's balance sheet, and so the book value of organization i becomes:

$$V_i = \sum_{j \neq i} C_{ij} V_j + \sum_k D_{ik} p_k - \beta_i I_{v_i < \underline{v}_i}$$

where $I_{v_i < \underline{v}_i}$ is an indicator variable taking value 1 if $v_i < \underline{v}_i$ and value 0 otherwise.

This leads to a new version of (2):

$$\mathbf{V} = (\mathbf{I} - \mathbf{C})^{-1}(\mathbf{D}\mathbf{p} - \mathbf{b}(\mathbf{v}, \mathbf{p})), \tag{4}$$

where $b_i(\mathbf{v}, \mathbf{p}) = \beta_i(\mathbf{p}) I_{v_i < \underline{v}_i}$.[23] Correspondingly, (3) is re-expressed as

$$\mathbf{v} = \widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1}(\mathbf{D}\mathbf{p} - \mathbf{b}(\mathbf{v})) = \mathbf{A}(\mathbf{D}\mathbf{p} - \mathbf{b}(\mathbf{v}, \mathbf{p})). \tag{5}$$

An entry A_{ij} of the dependency matrix describes the proportion of j's failure costs that i bears when j fails as well as i's claims on the primitive assets that j directly holds. If organization j fails, thereby incurring failure costs of β_j, then i's value will decrease by $A_{ij}\beta_j$.

2.5 A Simple Microfoundation

To help fix ideas, we discuss one simple microfoundation – among many – of the model and the value equations provided above.

Organizations are owner-operated firms. For simplicity, let each firm have a single proprietary asset: an investment project that generates a return. Our model is then simplified to the case $m = n$ and $\mathbf{D} = \mathbf{I}$. Firms have obligations to each other: for instance, promised payments for inputs or other intermediate goods. These obligations comprise the cross-holdings. Once a firm's value no longer covers the full promised value of its payments, all creditor organizations – who are of equal seniority – are rationed in proportion to V_i, with organization j claiming $C_{ij}V_i$ of i's value. Thus, even though the obligations might initially be in the form of debt, the relevant scenario for our cascades – and the one the model focuses on – is one in which the full promised amounts cannot be met by the organizations. This is a regime of "orderly write-downs" in which creditors are willing to take a fraction of the face value they are owed. Thus, the values of cross-holdings are simply linear in V_i, as in our equations. (Section A.2 in the Online Appendix illustrates this in detail.)

[23] The number $b_i(\mathbf{v}, \mathbf{p})$ reflects realized failure costs, and is zero when failure does not occur. It always depends on the asset values through the indicator $I_{v_i \leq \underline{v}}$, but the bankruptcy costs β_i may depend on underlying asset values, $\mathbf{p}$. See Section 2.5 below for an example. We will suppress the argument $\mathbf{p}$ when it is not essential.

The value left to the owner-operators is $v_i = \widehat{C}_{ii}V_i$. While the firm continues to operate, this amount must cover return on capital, wages, benefits, and pension obligations for the owner operators.[24] The share $\widehat{C}_{ii}$ can be thought of as all of the stock or equity held in the firm, while the C_{ij}'s are payment obligations from the firm to other firms. The $\widehat{C}_{ii}$ residual shares correspond to the control rights of the firm, while the C_{ij}'s simply represent obligations to other creditors. If the value left to the owner-operators/shareholders is sufficiently low (below some outside option value of their time or effort), they may choose to cease operations.[25] Indeed, we posit that there is a critical threshold $\underline{v}_i$ such that if the value available to the owner-operator falls below it, he or she chooses to cease operations and to liquidate the asset. In other words, once $v_i < \underline{v}_i$ the asset is liquidated.

Liquidation is irreversible and total: a firm cannot partially liquidate its proprietary asset. Liquidation is also costly: if i liquidates its proprietary asset, it incurs a loss of λ_i cents on the dollar.[26] In terms of our model, $\beta_i(\mathbf{p}) = \lambda_i p_i$. Recalling that $b_i(\mathbf{v}, \mathbf{p}) = \beta_i(\mathbf{p}) I_{v_i < \underline{v}_i}$, it follows that

$$\mathbf{v} = \mathbf{A}\left(\mathbf{p} - \mathbf{b}(\mathbf{v}, \mathbf{p})\right).$$

2.6 Equilibrium Existence and Multiplicity

A solution for organization values in equation (5) is an *equilibrium* set of values, and encapsulates the network of cross-holdings in a clean and powerful form, building on the dependency matrix $\mathbf{A}$.

There always exists a solution and there can exist multiple solutions to the valuation equation (multiple vectors $\mathbf{v}$ satisfying (5)) in the presence of the discontinuities. In fact, the set of solutions forms a complete lattice.[27]

There are two distinct sources of equilibrium multiplicity. First, taking other organizations' values and the values of underlying assets as fixed and given, there can be multiple possible consistent values of organization i that solve equation (5). There may be a value of v_i satisfying equation (5) such that $1_{v_i < \underline{v}_i} = 0$ and another value of v_i satisfying equation (5) such that $1_{v_i < \underline{v}_i} = 1$; even when all other prices and values are held fixed. This source of multiple equilibria corresponds to the standard story of self-fulfilling bank runs (see classic models such as Diamond and Dybvig (1983)). The second source of multiple equilibria is the interdependence of the values of the organizations: the value of i depends on the value of organization j, while the value of organization j depends on the value of organization i. There might then be two consistent valuation vectors for i and j: one in which both i

[24] Indirectly, the value v_i includes the cross-holdings that firm i has in others; that is, accounts receivable that can be used to meet payroll and other obligations.

[25] This can happen for various reasons. For example, in the case of Spanair, there was too little money to cover wages, fuel, and other basic maintenance costs, and the airline was forced to cease operations. It could also be that the owners no longer view it worthwhile to continue to devote efforts to this investment project.

[26] These losses involve time that the asset is left idle, costs of assessing values and holding sales of assets, costs of moving assets to another production venue, and loss of firm specific capital and knowledge.

[27] This holds by a standard application of Tarski's fixed point theorem, as failures are complements.

and j fail and another in which both i and j remain solvent. This second source of multiple equilibria is different from the individual bank run concept, as here organizations fail because people expect other organizations to fail, which then becomes self-fulfilling.

In what follows, we typically focus on the best case equilibrium, in which as few organizations as possible fail.[28] This allows us to isolate sources of *necessary* cascades, distinct from self-fulfilling sorts of failure, which have already been studied in the sunspot and bank run literatures. When we do discuss multiple equilibria, we will consider only the second novel source of multiplicity – multiplicity due to interdependencies between organizations – rather than the well-known phenomenon of a bank run on a single organization. With suitable regularity conditions (so that other equilibria are appropriately stable in some range of parameters), the results presented below should have analogs applying to other equilibria, including the worst case equilibrium.

2.7 Measuring Dependencies

The dependency matrix $\mathbf{A}$ takes into account all indirect holdings as well as direct holdings. The central insights of the paper are derived using this matrix. In this section we identify some useful properties of the dependency matrix $\mathbf{A}$ and explore its relation to direct cross-holdings $\mathbf{C}$.

2.7.1 An Example

To see how the dependency matrix $\mathbf{A}$ and direct cross-holdings matrix $\mathbf{C}$ might differ, consider the following example. Suppose there are two organizations, $i = 1, 2$, each of which has a 50% stake in the other organization. The associated cross-holdings matrix $\mathbf{C}$ and the dependency matrix $\mathbf{A}$ are as follows. (Recall that $\widehat{C}_{ii}$ is equal to 1 minus the sum of the entries in column i of $\mathbf{C}$.)

$$\mathbf{C} = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix} \qquad \widehat{\mathbf{C}} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \qquad \mathbf{A} = \widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

In this simple example, we can already see that direct claims – as captured by $\mathbf{C}$ and $\widehat{\mathbf{C}}$ – can differ quite substantially from the ultimate value dependencies described by $\mathbf{A}$. First, even though an organization 1's shareholders have a direct claim on 50% of its value, they are ultimately entitled to more than this – as they also have some claims on the value of organization 2, which includes part of the value of organization 1. Second, the ultimate dependence of each organization on the other is smaller than what is apparent from $\mathbf{C}$, by the fact that value is conserved.[29]

[28] As discussed in Section 3.2.3, in this best case equilibrium no organization fails that does not also fail in all other equilibria.

[29] A further (starker) illustration of $\mathbf{A}$ and $\mathbf{C}$ can differ is available in the Online Appendix (Section A.1).

Although $\mathbf{A}$ can differ substantially from the direct holdings captured by $\mathbf{C} + \widehat{\mathbf{C}}$, some general statements can be made about the differences.

Lemma **1.** $\widehat{C}_{ii}$ is a lower bound on A_{ii}, but A_{ii} can be much larger than $\widehat{C}_{ii}$.

1. $\frac{A_{ii}}{\widehat{C}_{ii}} \geq 1$ for each i, with equality if and only if there are no cycles of cross-holdings (i.e. directed cycles in $\mathbf{C}$) that include i.

2. For any n, there exists a sequence of n-by-n matrices $\left(\mathbf{C}^{(\ell)}\right)$ such that $\frac{A^{(\ell)}_{ii}}{\widehat{C}^{(\ell)}_{ii}} \to \infty$ for all i.

The magnitudes of the terms on the main diagonal of $\mathbf{A}$ turn out to be critical for determining whether and to what extent failures cascades (Section 3.1) and the size of a moral hazard problem we discuss in the Online Appendix. Lemma 1 demonstrates that the lead diagonal of $\mathbf{A}$ can be larger than the lead diagonal of $\widehat{\mathbf{C}}$, but can never be smaller. The potential for a large divergence comes from the fact that sequences of cross-holdings can involve cycles (i holds j, who holds k, who holds ℓ, ..., who holds i), so that i can end up with a higher dependency on its own assets than indicated by looking only at its outside investors' direct holdings $(\widehat{C}_{ii})$.

2.8 Avoiding a First Failure

Before moving on to our main results regarding diversification and integration, we provide a result that uses our model to show that there are necessarily tradeoffs in preventing the spark that ignites a cascade. Any fair trades of cross-holdings and assets that help an organization avoid failure in some circumstances must make it vulnerable to failure in some new circumstances. This is a sort of "no-free-lunch" result for avoiding first failures.

To state this result, it is helpful to introduce some notation. We write organization i's value assuming no failures at asset prices $\mathbf{p}$, cross-holdings $\mathbf{C}$ and direct holdings $\mathbf{D}$ as $v_i(\mathbf{p}, \mathbf{C}, \mathbf{D})$. An organization i is *closest to failing* at positive asset prices $\mathbf{p}$, cross-holdings $\mathbf{C}$, and direct holdings $\mathbf{D}$ if there exists a (necessarily unique) $\lambda > 0$ such that at asset prices $\lambda\mathbf{p}$, organization i is about to fail, $v_i(\lambda\mathbf{p}, \mathbf{C}, \mathbf{D}) = \underline{v}_i$, while all other organizations are solvent, $v_j(\lambda\mathbf{p}, \mathbf{C}, \mathbf{D}) > \underline{v}_j$ for $j \neq i$. Define $\mathbf{q}(\mathbf{p}, \mathbf{C}, \mathbf{D}) := \lambda\mathbf{p}$.

Before stating the result we also introduce the concept of *fair trades.*[30] Fair trades are exchanges of cross-holdings or underlying assets that leave the (market) values of the organizations unchanged at current asset prices.[31] More precisely, the matrices $(\mathbf{C}, \mathbf{D})$ and

[30]This definition takes prices of assets ($\mathbf{p}$) as given, but not necessarily the prices of organizations, valuing them based on their holdings. It does not incorporate the potential impact of failures of organizations on their values. Thus it is a benchmark that abstracts away from the failure costs, which is the right benchmark for the exercise of seeing the impact of trades on *first*-failures.

[31]So, absent failure, the values of organizations are the same before and after fair trades.

$(\mathbf{C}', \mathbf{D}')$ are said to be related by a fair trade at $\mathbf{p}$ if $\mathbf{v} = \mathbf{v}'$, where $\mathbf{v} = \mathbf{A}\mathbf{p}$ and $\mathbf{v}' = \mathbf{A}'\mathbf{p}$; the matrix $\mathbf{A}'$ is computed as in (5) with $\mathbf{C}'$ and $\mathbf{D}'$ playing the roles of $\mathbf{C}$ and $\mathbf{D}$.[32]

PROPOSITION **1.** Suppose an organization i is closest to failing at asset prices $\mathbf{p}$, cross-holdings $\mathbf{C}$, and direct holdings $\mathbf{D}$. Consider new cross-holdings and direct holdings $\mathbf{C}'$ and $\mathbf{D}'$ resulting from a fair trade at $\mathbf{p}$ so that row i of $\mathbf{A}'$ is different from that of $\mathbf{A}$. Then, for any $\varepsilon > 0$, there is a $\mathbf{p}'$ within an ε-neighborhood of $\mathbf{q}(\mathbf{p}, \mathbf{C}, \mathbf{D})$,[33] such that i fails at prices $\mathbf{p}'$ after the fair trade but not before: $v_i(\mathbf{p}', \mathbf{C}', \mathbf{D}') < \underline{v}_i < v_i(\mathbf{p}', \mathbf{C}, \mathbf{D})$.

It is conceivable that if an organization is at risk of eventual failure but not imminent failure there could exist some *fair* trades that would unambiguously make that organization safer: prone to failure at a smaller set of prices. An organization might hedge a particular risk. Proposition 1 shows that, at least when it comes to saving the most vulnerable organization, there are *always* tradeoffs: new holdings that avoid failure at one set of prices make failure more likely at another set of nearby prices. So, to fully avoid a failure (at nearby prices) once it is imminent requires some unfair trades or external infusion of capital.

3 Cascades of Failures: Definitions and Preliminaries

In order to present our main results, we need to first provide some background results and definitions regarding how the model captures cascades, which we present in this section. These preliminaries outline how failures cascade and become amplified, a simple algorithm for identifying the waves of failures in a cascade, and our distinction between diversification and integration.

3.1 Amplification through Cascades of Failures

A relatively small shock to even a small organization can have large effects by triggering a cascade of failures. The following example illustrates this. For simplicity, suppose that organization 1 has complete ownership of a single asset with value p_1. Suppose that $\mathbf{p}'$ differs from $\mathbf{p}$ only in the price of asset 1, such that $p_1' < p_1$. Finally, suppose $v_1(\mathbf{p}) > \underline{v}_1(\mathbf{p}) > v_1(\mathbf{p}')$ so that 1 fails after the shock changing asset values from $\mathbf{p}$ to $\mathbf{p}'$. Beyond the loss in value due to the decrease in the value of asset 1, organizations 2's value also decreases by a term arising from 1's failure cost, $A_{21}\beta_1$ (recall (5)). If organization 2 also fails, organization 3 absorbs part of both failure costs: $A_{31}\beta_1 + A_{32}\beta_2$, and so organization 3 may fail too, and so forth. With each failure, the combined shock to the value of each remaining solvent organization increases and organizations that were further and further from failure before the initial shock can get drawn into the cascade. If, for example, the first K organization end up failing in

[32] We show in Section A.3.1 of the Online Appendix that there are circumstances under which organizations may have incentives to undertake "unfair" trades because of the failure costs.

[33] I.e. $\mathbf{p}'$ such that $\|\mathbf{p}' - \mathbf{q}(\mathbf{p}, \mathbf{C}, \mathbf{D})\|_\infty < \varepsilon$, where $\|\cdot\|_\infty$ denotes the sup-norm.

the cascade, the the cumulative failure costs to the economy are $\beta_1 + \cdots + \beta_K$, which can greatly exceed the drop in asset value that precipitated the cascade.

3.2 Who Fails in a Cascade?

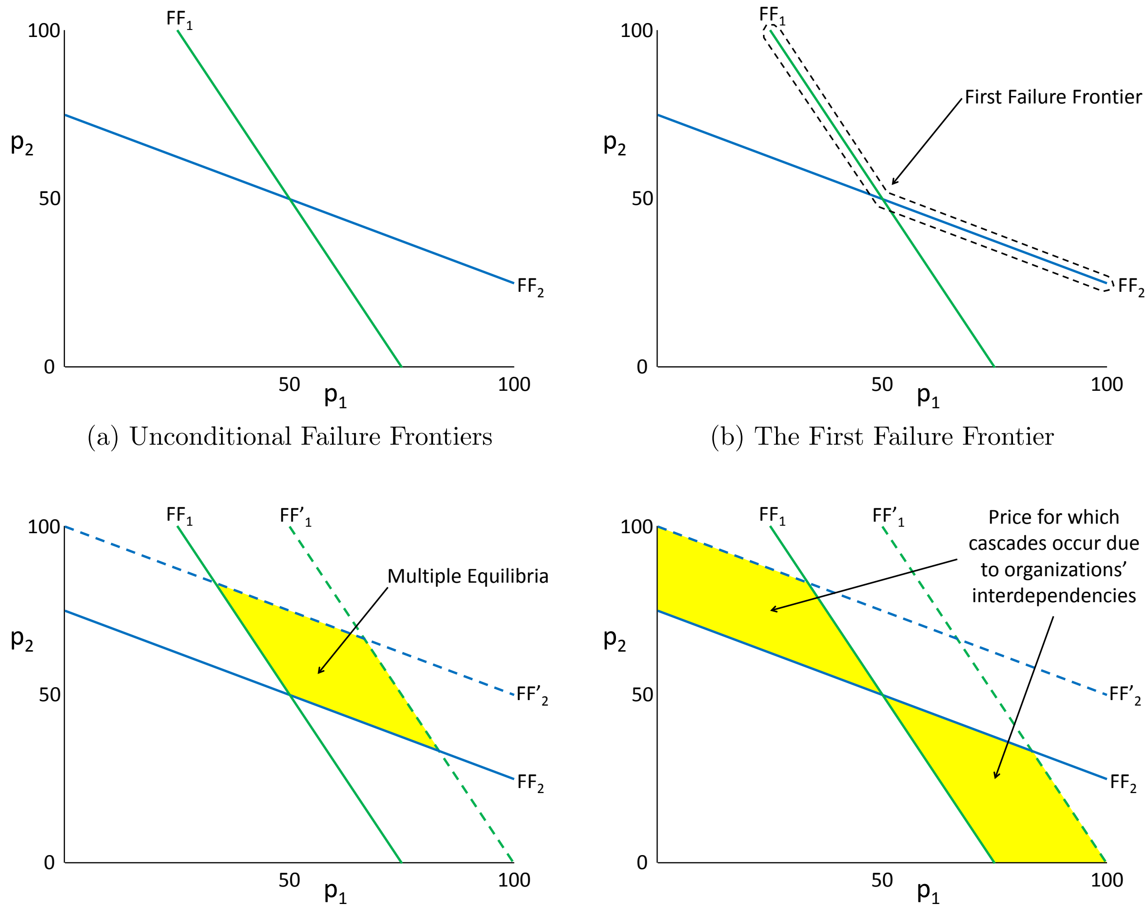
A first step towards understanding how susceptible a system is to a cascade of failures, and how extensive such a cascade might be, is to identify which organizations will fail following a shock. Again, we focus on the best-case equilibrium.[34] Studying the best case equilibrium following a shock identifies the minimal possible set of organizations that could fail. (Results for the worst-case equilibrium are easy analogs identifying the maximal possible set of organizations that will fail.)

3.2.1 Identifying Who Fails When

To understand how and when failures cascade we need to better understand when a fall in asset prices will cause an initial failure and whether the first failure will result in other failures. Utilizing the dependency matrix **A**, for each organization i we can identify the boundary in the space of underlying asset prices below which organization i must fail, assuming no other organization has failed yet. We can also identify how the failure of one organization affects the failure boundaries of other organizations and so determine when cascades will occur and who will fail in those cascades. We begin with an example that illustrates these ideas very simply, and then develop the more general analysis.

[34] This is the best case equilibrium across all possible equilibria; this statement remains true even when we consider multiplicity not arising from interdependencies among organizations.

3.2.2 Example Continued



(a) Unconditional Failure Frontiers

(b) The First Failure Frontier

(c) Multiple Equilibria

(d) Cascades of Failure

Figure 1: With positive cross-holdings the discontinuities in values generated by the failure costs can result in multiple equilibria and cascades of failure.

Let us return to the example introduced in Section 2.7.1, taking $\mathbf{D} = \mathbf{I}$, so each organization owns one proprietary asset. We suppose that organization i fails when its value falls below 50 and upon failing incurs failure costs of 50. Organization i therefore fails when $\frac{2}{3}p_i + \frac{1}{3}p_j < 50$. Figure 1a shows the failure frontiers for the two organizations. When asset prices are above both failure frontiers, neither organization fails in the best case equilibrium outcome. One object that we study is the boundary between this region and the region in which at least one organization fails in all equilibria. We call this boundary the first failure frontier and it is shown in Figure 1b.

The failure boundaries shown in Figure 1a are not the end of the story. If organization j fails, then organization i's value falls discontinuously. In effect, through i's cross-holding in j and the reduction in j's value, i bears 1/3 of j's failure costs of 50. Organization i then fails if $\frac{2}{3}p_i + \frac{1}{3}(p_j - 50) < 50$. We refer to this new failure threshold as i's failure frontier conditional on j failing and label it FF_i'. These conditional failure frontiers are shown in Figure 1c.

The conditional failure frontiers identify a region of multiple equilibria due to interdependencies in the value of the organizations. As discussed earlier, this is a different source of multiple equilibria from the familiar bank run story (we do not depict the multiple equilibria corresponding to this). The multiple equilibria arise because i's value decreases discontinuously when j fails and j's value decreases discontinuously when i fails. It is then consistent for both i to and j to survive, in which case the relevant failure frontiers are the unconditional ones, and consistent for both i and j to fail, in which case the relevant failure frontiers are the conditional ones.

Figure 1d identifies the regions where cascades occur in the best case equilibrium.[35] When asset prices move from being outside the first failure frontier to being inside this region, the failure of one organization precipitates the failure of the other organization. One organization crosses its *unconditional* (best-case) failure frontier and the corresponding asset prices are also inside the other organization's *conditional* failure frontier (which includes the costs arising from the other organizations failure).[36]

3.2.3 A Simple Algorithm for Identifying Cascade Hierarchies

Although all the relevant information about exactly who will fail at what asset prices can be represented in diagrams such as those in the previous section for simple examples, the number of conditional failure frontiers grows exponentially with organizations and while adding assets increases the dimensions making their geometric depiction infeasible. Thus, while the diagrams provide a useful device for introducing ideas, they are of less use practically. In this section, we provide an algorithm that traces the propagation of a specific shock that causes one organization to fail.[37] As before, we focus on the best-case equilibrium in terms of having the fewest failures and the maximum possible values v_i.

At step t of the algorithm, let the set Z_t be the set of failed organizations. Initialize $Z_0 = \emptyset$. At step $t \geq 1$:

1. Let $\widetilde{\mathbf{b}}_{t-1}$ be a vector with element $\widetilde{b}_i = \beta_i$ if $i \in Z_{t-1}$ and 0 otherwise.

[35] Compare with Figure 3 in Gouriéroux, Héam and Monfort (2012), which makes some of the same points.

[36] As hinted at above, the full set of multiple equilibria is more complex than pictured in Figure 1 and this is discussed in the Online Appendix (Sections A.7 and A.8). For example the worst-case equilibrium has frontiers further out than those in Figure 1c, as those are based on including failure costs arising from the other organization failing. The worst-case equilibrium is obtained by examining frontiers based on failure costs presuming that *both* fail, and then finding prices consistent with those frontiers. There are also additional equilibria that differ from both the best and worst case equilibria – ones that presume one organization's failure but not the other organization's, and find the highest prices consistent with these presumptions.

[37] This sort of algorithm is the obvious one for finding extreme points of a lattice, and so is standard (for instance, see Theorem Theorem 5.1 in Vives (1990)). Variations on it appear in the literature on contagions, as in Eisenberg and Noe (2001) and Blume et al (2011).

2. Let Z_t be the set of all k such that entry k of the following vector is negative:

$$\mathbf{A}\left[\mathbf{D}\mathbf{p} - \widetilde{\mathbf{b}}_{t-1}\right] - \underline{\mathbf{v}}.$$

3. Terminate if $Z_t = Z_{t-1}$. Otherwise return to step 1.

When this algorithm terminates at step T (which it will given the finite number of organizations), the set Z_T corresponds to the set of organizations that fail in the best case equilibrium.[38]

This algorithm provides us with *hierarchies* of failures. That is, the various organizations that are added at each step (the new entries in Z_t compared to Z_{t-1}) are organizations whose failures were triggered by the cumulative list of prior failures; they would not have failed if not for that accumulation and, in particular, if not for the failures of those added at the last step. Thus, Z_1 are the first organizations to fail, then $Z_2 \setminus Z_1$ are those whose failures are triggered by the first to fail, and so forth.

Note that the sets depend on $\mathbf{p}$ (and $\mathbf{C}$ and $\mathbf{D}$), and so each configuration of these can result in a different structure of failures. It is possible to have some $\mathbf{C}$ and $\mathbf{D}$ such that there are some organizations that are never the first to fail, and others who are sometimes the first to fail and sometimes not.

The hierarchical structure of failures has immediate and strong policy implications. If any level of the hierarchy can be made empty, then the cascade stops and no further organization will fail. This suggests that one cost effective policy for limiting the effect of failures should be to target high levels of the hierarchy that consist of relatively few organizations.[39] However, such policies may involve more intervention than is necessary. For example, within a wave there could be a single critical organization, the saving of which would prevent any further failure regardless of whether other organizations in the same level failed. Saving an entire level from failure is sufficient for stopping a cascade, but not necessary.

3.3 Defining Integration and Diversification

One of our contributions is a distinction between the roles of diversification and integration in cascades. Before presenting those results (in the next Section), we provide the essential distinction.

We say that a financial system becomes *more diversified* when the number of cross-holders in each organization i weakly increases and the cross-holdings of all original cross-holders of i weakly decrease.

Formally, cross-holdings $\mathbf{C}'$ are *more diversified* than cross-holdings $\mathbf{C}$ if and only if

[38] The same algorithm can be used to find the set of organizations that fail in the worst case equilibrium by instead initializing the set Z_0 to contain all organizations and looking for organizations that will not fail, and so forth.

[39] As considered in Section 2.8.

- $C'_{ij} \leq C_{ij}$ for all i, j such that $C_{ij} > 0$, with strict inequality for some ordered pair (i, j), and

- $C'_{ij} > C_{ij} = 0$ for some i, j.

Thus, diversification captures the spread in organizations' cross-holdings.

A financial system becomes *more integrated* if the external shareholders of each organization i have lower holdings, so that the total cross-holdings of the each organization by other organizations weakly increases.

Formally, cross-holdings $\mathbf{C}'$ are more integrated than cross-holdings $\mathbf{C}$ if and only if $\widehat{C}'_{ii} \leq \widehat{C}_{ii}$ for all i with strict inequality for some i. This is equivalent to the condition that

$$\sum_{j:j\neq i} C'_{ji} \geq \sum_{j:j\neq i} C_{ji},$$

for all i with strict inequality for some i.[40]

Thus, integration captures the depth or extent of organizations' cross-holdings. This can be viewed as an intensive margin. In contrast, diversification pertains to the number of organizations interacting directly with one another, and so is an extensive margin.

It is possible for a change in cross-holdings to both increase diversification and integration. There are changes in cross-holdings that increase diversification but not integration and other changes that increase integration but not diversification.

3.4 Essential Ingredients of a Cascade

To best understand the impact of diversification and integration on cascades it is useful to identify three ingredients that are necessary for a widespread cascade:

I. A First Failure: Some organization must be susceptible enough to shocks in some assets that it fails.

II. Contagion: It must be that some other organizations are sufficiently sensitive to the first organization's failure that they also fail.[41]

III. Interconnection: It must be that the network of cross-holdings is sufficiently connected so that the failures can continue to propagate and are not limited to some small component.

[40] This definition is simple and well-suited to our simulations as in these we will have symmetric values of underlying assets. However, when underlying asset values are asymmetric there may be changes in cross-holdings consistent with either increasing or decreasing integration that result in substantial changes in the relative values of organizations, and so a more complicated definition is needed. Thus, in our formal results we work with a definition that also holds organizations' market values constant.

[41] Note that it need not be an immediate cross-holder that is the sensitive one. Drops in values propagate through the network (as captured by the matrix $\mathbf{A}$), and so the second organization to fail need not be an immediate cross-holder, although that would typically be the case.

Keeping these different ingredients of cascades in mind will help us disentangle the different effects of changes in cross-holdings.

Let us preview of some of the ideas, which we will soon make precise in by imposing some additional structure on the model. As we increase integration (without changing each organization's counterparties), an organization becomes less sensitive to its own investments but more sensitive to other organizations' values, and so first failures can become less likely while contagion can become more likely conditional on a failure. This decreases the circumstances that lead to first failures, making things better with respect to I, while it increases the circumstances where there can be contagion, making things worse with respect to II. Interconnection (III) is not impacted one way or the other as the network pattern does not change (by assumption). As we increase diversification, organizations become less dependent on any particular neighbor, so contagions can be harder to start, but the network becomes more connected, and so the extent of a contagion broadens (at least up to a point where the network is fully connected). This decreases the circumstances where there can be contagion, making things better with respect to II, while increasing the potential reach of a contagion conditional upon one occurring, making things worse with respect to III.

Understanding this structure makes some things clear. First, integration and diversification affect different ingredients of cascades. Integration affects an organization's exposure to others compared to its exposure to its own assets, while diversification affects how many others one is (directly and indirectly) exposed to. Second, both integration and diversification improve matters with respect to at least one of the cascade ingredients above while causing problems along a different dimension. These tradeoffs result in nonmonotonic effects of diversification and integration on cascades, as we now examine in detail.

4 How Do Cascades Depend on the Diversification and Integration of Cross-Holdings?

We now turn to our main results.

We begin with some analytic results and then provide additional results via simulations for some random network structures.

4.1 The Consequences of Diversification and Integration: Analytic Results

4.1.1 A General Result on Integration

To begin, we prove a general result about how integration affects the extent of cascades. The result permits any initial cross-holdings $\mathbf{C}$, an arbitrary vector of costs $\boldsymbol{\beta}$, an arbitrary vector of threshold values $\underline{\mathbf{v}}$, any direct holdings of assets $\mathbf{D}$, and any underlying asset values $\mathbf{p}$.

Recall that the matrices $(\mathbf{C}, \mathbf{D})$ and $(\mathbf{C}', \mathbf{D}')$ are said to be related by a fair trade at $\mathbf{p}$ if $\mathbf{v} = \mathbf{v}'$, where $\mathbf{v} = \mathbf{A}\mathbf{p}$ and $\mathbf{v}' = \mathbf{A}'\mathbf{p}$; the matrix $\mathbf{A}'$ is computed as in (5) with $\mathbf{C}'$ and $\mathbf{D}'$ playing the roles of $\mathbf{C}$ and $\mathbf{D}$.[42]

PROPOSITION **2.** Consider $(\mathbf{C}, \mathbf{D})$ and $(\mathbf{C}', \mathbf{D}')$ that are related by a fair trade at $\mathbf{p}$,[43] and such that integration increases: $A'_{ij} \geq A_{ij}$ whenever $i \neq j$. Every organization that fails in the cascade at $(\mathbf{C}, \mathbf{D}, \mathbf{p})$ also fails at $(\mathbf{C}', \mathbf{D}', \mathbf{p})$.

Proposition 2 states that if we integrate cross holdings via fair trades, so that organizations end up holding more of each other's investments, then we face more failures in any given cascade that begins. Thus, benefits of integration comes *only* via avoiding first failures. There is a tradeoff: integrating can eliminate some first failures. However, given that a first failure occurs, it only exacerbates the resulting cascade.

The reasoning behind the proposition is as follows. As can be seen immediately from equation (5), when organization i fails and incurs failure costs β_i, it is the ith column of $\mathbf{A}$ which determines who (indirectly) pays these costs. Increasing A_{ij} for all i and $j \neq i$ increases the share of i's failure costs paid by each other organization. This increases the negative externality i imposes on each organization following its own failure. These other organizations are then more likely to also fail once i fails and so the number of organizations that fail in the cascade weakly increases.

4.1.2 A Result on Diversification and Integration

In order to bring diversification into the picture, we specialize the model a bit. Fixing any given level of diversification and integration a network can typically be rewired to make it more or much less susceptible to cascades of failures. This is an obstruction to analytical comparative statics in diversification that hold for every network. By working with a random graph model that imposes some structure on the distribution of possible cross-holdings matrices, we can overcome this challenge and make statements that hold with high probability.[44] The random graph model is tractable yet flexible with respect to degree distributions, making it well-suited to the study of diversification. Our analysis of it illustrates some basic intuitions. We then come back to verify, via simulations, that these intuitions generalize to random networks that are less analytically tractable.

Before introducing any randomness, suppose $\mathbf{G}$ is a fixed matrix with all entries in $\{0, 1\}$; we call this an *adjacency matrix* of an unweighted, directed graph. The interpretation is

[42] We show in the Online Appendix (Section A.3.1), that there are circumstances under which organizations may have incentives to undertake "unfair" trades because of the failure costs.

[43] The definition of a fair trade ignores any failure costs – i.e., the values before and after a trade are calculated as if failures do not occur. This offers a clear benchmark.

[44] When one allows the number of nodes to become arbitrarily large, then various techniques related to laws of large numbers can be applied to deduce connectedness properties of a random network. Thus, one can make statements that are likely to hold with high probability when the number of nodes is large. For surveys of techniques relevant to our analysis, see (Jackson, 2008, Chapter 4) and Newman (2010).

that G_{ij}=1 if organization i has a claim on organization j. To make it into a cross-holdings matrix, we posit that a fraction c of each organization is held by other organizations, spread evenly among the $d_i = \sum_j G_{ji}$ organizations that hold it. We call d_i the *out-degree* of i and analogously define *in-degree* by $d_i^{\text{in}} = \sum_i G_{ij}$ to be the number of organizations that i holds.[45]

Thus, for $i \neq j$

$$C_{ij} = \frac{cG_{ij}}{d_j}.$$

The remaining $1 - c$ of the organization is held by its external shareholders, so that $\widehat{C}_{ii} = 1 - c$.

Holding c fixed, as the out-degree d_j increases, the number of organizations having cross-holdings in j increases, but each of those organizations has lower cross-holdings in j. Thus, in this model, increasing d_j increases diversification but not integration.

Holding the underlying graph $\mathbf{G}$ fixed, as c increases each organization has lower self-holdings but higher cross-holdings in the other organizations it already holds. Thus increasing c increases integration but not diversification. This is made precise in the following lemma shows how increased integration weakly increases A_{ij} for all i and all $j \neq i$ and strictly increases at least one off-diagonal entry of $\mathbf{A}$ in each column.

LEMMA **2.** Suppose that $C_{ij} = cG_{ij}/d_j$ for some adjacency matrix $\mathbf{G}$, with $0 < c \leq \frac{1}{2}$ and each $d_i \geq 1$.[46] Then A_{ii} is decreasing in c and A_{ij} is increasing in c:

1. $\frac{\partial A_{ii}}{\partial c} < 0$ for each i;

2. $\frac{\partial A_{ij}}{\partial c} \geq 0$ for all $i \neq j$;

3. $\frac{\partial A_{ij}}{\partial c} > 0$ for all $i \neq j$ so that there is a path[47] from j to i in $\mathbf{G}$.

Next we introduce the random network model. Fix a *degree distribution* $\boldsymbol{\pi} = (\pi_{ij})$, where π_{ij} is the fraction of nodes that have in-degree i and out-degree j and the integer indices satisfy $0 \leq i, j \leq n-1$. Let $\mathcal{G}(\boldsymbol{\pi}, n)$ be the set of all directed graphs on n that have degree distribution $\boldsymbol{\pi}$. We say $\boldsymbol{\pi}$ is *feasible for n* when $\mathcal{G}(\boldsymbol{\pi}, n)$ is nonempty.[48] A *random network with degree distribution* $\boldsymbol{\pi}$ is a draw from $\mathcal{G}(\boldsymbol{\pi}, n)$ uniformly at random.

[45]Note that these terms are intuitive when viewed from the perspective of value flow: out-degree corresponds how many organizations receive the value that flows out from i by directly holding it. In-degree describes the number of organizations that i holds, and that therefore send value to i.

[46]Note that Lemma 2 does not impose any assumptions on the underlying graph $\mathbf{G}$ other than each organization being cross-held by at least one other. Interestingly, the monotonicity identified in Lemma 2 does not always hold for $c > 1/2$. For such c, there are graph structures where further increases in c result in the immediate neighbors of i depending less on i. The increase in A_{ij} for non-neighbors of i can come at the expense of both A_{ii} and A_{ij} for j such that $C_{ij} > 0$.

[47]Recall footnote 11.

[48]For $\mathcal{G}(\boldsymbol{\pi}, n)$ to be a nonempty set, some basic relations have to be satisfied by $\boldsymbol{\pi}$: (i) $n\pi_{ij}$ is always a (nonnegative) integer, since it must be a number of nodes; (ii) $\sum_{ij} i\pi_{ij} = \sum_{ij} j\pi_{ij}$, since each is equal to the number of directed edges in the graph divided by n.

For a given $\boldsymbol{\pi}$, we denote by $\overline{d} = \max\{i : \pi_{ij} > 0 \text{ or } \pi_{ji} > 0 \text{ for some } j\}$ the *maximum degree* of the network and by $\underline{d} = \min\{i : \pi_{ij} > 0 \text{ or } \pi_{ji} > 0 \text{ for some } j\}$ the *minimum degree.* Finally, we define the *average directed degree* d to be the expected out-degree of the vertex at the end of a link chosen uniformly at random from $\mathcal{G}(\boldsymbol{\pi}, n)$.[49] This is a basic measure of average diversification in the graph that overweights organizations held by many others, and turns out to be the right one for our purposes. Together, the three parameters $\underline{d}$, $\overline{d}$, and d operationalize the notion diversification in this random network model.

Each organization has a single asset of value 1 (so $\mathbf{D} = \mathbf{I}$ and $\mathbf{p} = (1, \ldots, 1)$). We set all organizations' thresholds $\underline{v}_i$ to a common $\underline{v} \in (0, 1)$, and set $\beta_i = p_i$, so that a failing organization has its proprietary asset completely devalued.

Define $\widetilde{v}_{\min} = \frac{1-c}{1-c\underline{d}/\overline{d}}$ and $\widetilde{v}_{\max} = \frac{1-c}{1-c\overline{d}/\max\{\underline{d},1\}}$.[50]

How does the degree distribution, $\boldsymbol{\pi}$, affect the extent of cascades? Let $\mathbf{G}$ be a random draw of a network with n nodes and degree distribution $\boldsymbol{\pi}$. Let $f(\boldsymbol{\pi}, n)$ be the expected fraction of organizations that fail if the network is given by $\mathbf{G}$ and one proprietary asset value p_i is devalued to 0, with i selected uniformly at random.

Proposition **3.**

If one proprietary asset fails (uniformly at random), a non-vanishing fraction of organizations fail if and only if there are intermediate levels of both integration and diversification.

In particular, consider a degree distribution $\boldsymbol{\pi}$ with associated average directed degree d, maximum degree $\overline{d}$, and minimum degree $\underline{d}$; and let (n_k) be an infinite sequence of natural numbers such that $\boldsymbol{\pi}$ is feasible for each n_k.

1. The fraction of failures tends to 0 ($f(\boldsymbol{\pi}, n_k) \to 0$) if *either* of the following conditions hold:

   (ii) $d < 1$ (diversification is too low), or

   (i) $\underline{d} > \frac{c(1-c)}{\widetilde{v}_{\min} - \underline{v}}$ (diversification is too high, or integration is too high or low).

2. The fraction of failures is nonvanishing ($\liminf_k f(\boldsymbol{\pi}, n_k) > 0$) if *both* of the following conditions hold:

   (i) $d > 1$ (diversification is not too low), and

   (ii) $\overline{d} < \frac{c(1-c)}{\widetilde{v}_{\max} - \underline{v}}$ (diversification is not too high and integration is intermediate).

Proposition 3 documents a non-monotonicity of failures in diversification and integration. Part (1) shows that if either integration or diversification is extreme (low or high), then there can be no substantial contagion: 1(i) is satisfied if diversification is too low, and 1(ii)

[49]This depends only on $\boldsymbol{\pi}$. To see this, let ϕ_i be the probability that a node of out-degree j is found by following a randomly chosen edge; we can see that $\phi_j = \sum_i i\pi_{ij} / \sum_{j,i} i\pi_{ij}$. Now note that $d = \sum j\phi_j$.

[50]These serve as lower and upper bounds, respectively, on organization values, as verified in the proof of Proposition 3.

is satisfied when diversification is high[51] or when integration is high or low (c is close to 0 or 1). In other words, contagion can occur *only* if both integration and diversification are intermediate. Part 2 then gives a sufficient condition: upper and lower bounds on the diversification parameters $\overline{d}$ and d, respectively,[52] specifying the intermediate range in which contagion occurs.[53]

The intuition for Proposition 3 is as follows. If c is very low, then no firm holds enough of its counterparties for contagion to propagate. If c is very high, then no firm is sufficiently exposed to its own asset for a first failure to happen. So consider the range where c is intermediate. For random graphs of the type we study here, once the average directed degree d crosses the threshold 1, the graph structure changes from many small isolated components of vanishing size to a giant component of non-vanishing size. It starts out small, but increases in size as d grows. Thus, if $d < 1$, contagion to a positive fraction of organizations following the failure of a single proprietary asset is impossible. At the other extreme, once $\underline{d} > \left\lceil \frac{c}{\widetilde{v}_{\min} - \underline{v}} \right\rceil$, a single organization's failure will not cause a sufficient decrease in the value of any other organization to induce a second failure. When integration and diversification are intermediate, so that none of these obstructions to contagion occur, part (2) of the proposition states that a (nonvanishing) fraction of organizations fail.

The reasoning above makes use of properties of large networks. Regardless of the parameter values, when there are only a small number of organizations, networks with intermediate connectedness are realized with non-trivial probability. Thus, in settings with very few critical organizations, one has to rely on direct calculations (e.g., see the core-periphery analysis in Section 5.1).

4.2 The Different Roles of Diversification and Integration: Simulations on Random Networks

We now show that the analytic results of the previous section hold in other classes of simulated random networks. We also derive some richer insights into comparative statics in various levels of diversification and integration.

4.2.1 Simulated Random Networks

To illustrate how increased diversification and increased integration affect the number of organizations that fail in a cascade following the failure of a single organization's assets, we specialize the model.

Each organization has exactly one proprietary asset, so that $m = n$ and $\mathbf{D} = \mathbf{I}$. This keeps the analysis uncluttered, and allows us to focus on the network of cross-holdings.

[51] Note that as $c(1-c) < 1/4$ for all $c \in (0, 1)$, 1(ii) is always satisfied for all $\underline{d} > 1/(4(\widetilde{v}_{\min} - \underline{v}))$

[52] Fixing a ratio $\overline{d}/\underline{d} < 1/c$, the right-hand side of 2(ii) is constant in $\overline{d}$; in this sense 2(ii) is a true upper bound on $\overline{d}$.

[53] Observe that when the graph is regular, so that $\overline{d} = d = \underline{d}$, then $\widetilde{v}_{\max}$ and $\widetilde{v}_{\min}$ become identical and the result becomes fully tight, with no distance between the necessary and the sufficient condition for contagion.

For simplicity, we also start with asset values of $p_i = 1$ for all organizations, and have common failure thresholds $\underline{v}_i = \theta v_i$, for a parameter $\theta \in (0, 1)$, where v_i is the starting value of organization i when all assets are at value 1. In case an organization fails it loses its full value, so that $\beta_i = \underline{v}_i$.

The cross-holdings are derived from an adjacency matrix $\mathbf{G}$ with entries in $\{0, 1\}$, where $G_{ij} = 1$ indicates that i has cross-holdings in j and we set $G_{ii} = 0$.

Again, a fraction c of each organization is held by other organizations, spread evenly among the $d_i = \sum_j G_{ji}$ organizations that hold it as in (4.1.2) The remaining $1 - c$ of the organization is held by its external shareholders, so that $\widehat{C}_{ii} = 1 - c$.

To illustrate the effects of increasing diversification and increasing integration on cascades we examine a setting where connections between organizations are formed at random, with each organization having cross-holdings in a random set of other organizations.

In particular, we form a directed random graph, with each directed link having probability $d/(n-1)$, so that the expected indegree and outdegree of any node is d. More precisely, the adjacency matrix of the graph is a matrix $\mathbf{G}$ (usually not symmetric), where G_{ij} for $i \neq j$ are i.i.d. Bernoulli random variables each taking value 1 with probability $d/(n-1)$ and 0 otherwise.

To examine the effects of increasing diversification (increasing d) and increasing integration (increasing c), we simulate an organization's proprietary asset failing and record the number of organizations that fail in the resulting cascade.

We follow a simple algorithm:

Step 1. Generate a directed random network $\mathbf{G}$ with parameter d as described above.

Step 2. Calculate the matrix $\mathbf{C}$ from $\mathbf{G}$ according to (4.1.2), where $\widehat{C}_{ii} = 0.5$.

Step 3. All organizations start with asset values of $p_i = 1$. Calculate organizations' initial values v_i and set $\underline{v}_i = \theta v_i$ for some $\theta \in (0, 1)$.

Step 4. Pick an organization i uniformly at random and drop the value (p_i) of i's proprietary asset to 0.[54]

Step 5. Assuming all other asset values (p_j for $j \neq i$) stay at 1, calculate the best equilibrium using the algorithm from Section 3.2.3.

The main outcome variable we track is the number of failures in the best-case equilibrium.

4.2.2 The Consequences of Diversification: It Gets Worse Before it Gets Better

For our simulations, we consider $n = 100$ nodes and work with a grid on expected degree d between 1 and 20 (varying it increments of 1/3). We work with values of $\theta \in [0.8, 1]$.

[54]Thus, we are focusing on a case where an organization's proprietary project is shut down upon failure. While clearly not the only case of interest, it is a common one in some bankruptcies.

Our first exercise is to vary the level of diversification (the expected degree d in the network) while holding other variables fixed and to see how the number of organizations (out of 100) that fail varies with the diversification.

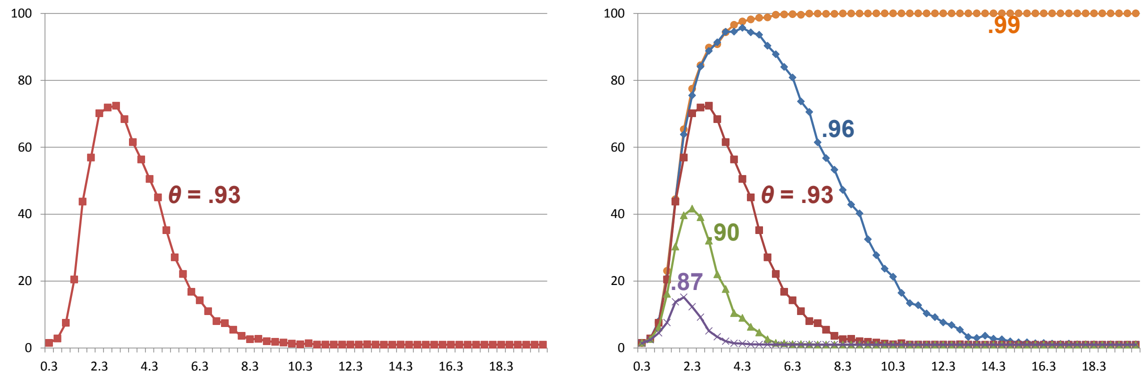
Figures 2a and 2b illustrate how the proportion of organizations that fail changes as the level of diversification (d) is varied (fixing integration at $c = 0.5$).

Figure 2a shows the result for a level of the failure threshold ($\theta = 0.93$) for which the curves display their typical nonmontonicities clearly. When d is sufficiently low, 1.5 or below, then we see the percentage of organizations that fail is less than 20. At that level, the network is not connected; a typical organization has direct or indirect connections through cross-holdings to only a small fraction of others, and any contagion is typically limited to a small component. As d increases (in the range of 2 to 6 other organizations) then we see substantial cascades affecting large percentages of the organizations. In this middle range, the network of cross-holdings has two crucial properties: it is usually connected[55], and organizations still hold large enough cross-holdings in individual other organizations so that contagion can occur. This is the "sweet spot" where ingredients II and III are present and strong – contagion is possible and there is enough interconnection for a cascade to spread. As we continue to increase diversification, the extent of cascades is falls, as diversification is now lowering the chance that contagion occurs. In summary, there is constantly a tradeoff between II and III, but initially III dominates as diversification leads to dramatic changes in the connectedness of the network. Then II dominates: once the network is connected, the main limiting force is the extent to which the failure of one organization sparks failures in others, which is decreasing with diversification. These three regimes are illustrated in Figure 3.

Figure 2b shows how these effects vary with θ. Higher values of θ correspond to higher failure thresholds, and so it becomes easier to trigger contagions. This leads to increases in the curves for all levels of diversification. Essentially, increasing θ leads to a more fragile economy across the board.

The main results in Section 4.1 provide analytical support for the non-monotonicity due to diversification identified in the simulations and helps identify the forces behind the non-monotonicity. With low levels of diversification, contagions are difficult to start and will frequently die out before affecting many organizations. Condition III is not met, as the network of cross-holdings is not connected. Even if all organizations directly or independently dependent on the failing organization i (those j such that $A_{ij} > 0$) also fail in the cascade, there are sufficiently few such organizations that the cascade dies out quickly and is small. As we increase diversification into intermediate levels, we see an increase in the number of organizations that fail in a cascade. Since network components are larger, the failure of any one organization infects more other organizations, and more organizations are drawn into the cascade. However, as we continue to diversify cross-holdings, eventually the increased diversification leads to a decrease in exposure of any one organization to any other, and so the necessary condition II is not met as no organization depends very much on any other.

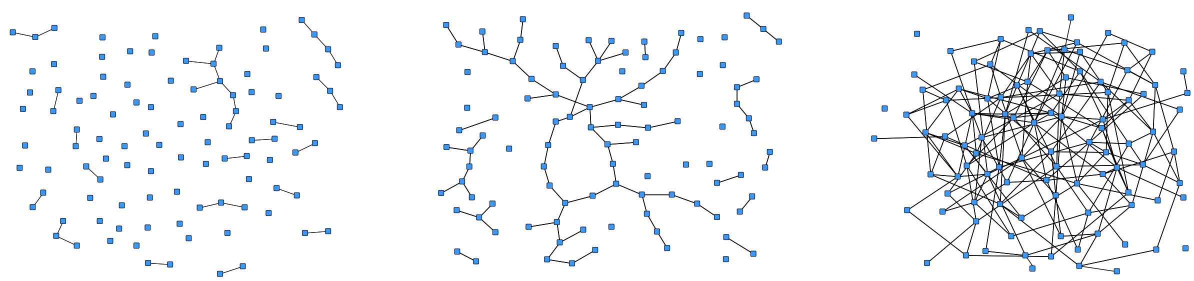
[55] That is, there is a path in **C** from any node to any other.



(a) Effects of diversification: the percentage of organizations failing as a function of expected degree for $\theta = 0.93$ ($c = 0.5$, $n = 100$).

(b) Effects of diversification for several failure thresholds: percentage of organizations failing as a function of expected degree for various levels of θ ($c = 0.5$, $n = 100$).

Figure 2: How diversification (the average number of other organizations that an organization cross-holds) affects the percentage of organizations failing, averaged over 1000 simulations. The horizontal axis corresponds to diversification in terms of the expected degree in the random network of cross-holdings.



(a) Low diversification

(b) Medium diversification

(c) High diversification

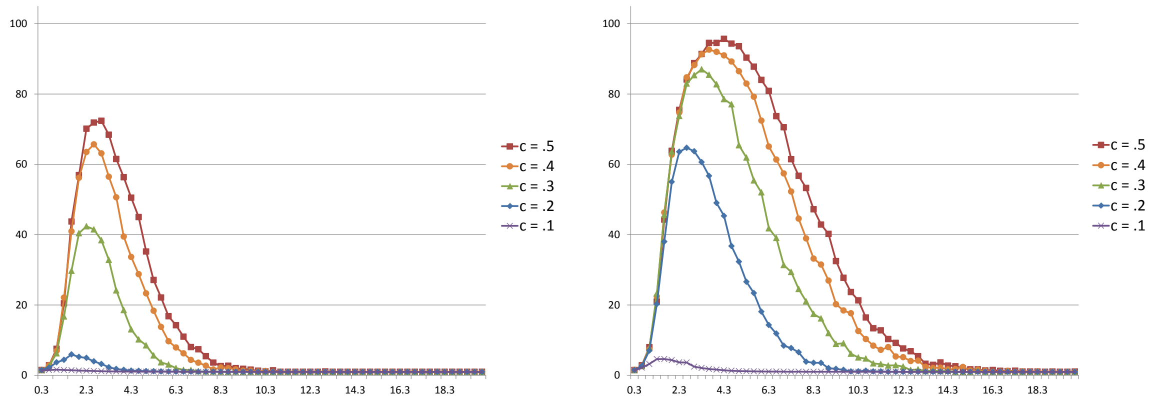
Figure 3: Example random networks (plotted here with undirected edges) for different levels of diversification. The transition from (a) many disconnected components to (b) a large component where each node has few neighbors to (c) a large component in which each node has many neighbors is clearly visible.

4.2.3 Cascades are Larger but Less Frequent in More Integrated Systems

Next, we consider the implications of increased integration in our simple model on the depth of cascades, as illustrated in Figure 4.

Figures 4a and 4b illustrate how the proportion of organizations that fail changes as the level of integration is varied from $c = 0.1$ to 0.5, for two different values of θ (the fraction of initial value that must be retained for an organization to avoid failure). As integration is increased the curves all shift upward and we see increased cascades.

Although the effects in Figures 4a and 4b show unambiguous increases in cascades as integration increases, they work with levels of $c \leq 0.5$ for which there is not so much of a tradeoff. In particular, for $c \leq 0.5$ the initial organization whose asset price is dropped



(a) Five levels of integration and the percentage of organizations failing as a function of expected degree ($\theta = 0.93$, $n = 100$).

(b) Five levels of integration and the percentage of organizations failing as a function of expected degree ($\theta = 0.96$, $n = 100$).

Figure 4: How integration (the fraction c of a typical portfolio held by other organizations) affects the percentage of organizations failing, averaged over 1000 simulations. The horizontal axis corresponds to the diversification level (the expected degree in the random network of cross-holdings). The two figures work with different failure thresholds and depict how the size of cascades varies with the level of integration c ranging from 0.1 to 0.5.

to 0 always fails (in the range of $\theta \geq 0.8$ considered in the simulations). As c is increased beyond 0.5, eventually the integration level begins to help avoid first failures, because each organization is less exposed to the failure of own proprietary asset. Then we see the tradeoff between I and II that is present as integration is varied (holding diversification constant, so III – having to do with the connectedness of the network – is not affected). We can see this in Figure 5.

Figure 5 shows that as integration increases to very high levels, the percentage of first failures drops: organizations are so integrated that the drop in the value of an organization's own investments is less consequential to it, and so there is no first failure.

To summarize, increasing integration (as long as it is not already very high) makes shocks more likely to propagate to neighbors in the financial network and increases contagion via the mechanism of II. For very high levels of integration, each organization begins to carry something close to the market portfolio, and so any first failure caused by the devaluation of a single proprietary asset becomes less likely.

5 Alternative Network Structures

Additional insights emerge from examining some other random graph models of financial interdependencies.

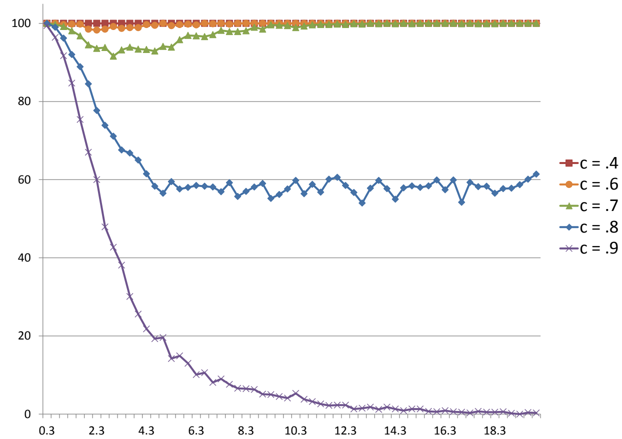


Figure 5: How integration affects the percentage of "first failures": the percentage of simulations with at least one organization failing, for various levels of integration c from 0.4 to 0.9, with the horizontal axis tracking diversification (expected degree) in the network. The failure threshold is constant at $\theta = 0.8$.

5.1 A Core-Periphery Model

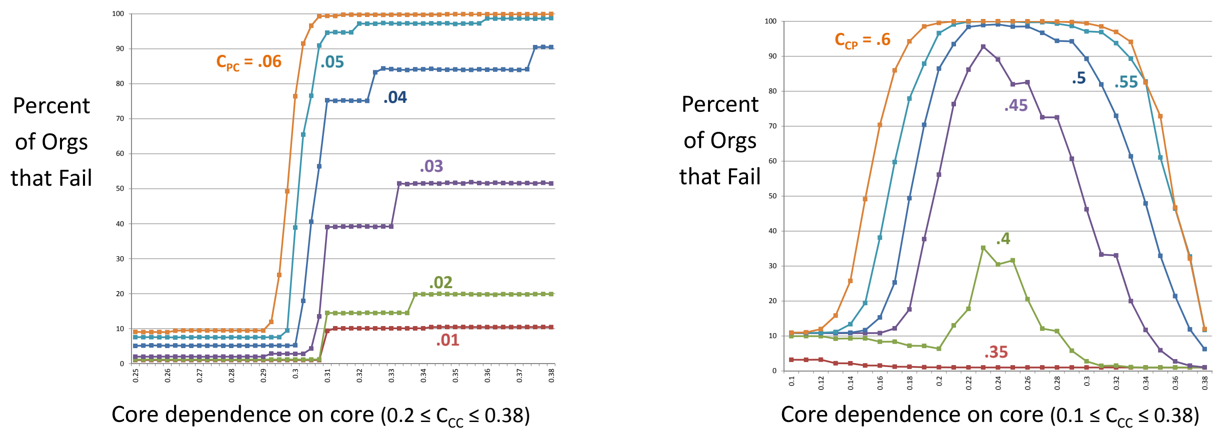
As a stylized representation of the interbank lending market, we examine a core-periphery model where 10 large organizations are completely connected among themselves, and each of 90 smaller organizations has one connection to a random core organization.[56] Each of the ten large core organizations has proprietary assets with an initial value of 8. Each of the 90 peripheral organizations has proprietary assets with an initial value of 1.

We then vary different facets of integration:[57] the level C_{CC} of cross-holdings of each core organization by other core organizations, the level C_{PC} of cross-holdings of each core organization by peripheral organizations, and the level C_{CP} of cross-holdings of each peripheral organization by core organizations. The remaining private holdings, $\widehat{C}_{ii}$, are as follows: $\widehat{C}_{ii} = 1 - C_{CC} - C_{PC}$ for a core organization, and $\widehat{C}_{ii} = 1 - C_{CP}$ for a peripheral one.

We first explore what happens when a core organization fails. As we see in the left-hand part of Figure 6a, the fraction of peripheral organizations that fail along with the core organization is increasing in C_{PC}. Once the core organizations become sufficiently integrated among themselves, starting around $C_{CC} = .29$, the core organization's failure begins to cascade to other core organizations, and then wider contagion occurs. How far this ultimately spreads is governed by the combination of integration levels.

[56] Soromaki et al. (2007) map the US interbank network based on the Fedpayments system. They identify a clique of 25 completely connected banks (including the very largest ones), and thousands of less connected peripheral regional and local banks.

[57] Note that in this model the diversification (degree) structure is essentially fixed given the structure of ten completely inter-connected organizations and the peripheral ones each having one connection; the only randomness comes from the random attachment of each peripheral organization to a single core organization.



(a) One core organization's asset initially fails (b) One peripheral organization's asset initially fails

Figure 6: The consequences of failure in the core-periphery model. The horizontal axis is the fraction of each core organization cross-held by other core organizations (integration of core to core). In Figure 6a, curves correspond to different levels of cross-holdings of each core organization by peripheral organizations. In Figure 6b, they correspond to different levels of cross-holdings of peripheral organizations by core ones. The failure threshold is $\theta = .98$.

The more subtle effects are seen in in Figure 6b. The curves are layered in terms of integration between the core and periphery C_{PC}, with increased integration leading to higher failure rates due to an initial failure of a peripheral organization. However, the magnitude of the failure rates is initially increasing in core integration ($C_{CC} < .25$) and then decreasing in core integration ($C_{CC} > .25$). Initial increases in core-integration enable contagion from one core organization to another, which leads to widespread cascades. Once core integration becomes high enough, however, core organizations become less exposed to their own peripheral organizations, and so then are less prone to fail because of the failure of a peripheral organization.

5.2 A Model with Segregation among Sectors

Second, we considered a model that admits segregation (homophily) among different segments of an economy: for instance among different countries, industries, or sectors. In this model, there are ten different groups of ten nodes each. The key feature being varied is the relative intensity of nodes' connections with others in their own group compared to other groups. This captures the difference between integration across industries and integration within industries. Varying this difference leads to the results captured in Figure 7. An obvious effect is that increasing homophily can eventually sever connections between groups of organizations and lead to lower contagion. However, as we see in Figure 7, the curves associated with different levels of diversification (expected degrees d) cross each other. With medium diversification (e.g., $d = 3$ or $d = 5$) there is initially a higher level of contagion than with higher diversification (e.g., $d = 7$ or $d = 9$). This is because organizations are more susceptible to each other with medium degrees than with high degrees and the network is still connected enough to permit widespread contagion. However, lower-degree networks fragment at lower levels of homophily than high degree networks. So at high levels of homophily, lower-degree networks are actually more robust. For example, once at least 95 percent of relationships are within own group (in expectation), then we see lower contagion rates with diversifications $d = 3, 5$ than with $d = 7, 9$.

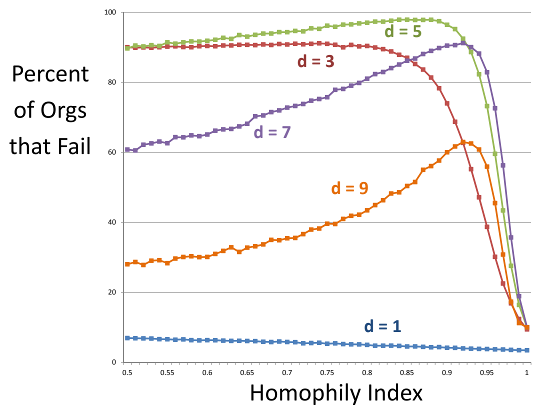


Figure 7: Ten groups of ten organizations each. The vertical axis is the fraction of organizations that fail as a function of the homophily. The horizontal axis is the fraction of expected cross-holdings in same-type organizations. Curves correspond to different diversification levels (expected degrees d). The failure threshold is $\theta = .96$.

5.3 Power Law Distributions

We also examined networks with more extreme degree distributions, such as a power-law distribution. Those results are described in detail in Section A.4.1 in the Online Appendix and are in line with the original regular networks. More extreme exponents in the power law actually lead to smaller contagions on average, but larger contagions conditional on some high-degree organization's failure.

5.4 Correlated and Common Assets

An important concern that emerged from the recent financial crisis is that many organizations may have investments with correlated payoffs, which could potentially exacerbate contagions, as many organizations' values may be low at the same time. In Sections A.4.2 and A.4.3 we examine two variations with correlated values. As one might expect, increasing correlation increases the failure rate. The more interesting part is that the increase occurs abruptly at a particular level of correlation.

We also examine a model in which organizations have some holdings of both an idiosyncratic and a common asset, with the possibility of leverage in holdings of the common asset. Some organizations are long the asset and others can be short. This results in some interesting patterns in cascades: even low leverage levels can lead to increased cascades by increasing organizations' exposures. However, organizations that are short the common asset might escape a cascade triggered by a shock to that asset.

6 Illustration with European Debt Cross-Holdings

We close the paper with an illustration of the model with data on the cross-holdings of debt among six European countries (France, Germany, Greece, Italy, Portugal and Spain). We include this as a proof of concept, and emphasize that the crude estimates that we use for cross-holdings make this noisy enough that we do not see the conclusions as robust, but merely as illustrative of the methodology.[58]

We take the fundamental asset owned by each country to be its fiscal stream; by exchanging cross-holdings, countries acquire holdings whose value depends on the value of others' fiscal streams as well as on their own. We model failure as being triggered by a certain percentage loss in the value of a country's aggregate holdings. In the simulations, when a country "fails," it defaults on 50% of its obligations to foreign countries – an arbitrary choice, but not unfounded, as we see from the writedown of Greek debt. Such losses may arise for various reasons: discontinuous changes in government policies of how to make use of fiscal streams; government decisions not to honor obligations (at which point it makes sense to do so discontinuously); discontinuities in the fiscal streams themselves (due to strikes, discontinuous changes in foreign investments, bank runs, and so forth). Indeed, all of these phenomena were observed in the recent Greek crisis. Finally, for the purposes of this illustrative exercise, we treat these countries as a closed system with no holdings by other countries outside of these six.

6.1 The Data

Data on the cross-holdings are for the end of December 2011 from the BIS (Bank for International Settlements) Quarterly Review (Table 9B). The data used for this exercise are the consolidated foreign claims of banks from one country on debt obligations of another

[58]See Upper (2011) for a nice review of the empirical literature simulating the effects of shocks to financial systems. Explicit losses due to bankruptcy are not usually considered in this literature, but an important exception is Elsinger et al. (2006), who find that these costs can make a large difference to the extent of contagion in simulation analysis. Our approach is well-suited to developing a deeper analysis of the propagation of discontinuities, as we examine the various levels of a cascade – which failures cause which others. This is illustrated in this section.

country. The data looks at the immediate borrower rather than the final borrower[59] when a bank from a country different from the final borrower serves as an intermediary.[60]

This gives following *raw* cross-holdings matrix, where the *column* represents the country whose debt is being held and the row is the country which holds that debt. So, for example, through their banking sectors Italy owes France \$329,550M, while France only owes Italy \$40,311M.

$$\begin{pmatrix} & \text{(France)} & \text{(Germany)} & \text{(Greece)} & \text{(Italy)} & \text{(Portugal)} & \text{(Spain)} \\ \text{(France)} & 0 & 174862 & 1960 & 40311 & 6679 & 27015 \\ \text{(Germany)} & 198304 & 0 & 2663 & 227813 & 2271 & 54178 \\ \text{(Greece)} & 39458 & 32977 & 0 & 2302 & 8077 & 1001 \\ \text{(Italy)} & 329550 & 133954 & 444 & 0 & 2108 & 29938 \\ \text{(Portugal)} & 21817 & 30208 & 51 & 3188 & 0 & 78005 \\ \text{(Spain)} & 115162 & 146096 & 292 & 26939 & 21620 & 0 \end{pmatrix}.$$

To convert the above matrix into our fractional cross-holdings matrix, $\mathbf{C}$, we then estimate the total amount of debt issued by each country. To do this, we estimate the ratio of foreign to domestic holdings by 1/3, in line with estimates of by Reinhart and Rogoff (2011). Then, the formula $\mathbf{A} = \widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1}$ implies:

$$\mathbf{A} = \begin{pmatrix} & \text{(France)} & \text{(Germany)} & \text{(Greece)} & \text{(Italy)} & \text{(Portugal)} & \text{(Spain)} \\ \text{(France)} & 0.71 & 0.13 & 0.13 & 0.17 & 0.07 & 0.11 \\ \text{(Germany)} & 0.18 & 0.72 & 0.12 & 0.11 & 0.09 & 0.14 \\ \text{(Greece)} & 0.00 & 0.00 & 0.67 & 0.00 & 0.00 & 0.00 \\ \text{(Italy)} & 0.07 & 0.12 & 0.03 & 0.70 & 0.03 & 0.05 \\ \text{(Portugal)} & 0.01 & 0.00 & 0.02 & 0.00 & 0.67 & 0.02 \\ \text{(Spain)} & 0.03 & 0.03 & 0.02 & 0.02 & 0.14 & 0.68 \end{pmatrix}.$$

The matrix $\mathbf{A}$ can be pictured as a weighted directed graph, as in Figure 8. The arrows show the way in which decreases in value flow from country to country. For example, the arrow from Greece to France represents the value of France's claims on Greek assets, and thus how much France is harmed when Greek debt loses value. The areas of the ovals represent the value of each country's direct holdings of primitive assets. All dependencies of less than 5% have been excluded from Figure 8 (but appear in the table above).

We treat the investments in primitive assets as if each country holds its own fiscal stream,

[59] Which basis is appropriate is discussed in section A.10 of the Online Appendix.

[60] For illustrative purposes, we examine holdings at a country level, so that all holdings of Italian debt by banks or other investors in France are treated as being held by the entity "France," and we suppose that substantial losses by banks and investors in France would lead to a French default on national debt. It would be more accurate to disaggregate and build a network of all organizations and investors, if such data were available.

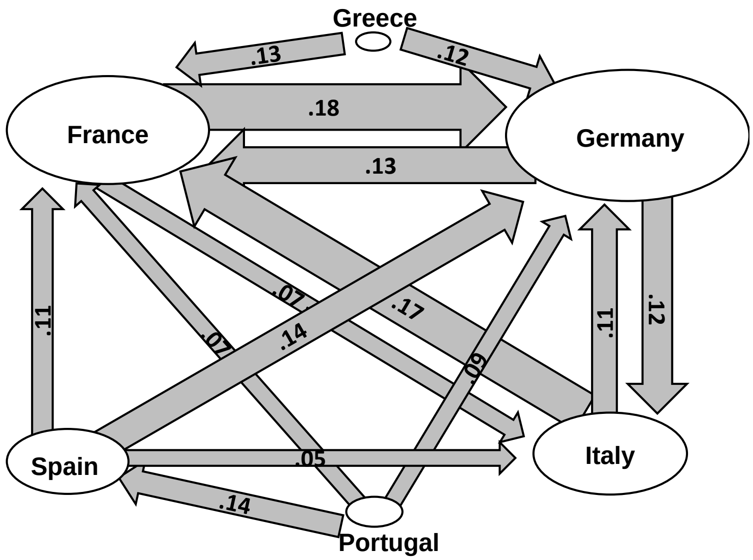


Figure 8: Interdependencies in Europe: The matrix $\mathbf{A}$, describing how much each country ultimately depends on the value of others' debt. The widths of the arrows are proportional to the sizes of the dependencies with dependencies less than 5% excluded; the area of the oval for each country is proportional to its underlying asset values.

which is used to pay for the debt, and presume that the values of these fiscal streams are proportional to GDP. Thus, $\mathbf{D} = \mathbf{I}$ and $\mathbf{p}$ is proportional to the vector of countries' GDPs.[61] Normalizing Portugal's GDP to 1, the initial values in 2011 are:

$$\mathbf{v}_0 = \mathbf{A}\mathbf{p} = \begin{pmatrix} 0.71 & 0.13 & 0.13 & 0.17 & 0.07 & 0.11 \\ 0.18 & 0.72 & 0.12 & 0.11 & 0.09 & 0.14 \\ 0.00 & 0.00 & 0.67 & 0.00 & 0.00 & 0.00 \\ 0.07 & 0.12 & 0.03 & 0.70 & 0.03 & 0.05 \\ 0.01 & 0.00 & 0.02 & 0.00 & 0.67 & 0.02 \\ 0.03 & 0.03 & 0.02 & 0.02 & 0.14 & 0.68 \end{pmatrix} \cdot \begin{pmatrix} 11.6 \\ 14.9 \\ 1.3 \\ 9.2 \\ 1.0 \\ 6.3 \end{pmatrix} = \begin{pmatrix} 12.7 & \text{(France)} \\ 14.9 & \text{(Germany)} \\ 0.8 & \text{(Greece)} \\ 9.4 & \text{(Italy)} \\ 0.9 & \text{(Portugal)} \\ 7.1 & \text{(Spain)} \end{pmatrix}.$$

6.2 Cascades

To illustrate the methodology, we consider a simple scenario. The failure thresholds are set to θ multiplied by 2008 values.[62] If a country fails, then the loss in value is $\underline{v}_i/2$, so that half the value of its debt is lost.

[61] We work in the scale of GDPs – that is, we do not carry around an explicit constant of proportionality relating the value of the fiscal streams $\mathbf{p}$ to the value of GDP; we simply take the entries of the vector $\mathbf{p}$ to be the GDP values.

[62] Those values are calculated in the same way as the values above, being proportional to 2008 GDP values instead of 2011 and again normalized by setting Portugal's 2011 GDP to 1.

We examine the best equilibrium values for various levels of θ. Greece's value has already fallen by well more than ten percent, and so it has hit its failure point for all of the values of θ. We then raise θ to various values and see which cascades occur.

Value of θ	.9	.93	.935	.94
First Failure	Greece	Greece	Greece	Greece, Portugal
Second Failure			Portugal	Spain
Third Failure			Spain	France
Fourth Failure			France, Germany	Germany, Italy
Fifth Failure			Italy	

Table 1: Hierarchies of Cascades in the Best Equilibrium Algorithm, as a Function of the Failure Threshold θ.

We see that Portugal is the first failure to be triggered by a contagion. Although it is not particularly exposed to Greek debt directly, the fact that its GDP has dropped substantially means that it is triggered once we get to $\theta = .935$. Once Portugal fails, then Spain fails due to its poor initial value and its exposure to Portugal. Then the large size of Spain, and the exposure of France and Germany to Spain cause them to fail. Pushing θ up to .94 causes Portugal to fail directly, and then leads to a similar sequence. (Increasing θ further would not change the ordering; it would just cause some countries to fail at earlier waves.) Interestingly, Italy is the last in each case: this is due to its low exposure to others' debts. Its GDP is not particularly strong, but it does not hold much of the dent of the other countries, with the exceptions of France and Germany.

Clearly the above exercise is based on rough numbers, *ad hoc* estimates for the default thresholds, and a closed (six country) world. Nonetheless, it illustrates the simplicity of the approach and makes it clear that much more accurate simulations could be run with access to precise cross-holdings data, default costs and thresholds.[63]

We re-emphasize that the cascades are (hopefully!) off the equilibrium path, but that understanding the dependency matrix and the hierarchical structure of potential cascades can improve policy interventions.

7 Concluding Remarks

Based on a simple model of cross-holdings among organizations that allows discontinuities in values, we have examined cascades in financial networks. We have highlighted several important features. First, diversification and integration are usefully distinguished as they have different effects on financial contagions. Second, both diversification and integration entail tradeoffs in how they affect contagion. These tradeoffs result in nonmonotonic effects

[63]Of course, a linear cross-holdings structure is also an important simplification. A further refinement would involve modeling the holdings in greater detail, and solving for the ultimate dependencies of organizations on assets (analogous to computing the **A** matrix) in that more complicated world.

where middle ranges are the most dangerous with respect to cascades of failures. The tradeoffs can also be related to important realistic aspects of a network, such as its core-periphery and segregation structure.

A fully endogenous study of the network of cross-holdings and of asset holdings is a natural next step.[64] We illustrate some moral hazard issues in the Online Appendix (Section A.3): organizations can have incentives to affect both bankruptcy costs and thresholds in socially inefficient ways. These considerations suggest that endogenizing the basic structures of our model will be delicate and that a simple general equilibrium approach will not suffice. This presents interesting challenges for future research.

The approach we have outlined could be used to inform policy. For example, counterfactual scenarios can be run using the algorithm. To determine the marginal effect of saving a set of organizations, the failure costs of those organizations can be set to zero and the algorithm run with and without their failure costs. This identifies a new set of organizations to fail in a cascade conditional on the intervention. This set of organizations can be compared to the set of organizations that fail under other interventions, including doing nothing. It is important to note that the aforementioned exercise must be repeated for any set of underlying asset prices that are of interest. As underlying asset prices change the differences between organizations' values and their failure thresholds change. These changes may be highly correlated depending on the underlying asset holdings. When many organizations have similar exposures to underlying assets, they will be relatively close to their failure frontiers at the same time, and so the first (and subsequent) waves of failures may change drastically for fairly small changes in asset prices.

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[64] For some analyses of network formation in other financial settings, see Babus (2013), Ibragimov, Jaffee and Walden (2011), Cohen-Cole, Patacchini and Zenou (2012), and Baral (2012). These can cut in either direction, as firms have some incentives to protect themselves (e.g., Babus (2013)), but might also wish to take excessively risky investments since they do not internalize the costs of others' exposures.

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8 Appendix: Proofs

Proof of Lemma 1:

One representation of $\mathbf{A}$ is as the following infinite sum, known as the Neumann series:

$$\mathbf{A} = \widehat{\mathbf{C}} \sum_{p=0}^{\infty} \mathbf{C}^p = \widehat{\mathbf{C}} + \widehat{\mathbf{C}} \sum_{p=1}^{\infty} \mathbf{C}^p \tag{6}$$

It follows immediately that $A_{ii} \geq \widehat{C}_{ii}$ and that there is equality if and only if there are no cycles involving i. Part (2) can be proved by considering $\widehat{\mathbf{C}}$ and $\mathbf{C}$ such that $\widehat{C}_{ii} = \epsilon$ for all i and $C_{ij} = (1-\epsilon)/(n-1)$ for all i and all j. Taking $\epsilon \to 0$, we have $\widehat{C}_{ii} \to 0$ but $\mathbf{A}$ tends to the matrix with all entries equal to $1/n$. ∎

Proof of Proposition 1.

As any trade involving organization i must change composition of i's dependency on underlying assets, after any trade there must exists a price vector $\mathbf{p}''$ within an ϵ neighborhood of $\lambda\mathbf{p}$, such that $v_i(\mathbf{p}'', \mathbf{C}', \mathbf{D}'|\mathbf{Z} = \emptyset) \neq v_i(\mathbf{p}'', \mathbf{C}, \mathbf{D}|\mathbf{Z} = \emptyset) = \underline{v}_i$. For the Proposition to be false, it must then be that $v_i(\mathbf{p}'', \mathbf{C}', \mathbf{D}'|\mathbf{Z} = \emptyset) > v_i(\mathbf{p}'', \mathbf{C}, \mathbf{D}|\mathbf{Z} = \emptyset)$. Define price $\mathbf{p}'$ such that $\frac{1}{2}\mathbf{p}'' + \frac{1}{2}\mathbf{p}' = \lambda\mathbf{p}$. As $||\mathbf{p}' - \lambda\mathbf{p}||_1 = ||\mathbf{p}'' - \lambda\mathbf{p}||_1$ and $\mathbf{p}''$ is within an ϵ neighborhood of $\lambda\mathbf{p}$, $\mathbf{p}'$ is also within an ϵ neighborhood of $\lambda\mathbf{p}$.

By the linearity of organizations' values, absent any failure, and as the trade was fair

$$\begin{aligned}
\frac{1}{2}v_i(\mathbf{p}'', \mathbf{C}', \mathbf{D}'|\mathbf{Z} = \emptyset) + \frac{1}{2}v_i(\mathbf{p}', \mathbf{C}', \mathbf{D}'|\mathbf{Z} = \emptyset) &= v_i(\lambda\mathbf{p}, \mathbf{C}', \mathbf{D}'|\mathbf{Z} = \emptyset) \\
&= \underline{v}_i \\
&= v_i(\lambda\mathbf{p}, \mathbf{C}, \mathbf{D}|\mathbf{Z} = \emptyset) \\
&= \frac{1}{2}v_i(\mathbf{p}'', \mathbf{C}, \mathbf{D}|\mathbf{Z} = \emptyset) + \frac{1}{2}v_i(\mathbf{p}', \mathbf{C}, \mathbf{D}|\mathbf{Z} = \emptyset)
\end{aligned}$$

Thus as $v_i(\mathbf{p}'', \mathbf{C}', \mathbf{D}'|\mathbf{Z} = \emptyset) > v_i(\mathbf{p}'', \mathbf{C}, \mathbf{D}|\mathbf{Z} = \emptyset)$,

$$v_i(\mathbf{p}', \mathbf{C}', \mathbf{D}'|\mathbf{Z} = \emptyset) < \underline{v}_i < v_i(\mathbf{p}', \mathbf{C}, \mathbf{D}|\mathbf{Z} = \emptyset).$$

∎

Proof of Proposition 2.

Following the failures of organizations $\mathbf{Z}_{k-1}$, the value of organization i is:

$$v_i(\mathbf{Z}_{k-1}) = \sum_{j \notin \mathbf{Z}_{k-1}}^{n} A_{ij} D_{jk} p_k + \sum_{j \in \mathbf{Z}_{k-1}}^{n} A_{ij}(D_{jk}p_k - \beta_j) = v_i(\emptyset) - \sum_{j \in \mathbf{Z}_{k-1}}^{n} A_{ij}\beta_j.$$

As fair trades hold constant $v_i(\emptyset)$, this equation shows that the value of organization i given failures $\mathbf{Z}_{k-1}$ is weakly decreasing in A_{ij} for all $j \neq i$. Holding fixed the hierarchies in which all other organizations fail, after a weak increase in A_{ij} for all i and all $j \neq i$, if organization i failed in hierarchy k it will now fail (weakly) sooner in hierarchy $k' \leq k$ and if organization i did not fail in any hierarchy it might now fail in some hierarchy.

Moreover, as failures are complementary, if organization i fails strictly sooner in hierarchy k' weakly more organizations will be included in all subsequent failure sets $\mathbf{Z}_{k''}$, for all $k'' > k'$.

This is because more failure costs are summed over in the above equation when calculating a organization's value in each failure hierarchy. ∎

Proof of Lemma 2:

Let $\overline{\mathbf{C}} = \mathbf{G}\mathbf{d}^{-1}$ and note that by the Neumann series we may write

$$\mathbf{A} = (1-c)\sum_{t=0}^{\infty} c^t \overline{\mathbf{C}}^t$$

$$\frac{\partial \mathbf{A}}{\partial c} = (1-c)\sum_{t=1}^{\infty} tc^{t-1}\overline{\mathbf{C}}^t - \sum_{t=0}^{\infty} c^t\overline{\mathbf{C}}^t = -\mathbf{I} + \sum_{t=1}^{\infty}(t(1-c)-c)c^{t-1}\overline{\mathbf{C}}^t.$$

Since $c \leq \frac{1}{2}$, every term in the summation over t is nonnegative. Moreover, $c^{t-1}\overline{\mathbf{C}}^t$ has a strictly positive entry whenever there is a path of length t from i to j in $\overline{\mathbf{C}}$, or equivalently in $\mathbf{G}$. This shows claims 2 and 3 in the proposition. To verify claim 1, note that every column of $\mathbf{A}$ sums to 1. Claim 3 along with the assumption that every node in $\mathbf{G}$ has at least one neighbor shows that every column has an off-diagonal entry that strictly increases in c; and no off-diagonal entry decreases by claim 2. So the diagonal entry strictly decrease in c. ∎

Proof of Proposition 3:

We begin the proof with a simple lemma, proved in Section A.11 of the Online Appendix.

LEMMA **3.** The values $\widetilde{v}_{\max}$ and $\widetilde{v}_{\min}$ are upper and lower bounds, respectively, for the value of any organization.

We also introduce some terminology. Recall from Section 2.1 that if $C_{ji} > 0$ there is an edge *from i to j* – corresponding to value flowing from i to j. We adopt the same convention for $\mathbf{G}$: we say there is an edge from i to j if $G_{ji} = 1$, and define paths analogously – recall footnote 11. Fixing a graph $\mathbf{G}$ and a node i, the *fan-out* of i, denoted $R^+(i)$, is the set of nodes j such that there is a directed path from i to j in $\mathbf{G}$. These are the j's that have direct or indirect cross-holdings in i. Throughout, $\mathbf{G}$ is drawn uniformly at random from $\mathcal{G}(\boldsymbol{\pi}, n_k)$, with n_k left implicit.

If 1(i) in the proposition's statement holds ($d < 1$), then by Theorem 1 of Cooper and Frieze (2004), for any $\varepsilon > 0$ and large enough k, with probability at least $1-\varepsilon$ there are at no nodes having a fan-out larger than εn_k. Since only nodes in $R^+(i)$ can fail following the failure of i, this proves that for large enough k, we have$f(\boldsymbol{\pi}, n_k) \leq \varepsilon$.

Suppose 1(ii) in the proposition's statement holds. Fix $\varepsilon > 0$. Suppose that proprietary asset i (belonging to organization i) is the one that is randomly selected to fail. Take any j such that $G_{ji} > 0$. The amount by which the value of organization j falls is A_{ji}. By the Neumann series (equation 6), $A_{ji} \leq (1-c)c/\underline{d} + R_{ji}$, where $R_{ji} = (1-c)\left(\sum_{p=2}^{\infty}\mathbf{C}^p\right)_{ji}$ accounts for the value flowing along paths from i to j in $\mathbf{C}$ other than the edge from i to j with weight C_{ji} – i.e., paths of length 2 or longer. The following is proved in Section A.11:

LEMMA **4.** For any ε, if k is large enough, then with probability at least $1-\varepsilon$, simultaneously for all j such that $G_{ji} = 1$, we have $R_{ji} = (1-c)\left(\sum_{p=2}^{\infty}\mathbf{C}^p\right)_{ji} \leq \varepsilon$.

By 1(ii) in the proposition's statement, and Lemma 3, $(1-c)c/\underline{d} < \widetilde{v}_{\min} - \underline{v} \leq v_j - \underline{v}$. So, for small enough ε, a failure of i, which reduces j's value by at most $(1-c)c/\underline{d} + \varepsilon$, is not enough to cause the failure of any counterparty j, and so there is no contagion.

Now suppose 2(i) and 2(ii) hold, and again fix $\varepsilon > 0$. Let i be the index of the first asset to fail. By Theorems 2 and 3 of Cooper and Frieze (2004), because $d > 1$, with probability at least ε, the node i has fan-out of size at least εn_k, for small enough ε and large enough k. Suppose that organization j has holdings in organization i (i.e., $G_{ji} > 0$) and recall that if organization i fails (losing all remaining value, since $\beta_i = \underline{v}_i$), organization j's value will decrease by A_{ji}. By the Neumann series (equation 6) $A_{ji} \geq \frac{c(1-c)}{\overline{d}}$, deterministically.[65] Organization j will therefore fail, following the failure of organization i if:

$$v_i - \frac{c(1-c)}{\overline{d}} < \underline{v},$$

which is guaranteed by $\overline{d} < \frac{c(1-c)}{\widetilde{v}_{max} - \underline{v}}$. This argument applies again to all the neighbors of j once it fails; iterating this argument, we find that the whole set $R^+(i)$ fails. Thus, in the event (probability $\geq \varepsilon$) that node i has fan-out of size at least εn_k, at least εn_k nodes fail, which establishes that $f(\boldsymbol{\pi}, n_k) \geq \varepsilon^2$ for large enough k.

This completes the proof of the proposition. ∎

[65] This lower bound on A_{ji} can be found by considering only the direct effect of j's cross-holdings in i and not any further feedbacks.

A Online Appendix: Financial Networks and Contagion

A.1 More on Cross-Holdings Matrices and the Induced Dependencies

We expand the illustration of the model.

Recall our simple example from Section 2.7. There are two organizations, $i = 1, 2$, each of which has a 50% stake in the other organization. The associated cross-holdings matrix $\mathbf{C}$ and the dependency matrix $\mathbf{A}$ are as follows. (Recall that $\widehat{C}_{ii}$ is equal to 1 minus the sum of the entries in column i of $\mathbf{C}$.)

$$\mathbf{C} = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix} \qquad \widehat{\mathbf{C}} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \qquad \mathbf{A} = \widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

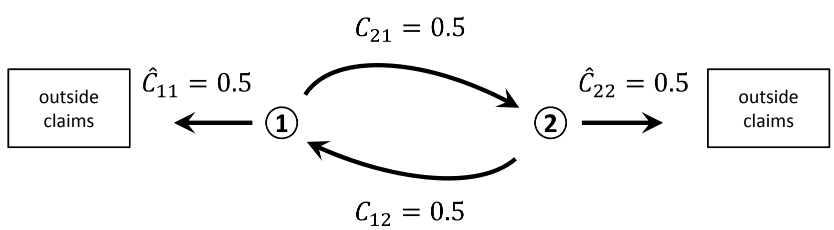


Figure 9: An illustration of cross-holdings in the two-organization example. The arrows indicate how a dollar of income arriving at one of the organizations is allocated between its direct holders and other organizations. Dollars that stay within the system are further split up. The $\mathbf{A}$ matrix describes how they are ultimately allocated.

A slightly richer example of potential differences between the cross-holdings and induced dependencies is as follows, with three organizations.

$$\mathbf{C} = \begin{pmatrix} 0 & 0.75 & 0.75 \\ 0.85 & 0 & 0.10 \\ 0.10 & 0.00 & 0 \end{pmatrix} \qquad \mathbf{A} = \widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1} = \begin{pmatrix} 0.18 & 0.13 & 0.15 \\ 0.77 & 0.83 & 0.66 \\ 0.05 & 0.04 & 0.19 \end{pmatrix}$$

The weighted graphs of the matrix $\mathbf{C} + \widehat{\mathbf{C}}$ and the associated $\mathbf{A}$ are shown in Figure 10, illustrating the substantial differences.

First, note that organization 1 is almost a holding company: it is mostly owned by other organizations, and so the second two entries of the corresponding row in $\mathbf{A}$ are much smaller than the corresponding entries in $\mathbf{C} + \widehat{\mathbf{C}}$.

(a) Weighted graph of $\mathbf{C} + \widehat{\mathbf{C}}$.

(b) Weighted graph of $\mathbf{A}$.

Figure 10: The widths of the edges are proportional to the sizes of cross-holdings; the arrows point in the direction of the flow of assets: from the organization that is held and to the holder. Outgoing edges in (a) reflect the private (final) shareholders' holdings. The cross-holdings and outside holdings measured by $\mathbf{C} + \widehat{\mathbf{C}}$ can be very different from the dependency matrix $\mathbf{A}$, which measures how each organization's market value ultimately depends on the assets held by each organization.

Also, we see that the outside shareholders of organization 2 have direct and indirect claims on 66% of organization 3's direct asset holdings, even though the organization has only 10% of the shares of organization 3 directly in cross-holdings. Intuitively, as organization 2 directly owns 85% of organization 1, its outside shareholders indirectly have claims to organization 1's large direct stakes in both organization 2 and organization 3.

A.2 Debt and Other Liabilities

Throughout the paper we suppose that organizations' values depend linearly on the organizations they have holdings in, with positive slope coefficients. Debt contracts do not have this form, provided organizations can meet the face values of their obligations. But, as we emphasize in Section 2.5, the center of our analysis is on situations in which organizations cannot meet the face values of their obligations and must ration their counterparties. In this region, our linear cross dependencies approximate cross-holding of debt.[66]

Also, we emphasize that the discontinuous failure costs need not be triggered at the point where the value of an organization is below the face value of debt. There can be some regime of orderly write-down until a threshold where there is a disruption in the ability for the organization to operate, below which its value is reduced discontinuously, entering a

[66] It is not essential that debt all organizations be in the linear regime. If there are organizations that are "safe" and are able to pay the face value of their debts, one can model claims on them as claims on just another fundamental asset. Claims that a "safe" organization i has on an organization j in the write-downs regime can be viewed as j's obligation to an outside shareholder. In other words, since reductions in value do not feed through safe organizations, they can be treated as exogenous.

regime of disorderly default. This is illustrated in Figure 11.

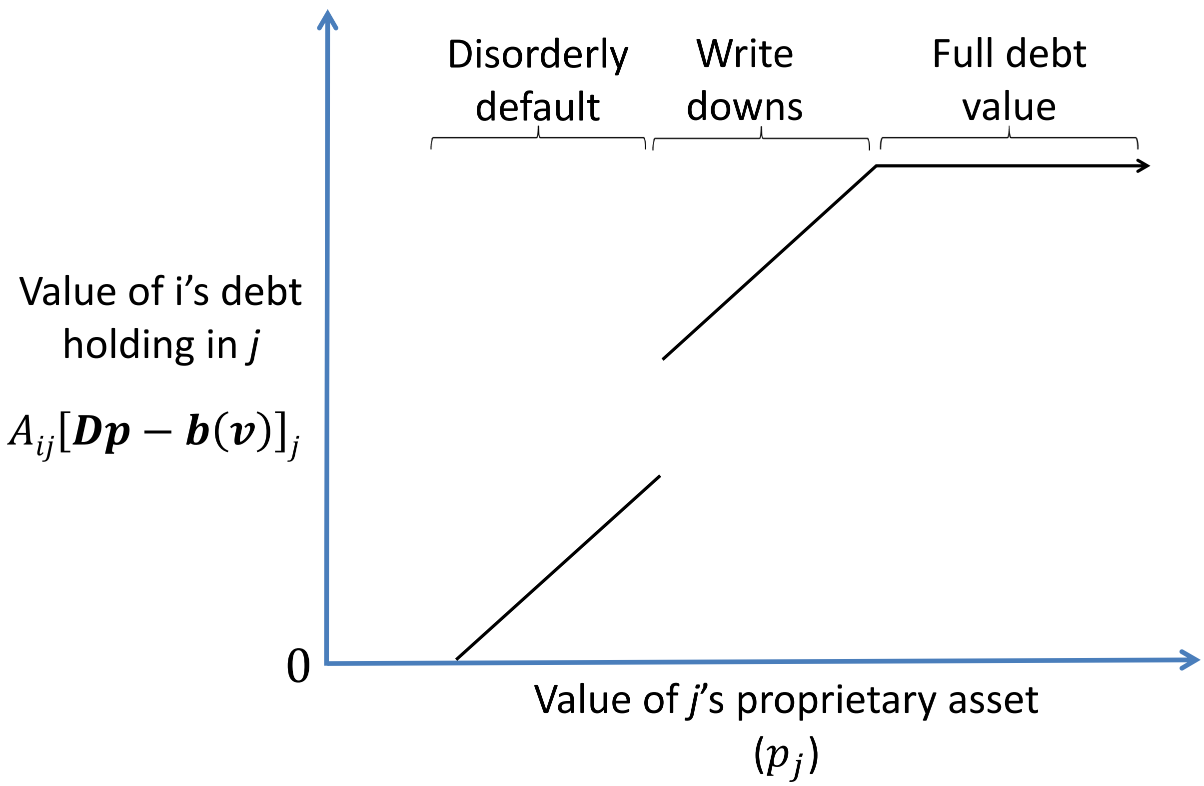


Figure 11: As the value of organization j's proprietary asset p_j decreases, a first threshold is reached at which the organization cannot meets its liabilities. We focus on values of p_j below this threshold. After the threshold comes a region of orderly default in which debt holders accept write-downs on the value of their debt. As remaining value is rationed i's value decreases linearly in p_j until a second threshold is crossed which we refer to as j's failure threshold. This can be interpreted as the point at which j's assets are liquidated and ongoing operations cease. The resulting failure costs cause a discontinuous decrease in the value of debt holdings in j.

More generally, the model is easily adapted to other sorts of cross-liabilities in addition to the linear cross-holdings. These could reflect any sort of debt or other contractual agreement, which could be contingent on the market value of the organizations (for instance, the debt cannot exceed the organization's market value if there is limited liability). If we let $L_{ji}(\mathbf{V})$ be the amount owed to j by i as a function of book value, and $\mathbf{L}$ the corresponding matrix (with 0's on its diagonal, as an organization cannot have debt to itself) book values become:

$$V_i = \sum_{j \neq i} C_{ij} V_j + \sum_{j \neq i} (L_{ij}(\mathbf{V}) - L_{ji}(\mathbf{V})) + \sum_k D_{ik} p_k - \beta_i I_{v_i < \underline{v}_i}.$$

This leads to book values of

$$\mathbf{V} = (\mathbf{I} - \mathbf{C})^{-1} (\mathbf{Dp} - (\mathbf{L}(\mathbf{V}) - \mathbf{L}^T(\mathbf{V}))\mathbf{1} - b(\mathbf{v})). \tag{7}$$

where a superscript T indicates transpose, and correspondingly market values are then

$$\mathbf{v} = \widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1}(\mathbf{Dp} - (\mathbf{L}(\mathbf{V}) - \mathbf{L}^T(\mathbf{V}))\mathbf{1} - b(\mathbf{v})). \tag{8}$$

A.3 Endogenously High Failure Costs and Thresholds due to Moral Hazard

Whether an organization fails depends on its failure threshold. The impact that its failure has on other organizations depends on its failure costs. If organizations have some control over their failure thresholds and costs, then we might hope that they would choose to limit these. We show in this section that organizations can actually have incentives to increase both their failure costs and thresholds.

A.3.1 Organization Values Can Be Endogenous

Our previous analysis has assumed that exchanges of cross-holdings or assets between organizations occur through fair trades at the current asset prices (recall Section 4.1). That was useful for illustrating the workings of the model and identifying the general effects of diversification and integration. However, the value to an organization of a trade depends not only on the value of the bundle of assets being received, but also on the implications of the trade for ensuing failures. Solvent or liquid organizations may have incentives to bail out insolvent or illiquid ones in order to avert a contagion (as pointed out, e.g., by Leitner 2005).[67] For instance, it can be that by relinquishing some holdings (in either assets or another organization) an organization's value actually increases! This means that we cannot value organizations solely based on their implied underlying asset holdings, but need also to consider the solvency of all other organizations. Trades can be "incentive compatible" when they are not "fair" (as evaluated by pricing the traded assets at the prices $\mathbf{p}$ and neglecting failure costs).

We first illustrate the endogeneity of values through a simple example, and then explore the associated moral hazard issues.

A.3.2 An Example

Consider a world with two assets and two organizations. We begin with a case where asset holdings are $D_1 = (1, 0)$, $D_2 = (0, 1)$. Initial cross-holdings are $C_1(0) = (0, 1/2)$ and $C_2(0) = (1/2, 0)$, such that each organization has a one half stake in the other (so $\widehat{C}_{ii} = 1/2$).

[67] Leitner 2005 argues that incentives for interconnected organizations to bail one another out can help them provide insurance to each other when they otherwise would not be able to commit to doing so, and that this provides an efficiency benefit from financial interconnections that can be traded off against increased systemic risk.

From equation (5) it is easily verified that the organizations' indirect holdings of the underlying assets are given by

$$\mathbf{A} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

With the initial cross-holdings organization 1 receives 2/3 of asset 1's value while organization 2 receives 1/3. The opposite is true for the asset 2.

Let both asset 1 and asset 2 have price $p_1 = p_2 = 10$. Thus, without any failure costs, the values of the organizations would be $v_1 = v_2 = 10$.

We let $\underline{v}_1 = 0$ and $\underline{v}_2 = 11$; let organization 2's failure costs be $\beta_2 = 6$. This means that if there are no changes in cross-holdings, from (5) the values of the two organizations are 8 and 6.[68] Suppose now that organization 1 can make a transfer to organization 2. If organization 1 were to make a transfer of 1 to organization 2, organization 2 would not fail and the values of the two organizations would be 9 and 11. Thus by making a transfer to organization 2, organization 1 is able to increase its value from 8 to 9! Such a payment might be a direct transfer of cash or implemented through a trade in underlying assets or cross-holdings. For example, organization 1 might simply give organization 2 an increased stake in itself.[69] Organization 1 is incentivized to "save" organization 2.[70]

Suppose we now extend the above example to permit organization 2 to have some control over its failure costs β_2 and failure threshold $\underline{v}_2$. For simplicity we suppose that organization 2 can choose from $\beta_2 \in \{0, 5, 10\}$ and from $\underline{v}_2 \in \{10, 11, 12, 13, 14\}$. Note that organization 2 can avoid failure without any intervention from organization 1 by choosing $\underline{v}_2 = 10$. However, such a choice is not in the best interest of organization 2.

We assume organization 1 will 'save' organization 2 if doing so weakly increases its value. If organization 2 needs saving ($\underline{v}_2 > 10$), 1's value after just saving 2 will be $v_1' = 10 - (\underline{v}_2 - 10)$ while its value will be $10 - (\beta_2/3)$ if it does not save organization 2. Organization 1 will therefore save organization 2 if and only if $\underline{v}_2 > 10$ and:

$$\frac{\beta_2}{3} > (\underline{v}_2 - 10).$$

The left hand side is the increase in value 1 receives from 2 remaining solvent and the right hand side is the cost of saving 2 – the transfer 1 must make to 2 for 2 to remain solvent. Table 2 below shows the transfers that organization 1 will make to organization 2 for the different values of $\underline{v}_2$ and β_2 that organization 2 can choose. These choices of $\underline{v}_2$ and β_2 then result in different values for organization 2 as shown in Table 3:

[68] Values before failure costs are 10 for both organizations. Organization 2 therefore fails and its failure cost of 6 reduces the effective value of its proprietary asset from 10 to 4. Organization 2 ultimately incurs 2/3 of this loss while organization 1 incurs 1/3.

[69] One of the lower cost ways in which organization 1 might "save" organization 2 is to simply take over organization 2.

[70] All the parameter values in the example can be varied slightly without generating a discontinuous change in the equilibrium. In this sense the example presented is not a knife-edge case.

	Failure Costs β_2		
Failure Threshold $\underline{v}_2$	0	5	10
10	0	0	0
11	0	1	1
12	0	0	2
13	0	0	3
14	0	0	0

Table 2: Transfer made from 1 to 2.

	Failure Costs β_2		
Failure Threshold $\underline{v}_2$	0	5	10
10	10	10	10
11	10	11	11
12	10	6 2/3	12
13	10	6 2/3	13
14	10	6 2/3	3 1/3

Table 3: Value of 2 after the transfer.

As can be seen in Tables 2 and 3, for a fixed failure threshold, organization 2 is only saved when its failure costs are sufficiently large. Conditional on being saved 2's value is increasing in his failure threshold and conditional on not being saved, organization 2's value is weakly decreasing in his failure threshold. For sufficiently high failure thresholds organization 2 is never saved and for sufficiently low failure threshold organization 2 doesn't fail. To maximize its utility after a bailout, organization 2 must set the highest failure costs it can and then carefully choose its failure threshold so that organization 1 is just incentivized to save it. In this example, this requires organization 2 choosing a failure threshold of 13 and failure costs of 10.

Of course, if organizations can commit not to bail each other out, then these moral hazard problems can be avoided. However, firms have a fiduciary obligation to maximize shareholder value, even if this involves bailing out a failing organization they have a stake in. This can make it difficult for organizations to commit not to bail out one another, and absent such a commitment device, organizations can have strong incentives to increase their failure costs and manipulate their failure thresholds.

The moral hazard problem in this example occurs absent any intervention by the government. Failure costs alone are sufficient for moral hazard problems to arise.[71] It arises because organizations do not fully bear their failure costs. As other organizations pay (indirectly through the devaluation of holdings in i) some of organization i's failure costs (β_i), these other organizations will be prepared to expend resources bailing out i. As the proportion of i's failure costs that i pays is given by A_{ii}, a natural measure of the severity of the moral hazard problem is $1-A_{ii}$. When $1-A_{ii}=0$ there is no moral hazard problem and the extent of the moral hazard problem is monotonic in $1-A_{ii}$ in the following sense: If $1-A_{ii}$ is increased by redistributing shares of i from outside shareholders to other organizations, such that all other organizations' claims on i weakly increase, any organization that previously would have bailed out i faces weakly stronger incentives to bail out i while organizations who previously would not have found it profitable to bail out i may now find it profitable to do so.

We saw in Section 3.1 that cascades of failure can occur, amplifying and propagating

[71]This moral hazard problem also distorts organizations' investment decisions, both in terms of their investments in risky projects and their investments in cross-holdings.

shocks if failure costs are sufficiently large and failure thresholds are sufficiently high. The analysis in this section has identified an endogenous mechanism through which organizations are willing to invest in increasing their failure costs and possibly their failure thresholds. Although such investments are valuable to an organization only in the event that it is bailed out, and in an uncertain world such bailouts may or may not be forthcoming, the misalignment of incentives due to the moral hazard problem can nevertheless result in systems endogenously conducive to cascades of failure.

A.4 Additional Simulations

In this section we describe some additional simulations similar to those reported in section 4, but with a couple of alterations.

A.4.1 Power Law Distributions

First we let the out-degree distribution for the organizations follow a (truncated) power law instead of modeling Erdos-Renyi random graphs. Specifically we let the outdegree d_{out} of each organization be drawn independently from a distribution $p(d_{out}) = a * d_{out}^{-\gamma}$, where γ is the power law parameter and a is a normalizing constant that ensures $p(d_{out})$ is a probability distribution. So, if according to a draw from this power law distribution organization i has a degree of 6, we randomly gave six other organizations a $c/6$ share of i.

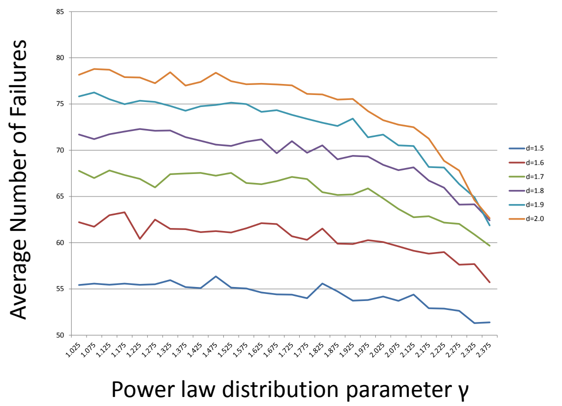
The objective of these simulations is to study the affect of the parameter γ on the number of failures. However, to prevent the effect of γ being conflated with changes to the expected degree d, we hold the expected degree constant by truncating the degree distribution. In other words we pick a maximum possible degree and adjust it for different levels of gamma to hold the expected degree d constant.[72]

As γ increases the number of failures decrease, but there are typically larger effects for even small changes in the expected degree d. This is true both when the out-degree follows a power law and when the in-degree follows a power law.

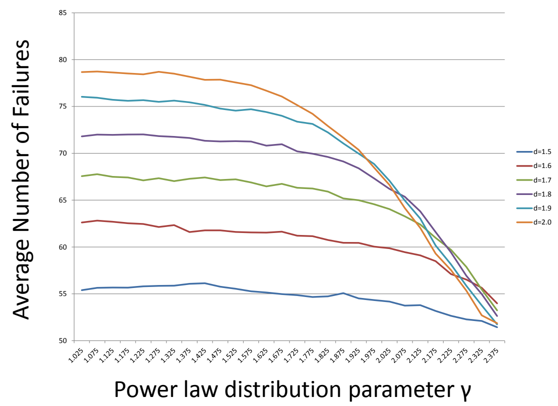
A.4.2 Correlated Asset Holdings

To explore the impact of organizations' asset holdings being correlated, we run simulations where instead of simply sending one organization's underlying asset value to zero and keeping all others at value 1, we do the following. We model drop one organization's direct asset holdings by $s\%$, *and* we also decrease some other organizations' assets by $s\%$ where we pick

[72]As the truncation can only occur at integer maximum degrees we vary the maximum degree between the floor and ceiling of the ideal truncation point. In all cases the normalizing constant adjusts to ensure $p(d_{out})$ is a probability distribution.



(a) Out Degree: Average failures of 100 organizations with out degrees drawn from a power distribution.



(b) In Degree: Average failures of 100 organizations with in degrees drawn from a power distribution.

Figure 12: How the average number of failures changed with the power law parameter γ for different expected degrees, averaged over 10000 simulations. The failure threshold is constant at $\theta = 0.95$ and the degree of integration is $c = 0.4$.

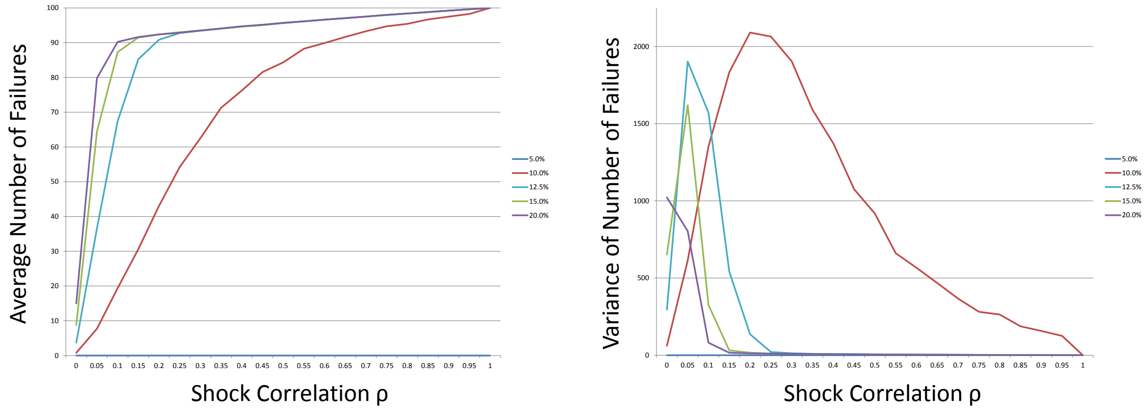
those other organizations each with a probability ρ. As ρ nears 1, all the assets drop together, whereas when ρ nears 0 then only the one organization fails. As we increase ρ we increase the number of organizations that fail together.[73]

From Figures 13a and 13b, when there is a network of interdependent organizations increasing the correlation of asset holdings to even a low level from a baseline of an uncorrelated system can result in relatively small shocks having highly uncertain outcomes that often result in very many failures.

A.4.3 Common Asset Holdings

We begin with baseline simulation model with average degree $d = 3$ and integration of $c = 0.4$, and make the following adjustments. First, each organization has holdings in two assets, a proprietary asset and a common asset. The total value of the common asset is set to 1 and the total value of all proprietary assets is set to 99, so that the relative value of the common asset is relatively low. Next each organization has holdings of the common asset equal to a $1/n$-th share. However, this share was then adjusted in the following way. One organization has an additional share equal to ℓ times a uniform[0,1] draw, and another organization is the counter-party to this position and reduces their holdings by the same

[73] This is a very simple way of introducing correlated shocks. A more detailed but nonetheless straightforward way of incorporating correlated positions would be to model holdings of many different assets that are held by multiple organizations. We could even permit people to hold negative amounts of an asset to represent shorting, although the total net position in the system must remain constant. See Section A.4.3.



(a) Percentage of organizations failing by correlation of asset holdings for different initial shocks for $\theta = 0.95$, $c = 0.4$, $d = 3$ and $n = 100$

(b) Variance in percentage of organizations failing by correlation of asset holdings for different initial shocks, $\theta = 0.95$, $c = 0.4$, $d = 3$ and $n = 100$

Figure 13: How correlated asset holdings affects the percentage of organizations failing, averaged over 5000 simulations. The x-axis lists the correlation in asset holdings measure by the proportion of organizations that suffer the different shocks.

amount. We continue in this way until each organization receives one positive or negative adjustment. The parameter ℓ is intended to capture *leverage* – and note that for $\ell > 1/d$ negative holdings of the common asset are possible. We then adjust the value of each organization's proprietary asset so that their total initial asset value is 1 – as before.

Next, we group the organizations into 10 groups of 10, as in our homophily simulations. However, unlike before, this grouping was not entirely random. A parameter ρ governs the extent to which the grouping is random versus based on organizations' positions in the common asset. When $\rho = 0$ it is entirely random. When $\rho = 1$ it is based entirely on holdings of the common asset. As before a homophily parameter h governs the relatively likelihood of links within groups versus across groups.

Thus, when $h > 0$ and $\rho > 0$ organizations with similar exposures to the common asset are more likely to be linked to each other. We can now look at the effect of correlating risks in a system with homophily/segregation by holding h constant and comparing $\rho = 0$ to $\rho > 0$. And in a system with no homophily/segregation, we can see the effect of correlated risks by reducing the leverage parameter – exposure to the common asset becomes more correlated as the parameter ℓ decreases, with perfect correlation for $\ell = 0$.

Interestingly, in this model, correlating risks by adjusting the ρ parameter has a minimal impact regardless of homophily. The key parameter that had a very substantial impact is the leverage parameter. For even small shocks to the common asset of 5 percent, large cascades occur (across the range of other parameters) for $\ell > 1.5$. Note that for these higher levels of leverage, the correlation in exposure to the common asset is actually lower. The threshold value of the parameter ℓ for which a large cascade occurs decreases in the size of the shock. However, for large shocks to the common asset of 20 percent, increasing the parameter ℓ reduces the extent of the cascade. Intuitively, a large parameter ℓ means that some organizations have negative holdings of the common asset (short positions - e.g., Goldman Sachs in the 2008 crisis) and their value can increase sufficiently for them to survive the failure of many other organizations.

A.5 Using the Dependency Matrix

This section validates the direct use and manipulation of the dependency matrix $\mathbf{A}$. Proposition 4 shows that absent any discontinuities (i.e. with failure costs of zero for all organizations), any change in $\mathbf{C}$ or $\mathbf{A}$ can be represented as changes in $\mathbf{D}$ alone. Proposition 5 then identifies a simple necessary and sufficient condition for the $\mathbf{A}$ to be valid – that is for there to exist direct cross-holdings $\mathbf{C}$ it can be derived from. This second result allows one to directly manipulate $\mathbf{A}$.

Proposition 4. Assuming there are no failures, for any $\mathbf{D}, \mathbf{C}$ there is a $\mathbf{D}', \mathbf{C}'$ with $\mathbf{C}$ being the matrix of zeros and $\widehat{\mathbf{C}}$ being the identity that results in the same organization values for any underlying asset prices $\mathbf{p}$. Similarly, for any $\mathbf{A}, \mathbf{D}$ there exists $\mathbf{D}'$ with $\mathbf{C}$ being the matrix of zeros and $\widehat{\mathbf{C}}$ being the identity that results in the same organization values for any underlying asset prices $\mathbf{p}$.

Proposition 4 follows directly from letting

$$\mathbf{D}' = (\widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1})\mathbf{D} = \mathbf{A}\mathbf{D}.$$

Thus, in the absence of failure, it is simply the indirect holdings of underlying assets that matter, and so one can equivalently work with them in understanding organizations' values.

The proposition implies that instead of considering trades in cross-holdings, when we are working to understand what might trigger a *first* failure (so that none have yet occurred) there is always some trade in underlying assets that replicates the trade in cross-holdings.

However, in practice, at least some of the underlying assets are non-tradeable and so can only be held through cross-holdings.[74] To work in the underlying asset space we therefore want to know when trades of underlying assets can be replicated through an exchange of cross-holdings, keeping the organizations' asset holdings ($\mathbf{D}$) constant. Proposition 5 provides necessary and sufficient conditions on $\mathbf{A}$ for it to be a valid representation of some $\mathbf{C}$.

Proposition 5. There exists a valid cross-holdings matrix $\widehat{\mathbf{C}} + \mathbf{C}$ (i.e. one that is column stochastic, contains non-negative entries and has strictly positive entries on the lead diagonal) that generates $\mathbf{A}$ if and only if $\mathbf{A}^{-1}_{ii} > 0$ for all i and $\mathbf{A}^{-1}_{ij} \leq 0$ for all i and all $j \neq i$.

[74] If all underlying assets were freely tradeable then there would be no reason for any cross-holdings. Any portfolio of claims to underlying assets held through cross-holdings could be replicated as direct holdings and without any risk of devaluation through failure.

Proof of Proposition 5: Recall from (5) that

$$\mathbf{A} = \widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1}.$$

If $\mathbf{A}$ is invertible, manipulating this equations yields that:

$$\begin{aligned}
\mathbf{A}^{-1} &= (\widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1})^{-1} \\
\mathbf{A}^{-1} &= (\mathbf{I} - \mathbf{C})\widehat{\mathbf{C}}^{-1} \\
\mathbf{A}^{-1}\widehat{\mathbf{C}} &= \mathbf{I} - \mathbf{C} \\
\mathbf{C} &= \mathbf{I} - \mathbf{A}^{-1}\widehat{\mathbf{C}}
\end{aligned} \tag{9}$$

If we can represent the right hand side of this equation just in terms of the $\mathbf{A}$ matrix, we will have found a way to map an $\mathbf{A}$ matrix into a $\mathbf{C}$ matrix. We will then just need to find conditions under which the $\mathbf{C}$ matrix we are deriving is column stochastic and has all non-negative elements (and strictly positive elements on the lead diagonal) when added to $\widehat{\mathbf{C}}$. When these conditions are met, the $\mathbf{A}$ matrix will have an associated valid $\mathbf{C}$ matrix it can be derived from and we can work directly with it.

Considering entry (i, i) of this matrix equation, and recalling that $\widehat{\mathbf{C}}$ is a diagonal matrix:

$$C_{ii} = 1 - (\mathbf{A}^{-1})_{ii}\widehat{C}_{ii}.$$

Since $C_{ii} = 0$ by assumption, we find $\widehat{C}_{ii} = 1/(\mathbf{A}^{-1})_{ii}$. This puts the left hand side of (9) in terms of just $\mathbf{A}$. Letting $\widehat{\mathbf{C}}$ be the matrix thus defined, set

$$\mathbf{S} = \mathbf{I} - \mathbf{A}^{-1}\widehat{\mathbf{C}}. \tag{10}$$

Thus the matrix $\mathbf{A}$ can be derived from a valid $\mathbf{C}$ (equal to the $\mathbf{S}$ matrix in equation 10) if and only if (i) $\mathbf{S} + \widehat{\mathbf{C}}$ is column stochastic such that column j of $\mathbf{S}$ sums to $1 - \widehat{C}_{jj}$ and (ii) all entries of $\mathbf{S} + \widehat{\mathbf{C}}$ are non-negative and the lead diagonal is strictly positive.

First we prove that $\mathbf{S}+\widehat{\mathbf{C}}$ is column stochastic. All valid $\mathbf{A}$ matrices are column stochastic and so $\mathbf{A}^{-1}$ is also column stochastic. To see this let $\mathbf{1}$ be the vector of ones such that $\mathbf{1A} = \mathbf{1}$. This is the definition of $\mathbf{A}$ being column stochastic. Now post multiply by $\mathbf{A}^{-1}$. We then find that $\mathbf{1} = \mathbf{1A}^{-1}$ and so $\mathbf{A}^{-1}$ is also column stochastic.

As $\mathbf{A}^{-1}$ is column stochastic, $\sum_{i=1}^{n}(\mathbf{A}^{-1})_{ij}\widehat{C}_{jj} = \widehat{C}_{jj}\sum_{i=1}^{n}(\mathbf{A}^{-1})_{ij} = \widehat{C}_{jj}$. Adding $\widehat{\mathbf{C}}$ to both sides of equation 10 we then have that:

$$\sum_{i=1}^{n} S_{ij} + \widehat{C}_{ij} = \sum_{i=1}^{n} I_{ij} - (\mathbf{A}^{-1})_{ij}\widehat{C}_{jj} + \widehat{C}_{ij} = 1 - \widehat{C}_{jj} + \widehat{C}_{jj} = 1$$

As $\mathbf{S} + \widehat{\mathbf{C}}$ is always column stochastic, there exists a valid $\mathbf{C}$ representation of $\mathbf{A}$ if and only if all entries of $\mathbf{S}+\widehat{\mathbf{C}}$ are non-negative and all entries of $\widehat{\mathbf{C}}$ are strictly positive.

From equation 10 the elements of $\mathbf{S}$ are:

$$S_{ii}+\widehat{C}_{ii} = 1-\frac{(\mathbf{A}^{-1})_{ii}}{(\mathbf{A}^{-1})_{ii}}+\frac{1}{(\mathbf{A}^{-1})_{ii}} = \frac{1}{(\mathbf{A}^{-1})_{ii}} \qquad \text{and} \qquad S_{ij}+\widehat{C}_{ii} = -\frac{(\mathbf{A}^{-1})_{ij}}{(\mathbf{A}^{-1})_{jj}},$$

for all i and all $j \neq i$. Thus all elements of $\mathbf{S}$ are well-defined and weakly positive if and only if $(\mathbf{A}^{-1})_{ii} > 0$ and $(\mathbf{A}^{-1})_{ij} \leq 0$ for all i and all $j \neq i$. ∎

A.6 Bounds on the Dependency Matrix

We provide some useful upper bounds on the possible values of the dependency matrix $\mathbf{A}$.

Let $\bar{c} = \max_k 1-\widehat{C}_{kk}$, and

$$\overline{A}_{ij} = \widehat{C}_{ii}\frac{\bar{c}}{1-\bar{c}}\max_{k\neq i}\frac{C_{ik}}{1-\widehat{C}_{kk}}$$

and

$$\overline{A}_{ii} = \widehat{C}_{ii}\left(1+\frac{\bar{c}}{1-\bar{c}}\max_{k\neq i}\frac{C_{ik}}{1-\widehat{C}_{kk}}\right).$$

Lemma 5. $\overline{A}_{ij}$ is an upper bound on A_{ij} for all i and j. Therefore, if $\widehat{C}_{ii} = 1-c$ for all i, so that each organization holds c of its holdings in other organizations and $1-c$ in itself, then $A_{ij} \leq \max_{k\neq i} C_{ik}$ for each i and $j \neq i$, and $A_{ii} \leq (1-c)+\max_{k\neq i} C_{ik}$.

Proof. Recall that

$$\mathbf{A} = \widehat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-1},$$

or alternatively that

$$\mathbf{A} = \widehat{\mathbf{C}}\sum_{t=0}^{\infty}\mathbf{C}^t.$$

Let $\overline{\mathbf{C}}$ be the matrix for which we set $\overline{C}_{ij} = \frac{C_{ij}}{1-\widehat{C}_{jj}}$.

Then,

$$\mathbf{A} \leq \widehat{\mathbf{C}}\sum_{t=0}^{\infty}\bar{c}^t\overline{\mathbf{C}}^t.$$

Note that $\overline{\mathbf{C}}$ is a column stochastic matrix. It follows that $\overline{\mathbf{C}}^{t-1}$ is also a column stochastic for any $t \geq 1$ (because it is a column-stochastic matrix raised to a power). Write $\overline{\mathbf{C}}^t = \overline{\mathbf{C}}\overline{\mathbf{C}}^{t-1}$. From this, given the fact that $\overline{\mathbf{C}}^{t-1}$ is column stochastic for each t, it follows that the ij-th entry of $\overline{\mathbf{C}}^t$ is no more than $\max_{k\neq i}\max_{k\neq i}\frac{C_{ik}}{1-\widehat{C}_{kk}}$. Also, note that for $t=0$, the ij-th entry of $\overline{\mathbf{C}}^t$ when $j \neq i$ is 0. Thus, for $i \neq j$,

$$A_{ij} \leq \widehat{C}_{ii} \sum_{t=1}^{\infty} \overline{c}^t \max_{k \neq i} \overline{C}_{ik}.$$

Then since $1/\sum_{t=1}^{\infty} \overline{c}^t = \overline{c}/(1-\overline{c})$ it follows that

$$A_{ij} \leq \widehat{C}_{ii} \frac{\overline{c}}{1-\overline{c}} \max_{k \neq i} \overline{C}_{ik},$$

This is the claimed expression for $j \neq i$. For $j = i$ we also have the ii-then entry of $\overline{\mathbf{C}}^0$ being 1. The simplifications for $\widehat{C}_{ii} = 1 - c$ for all i follow directly. □

A.7 Multiple Equilibria and Discontinuities in Organizations' Values

In the absence of any failure issues, equation (5) is a standard pricing equation describing how the values of organizations depend on the primitive asset values $\mathbf{v} = \mathbf{A}[\mathbf{D}\mathbf{p}]$. The novel and interesting part of equation (5) comes from the failure costs $\mathbf{b}(\mathbf{v})$. These terms generate several complexities that equation (5) illuminates.

In particular, the presence of failure introduces several forms of discontinuity which result in multiple equilibria. Discontinuities in the value of a given organization i can come from two sources. The basic form is that the failure costs of organization i can be triggered when the values of other organizations or underlying assets fall which then lead i to hit its failure threshold. The other form is due to another organization, in which i has cross-holdings, hitting its failure threshold, which then leads to a discontinuous drop in the value of i's holdings and consequently its value.

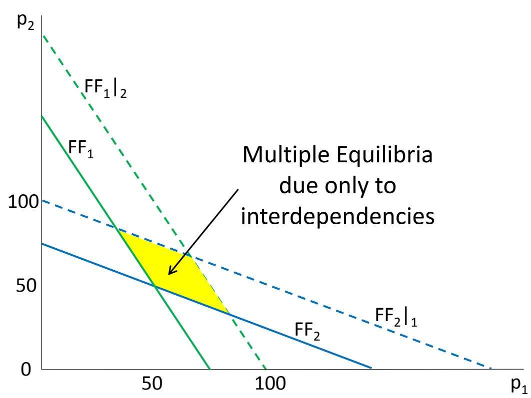
In terms of multiplicities of equilibria, there are also different ways in which these can occur. The first is that taking other organizations' values and the value of underlying assets as fixed and given, there can be multiple possible consistent values of organization i that solve equation (5). There may be a value of v_i satisfying equation (5) such that $1_{v_i \leq \underline{v}_i} = 0$ and another value of v_i satisfying equation (5) such that $1_{v_i \leq \underline{v}_i} = 1$; even when all other prices and values are held fixed. This generates the a first source of multiple equilibria corresponding to the standard story of self-fulfilling bank runs (such as those in classic models such as Diamond and Dybvig (1983)).

The second is the interdependence of the values of the organizations: the value of i depends on the value of organization j, while the value of organization j depends on the value of organization i, and given the discontinuities possible in prices due to failure costs, there can be multiple solutions. There might then be two consistent joint values of i and j: one consistent value in which both i and j fail and another consistent value in which both i and j remain solvent. This second source of multiple equilibria is different from the individual bank run concept, as here organizations fail because people expect other organizations to fail, which then becomes self-fulfilling.

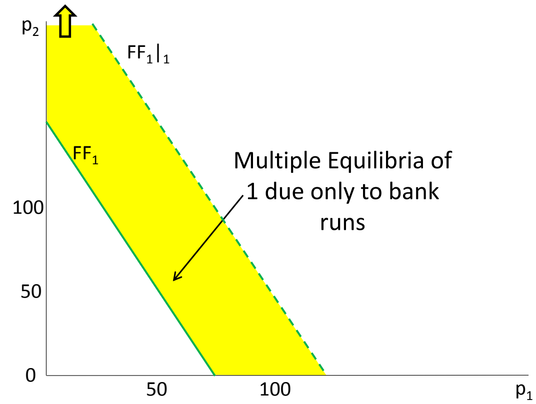
Although governments may be able to give assurances such as insuring deposits that manipulate expectations regarding the self-fulfilling value of a single organizations, it seems more difficult to control expectations when an organization's value depend on the expected values of many other organizations. For example, an organization's value can depend on the expected value of an organization that falls under the regulatory oversight of another government. Suppose organizations A and B have cross-holdings in each other and organization B also has cross-holdings in organization C. Investors in organization A may then become less confident investors will keep their money in organization B, or less confident the investors in B have confidence in them or in the investors in C, and so on.

A.8 Including Multiple Equilibria Due to Bank Runs

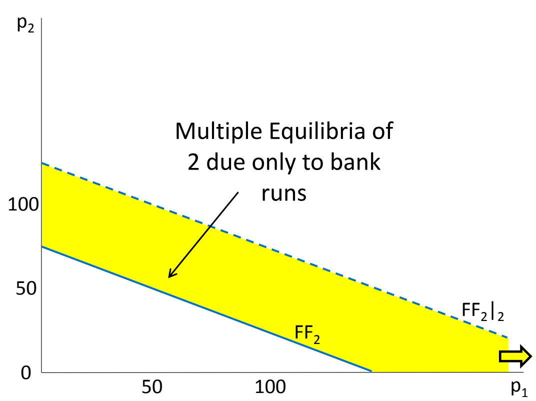
This section extends the example in section 3.2.2. The same parameter values are used in Figure 14 as were used in section 3.2.2 and Figure 1, although the scale of the axis has been adjusted. As can be seen the scope for multiple equilibria increases a great deal once bank runs are permitted. Note i's failure threshold conditional on i failing is shift out twice as far as i's failure threshold conditional on j failing because i effectively pays 2/3 of his failure costs but only 1/3 of j's. As shown in Figure 14d there is a large set of prices for which it is consistent for both 1 and 2 to both fail such that total failure costs of 100 are incurred and failure costs of 50 are paid by each organization.



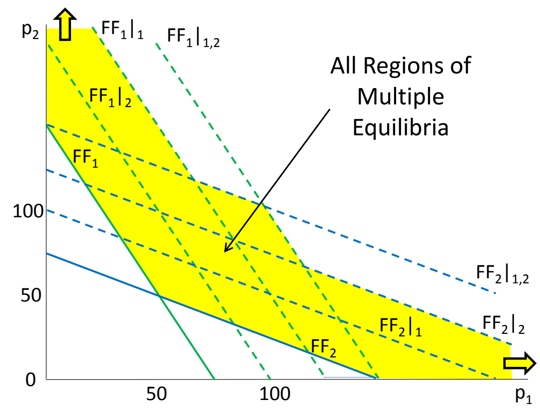
(a) Multiple Equilibria due only to interdependencies and without bankruns



(b) Multiple Equilibria of 1 due to bank runs



(c) Multiple Equilibria of 2 due to bank runs



(d) All Multiple Equilibria

Figure 14: The total set of multiple equilibria is much larger once bank runs are permitted. Nevertheless, the interdependencies provide an additional source of multiplicity even when bank runs are permitted.

A.9 Best-Case and Worst-Case Tradeoffs

We now return to considering multiplicity of equilibria due to the interdependencies between organizations. We identify a the tension between limiting failures in the best case equilibrium and worst case equilibrium. Trades that prevent any organizations failing in the best case outcome can also make more organizations fail in the worst case outcome.

We say that cross-holdings are *best-case safest* when they maximize the percentage decrease in asset prices that would be necessary for a first organization to fail. More formally, cross-holdings are *best-case safest* at $\mathbf{D}, \mathbf{p}$ if in the best equilibrium all organizations survive and the cross-holdings solve the following maximization problem:

$$\max_{\mathbf{C}} \min_{i} \frac{v_i(\mathbf{C}, p) - \underline{v}_i}{v_i(\mathbf{C}, p)}$$

It is *possible for all organizations to fail* if the total value of primitive assets less all failure costs can be allocated in a way that leaves all organizations below their failure thresholds. Such an allocation exists if and only if: $\sum_k \sum_i D_{ik} p_{ik} - \sum_i \beta_i < \sum_i \underline{v}_i$.

PROPOSITION **6.** Suppose organizations' failure costs are a proportion γ of the value of their direct asset holdings such that $\beta_i = \gamma \sum_k D_{ik} p_k$ and it is possible for all organizations to fail. Then all asset holdings that are best-case safest at prices $\mathbf{p}$ also result in all organizations failing in worst-case equilibrium.

Proof. If no organization fails, then their market values are:

$$\mathbf{v} = \mathbf{ADp}.$$

In order to be *best case safest*, we need to maximize the percentage loss that any organization can suffer without failing. As all assets have positive value, this requires equalizing the proportional loss in value each organization must suffer to fail. If this was not equalized, reallocating assets at the margin from the set of organizations furthest from their failure constraints to those organizations closest to them would increase the percentage loss in value that any organization can suffer without failing. Thus, in a best case safest asset allocation:

$$\mathbf{v} = \mathbf{ADp} = \theta \underline{\mathbf{v}}$$

for some scalar θ.

As by assumption it is possible for all organizations to fail at the same time and so:

$$\sum_i \sum_k D_{ik} p_k - \sum_i \beta_i < \sum_i \underline{v}_i$$

As failure costs are a constant proportion of the value of organizations' direct asset holdings and as $\mathbf{A}$ is column stochastic:

$$\sum_j \sum_i \sum_k (1-\gamma) A_{ij} D_{ik} p_k < \sum_i \underline{v}_i$$

Using equation A.9:

$$(1-\gamma)\theta \sum_i \underline{v}_i < \sum_i \underline{v}_i$$

and so $(1-\gamma)\theta < 1$.

Suppose now all organizations fail. In this case:

$$\mathbf{v} = \mathbf{A}(\mathbf{D}\mathbf{p} - \boldsymbol{\beta}) = \mathbf{A}\mathbf{D}\mathbf{p}(1-\gamma) = (1-\gamma)\theta\underline{\mathbf{v}} < \underline{\mathbf{v}}$$

Thus, in the worst case equilibrium, all organizations fail. ∎

□

Proposition 6 illustrates that if trades are undertaken with the sole purpose of achieving the best case safest outcome, these same trades can also result in the worst possible outcome occurring in the worst-case equilibrium – all organizations failing.

A.10 Details: Cascades of Default in Europe

We first discuss the data used and then provide the calculations of the $\underline{v}_i$s. There is data available from the Bank of International Settlements on aggregated cross-liabilities between countries on both an immediate borrower basis (which reports all contracts) and a final borrower basis (which nets out contracts with intermediaries replacing them with contracts between the final parties). If two parties trade through an intermediary we assume that intermediary writes separate contracts with the two parties (or acts as some kind of guarantor). In this case default by the intermediary would affect both parties and it is appropriate to use the intermediate borrower basis data.[75]

The calculations of the $\underline{v}_i$s are based on the peak GDPs from 2008. The normalized GDPs (relative to Portugal's GDP in 2011) are:

$$\begin{pmatrix} 12.0 \\ 15.3 \\ 1.5 \\ 9.7 \\ 1.1 \\ 6.7 \end{pmatrix}.$$

This leads to values based on the A matrix of:

$$\mathbf{v}_0 = \mathbf{A}\mathbf{p} = \begin{pmatrix} 0.71 & 0.13 & 0.13 & 0.17 & 0.07 & 0.11 \\ 0.18 & 0.72 & 0.12 & 0.11 & 0.09 & 0.14 \\ 0.00 & 0.00 & 0.67 & 0.00 & 0.00 & 0.00 \\ 0.07 & 0.12 & 0.03 & 0.70 & 0.03 & 0.05 \\ 0.01 & 0.00 & 0.02 & 0.00 & 0.67 & 0.02 \\ 0.03 & 0.03 & 0.02 & 0.02 & 0.14 & 0.68 \end{pmatrix} \cdot \begin{pmatrix} 12.0 \\ 15.3 \\ 1.5 \\ 9.7 \\ 1.1 \\ 6.7 \end{pmatrix} = \begin{pmatrix} 13.1 & \text{(France)} \\ 15.4 & \text{(Germany)} \\ 1.0 & \text{(Greece)} \\ 9.8 & \text{(Italy)} \\ 1.0 & \text{(Portugal)} \\ 7.5 & \text{(Spain)} \end{pmatrix}.$$

[75] Note that calculating the **A** matrix is far more involved than just looking at the final borrower basis data.

Thus

$$\underline{v} = \theta \begin{pmatrix} 13.1 & \text{(France)} \\ 15.4 & \text{(Germany)} \\ 1.0 & \text{(Greece)} \\ 9.8 & \text{(Italy)} \\ 1.0 & \text{(Portugal)} \\ 7.5 & \text{(Spain)} \end{pmatrix}, \quad \text{and} \quad \beta = \frac{\theta}{2} \begin{pmatrix} 13.1 & \text{(France)} \\ 15.4 & \text{(Germany)} \\ 1.0 & \text{(Greece)} \\ 9.8 & \text{(Italy)} \\ 1.0 & \text{(Portugal)} \\ 7.5 & \text{(Spain)} \end{pmatrix}.$$

A.11 Lemmas in the Proof of Proposition 3

Here we prove Lemmas 3 and 4. We maintain the notation of that proof.

Proof of Lemma 3: By the Neumann series (equation 6) applied to the structure of the present random graph, we have (absent any failures)

$$\mathbf{v} = (1-c)\sum_{p=0}^{\infty} \mathbf{C}^k \mathbf{1} \leq (1-c)\sum_{p=0}^{\infty} c^k (\underline{d}^{-1}\mathbf{G})^k \mathbf{1} \leq (1-c)\sum_{p=0}^{\infty} \left(c\frac{\overline{d}}{\underline{d}}\right)^k \mathbf{1}$$

where in the first inequality we have used a bound $C_{ij} \leq A_{ij}/\underline{d}$ on the entries of $\mathbf{C}$, and in the second we have used the fact that $\mathbf{G}^k\mathbf{1} \leq \overline{d}^k\mathbf{1}$, which is easy to verify by induction and the fact that $\overline{d}$ is the maximum degree of k. This establishes that $v_i \leq \widetilde{v}_{\max}$ for each i. The argument for the inequality $v_i \geq \widetilde{v}_{\min}$ is analogous: we use that $C_{ij} \geq A_{ij}/\overline{d}$ and then the fact that $\mathbf{G}^k\mathbf{1} \geq \underline{d}^k\mathbf{1}$. ∎

Proof of Lemma 4: Fix a j as defined in the statement. We will prove the lemma by translating into a statement about the probability of a certain event in a suitably defined Markov chain, which turns out to be more intuitive to establish. Let $\overline{\mathbf{C}}$ be defined by $\overline{C}_{xy} = G_{xy}/d_y$. Consider a Markov process (X_t) with state space $\{0, 1, 2, \ldots, n\}$ and initial state i. The state 0 is an absorbing state. From state $x \geq 1$, with probability $1-c$ a transition occurs to state 0, and otherwise the probability of moving to any state $y \geq 1$ is given by $\overline{C}_{yx}$. Observing that $\mathbf{C} = c \cdot \overline{\mathbf{C}}$, it is easy to verify that $Q_{ji} = \left(\sum_{p=2}^{\infty} \mathbf{C}^p\right)_{ji}$ is the probability of the following event E_j: there is some $t \geq 2$ such that $X_t = j$.

We will show that the probability of E_j is at most $\varepsilon/\overline{d}$ for each j such that $G_{ji} = 1$; since there are at most $\overline{d}$ such j, we then conclude by the union bound that the probability of $\bigcup_{j:A_{ji}=1} E_j$ is at most ε. Let T be the (random) set of nodes reached with positive probability from i in exactly two steps. For a fixed constant a, let M be the (random) set of nodes with a directed path of length at most a to j. Clearly, $|M| \leq \sum_{k=0}^{a} \overline{d}^a \leq \overline{d}^{a+1}$ (recall that the maximum degree of any node is $\overline{d}$). In other words, M constitutes a very small fraction of the nodes in the graph. Applying the Bollobás configuration model as outlined in Section 2.1 of Cooper and Frieze to make precise the fact that T and M are essentially independent conditional on i, we deduce that we can find n large enough so that the probability that $T \cap M$ is nonempty is at most $\varepsilon/(2\overline{d})$. Thus, $Q_{ji} \leq \varepsilon/(2\overline{d}) + (1-c)^a$: to return to j via a path of length at least 2, the Markov process has to take at least a steps, and has a probability $1-c$ of being absorbed at 0 at every step. Taking a large shows that $Q_{ji} \leq \varepsilon/\overline{d}$.