Stability and dynamics of offshore single point mooring systems

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STABILITY AND DYNAMICS OF OFFSHORE SINGLE POINT MOORING SYSTEMS

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ABSTRACT

The Floating Production Storage and Offloading System (FPSO's) is a modern concept for floating offshore oil exploration units, moored in deep water. 'Turret' and 'Mono-Buoys' are similar types of Single Point Mooring systems (SPM) envisaged for the stationkeeping task. Neverthless, the highly non-linear dynamic nature of this kind of system may give rise to a rich behaviour scenario that may comprise from simple pitchfork point equilibrium bifurcations to Hopf bifurcations (limit cycles), or even chaotic regimes. Standard linearised stability analysis may be not sufficient anymore to deal with the design problem. Bifurcation theory and modern system dynamics form then a proper theoretical basis for the analysis. This paper addresses the stability problem and discusses a number of interesting dynamic behaviors that arise in steady current. A self-excited autonomous and dissipative non-linear system of equations governs the system dynamics. A classical 'hydrodynamic derivatives' model form the core for hydrodynamic forces description. Following Papoulias & Bernitsas, 1988, some classical results on the stability problem are recovered. Then, reinterpreting the equilibrium analysis, it is also shown that bifurcation theory enables one not only to predict but also to qualify equilibrium pitchfork bifurcation scenarios, if super- or sub-critical. It is shown that the algebraic sign of the third-order derivative of the lateral force with respect to the lateral component of relative velocity governs the type of bifurcation scenario. When super-critical pitchfork bifurcation scenario is present a condition for structural stability loss is established and discussed. Hopf bifurcations (limit cycles) are also presented and discussed.

INTRODUCTION

The Floating Production Storage and Offloading System (FPSO's) is a modern concept for floating offshore oil exploration units, moored in deep water. A tanker is moored offshore and oil is stored before being transported by shuttle tankers that periodically are connected to the mother ship in a tandem formation. The vessels are subject to the environmental loads, due to the concomitant action of ocean currents, waves and wind. Single Point Mooring (SPM) systems are alternatives to conventional spread systems, envisaged for the stationkeeping task. The pimer motivation of such systems is to allow the ship to be aligned with the 'resultant' of the environmental forces, diminishing motions and structural loads on the mooring lines, hawsers[3] and risers.

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[1] Prof. Dr.

[2] CAPES-PET scholarship

[3] the cable through which the ship is attached to the mono-buoy

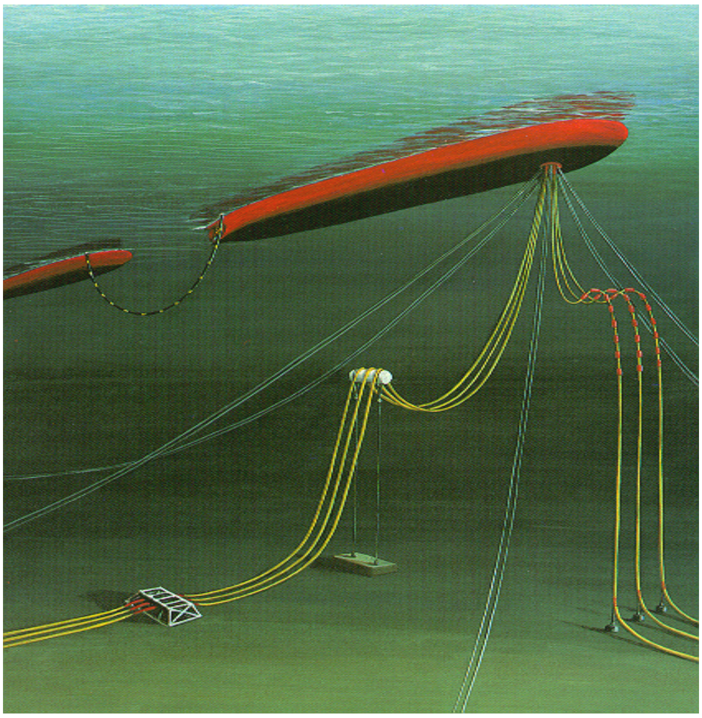

Figure 1 - FPSO moored at a 'Turret' System with a shuttle tanker in tandem formation[4]

The conventional SPM system is a mooring system composed by a mono-buoy, moored to the sea bed by means of cables and chains and to which the FPSO, a specially converted tanker, is attached through a 'hawser cable'. Not only the length but also the elastic characteristics of the cable and the attachement point position are important control parameters, concerning the stability and the dynamics of the system. In this paper we shall refer to mono-buoy systems (or 'hawser cable' systems) symply as SPM.

The 'Turret' type is a special kind of single point mooring system composed by a huge bearing system, fixed directly to the ship, the hawser cable being eliminated, and moored to the sea-bed, as shown in Figure 2 .

If wind and waves actions are not considered and if ocean current is taken as steady, a self-excited autonomous non-linear system of ordinary differential equations can be shown to govern the system dynamics. The highly non-linear dynamic nature of this kind of system gives rise to a rich behaviour scenario that may comprise from simple pitchfork point equilibrium bifurcations to Hopf bifurcations (limit cycles), or even chaotic regims. Bifurcation theory and modern system dynamics may form a proper theoretical basis for the analysis, although standard linearised stability analysis remains suitable for preliminary design purposes.

Under the bifurcation theory approach some research work has been done in this subject, primarily motivated by a close-related prolem of a towed ship in either a straigth course or maneuvering in a port site. Bernitsas & Kekridis, 1985, treat the towed-ship problem. Papoulias & Bernitsas, 1988, take the single point mooring problem into attention. Recently, Bernitsas & Garza-Rios, 1995, have studied the dynamics of some offshore spread but slacken mooring systems, that exhibit the same sort of dynamic behavior. Under such an approach the general equation of motion is derived on a "hydrodynamic derivative" model basis, in which

[4] Cortesia: Orcina Cable Protection

the hydrodynamic forces due to the relative motion with respect to the water are represented through Taylor series expansions given in terms of relative velocity components.

Following Papoulias & Bernitsas, 1988, this paper recovers and enlarges the analysis, addressing the stability problem again and discussing a number of interesting dynamic behaviors that arise for a 'Turret'or a SPM in steady current. Under a third-order model of the "hydrodynamic derivatives" type, it is shown that bifurcation theory enables one not only to predict but also to qualify point equilibrium instability. Two excludent *pitchfork bifurcation* scenarios are shown to exist: *sub*- and *super-critical*. As well known, the position of mooring line attachement at the ship is the control parameter governing equilibria bifurcation. It is also shown, in this paper, that this parameter can be responsible for a loss of structural stability of the system, switching bifurcation scenarios, whose type is controled primarily by the 'hydrodynamic derivatives' coefficients. The sign of the third derivative of the lateral force with respect to the lateral velocity, for instance, is shown to govern the type of bifurcation scenario that would appear. The occurrence of Hopf bifurcations (limit cycles) are also exemplified and discussed.

THE GOVERNING EQUATIONS

We follow closely Papoulias & Bernitsas, 1988. A classical "hydrodynamic derivatives model", extracted from maneuvering theory, see, e.g., Abkowitz, 1972, is used in order to simulate the action of the relative current on the ship. Waves and wind are not considered in the present paper, neither are the hydrodynamic forces acting directly on the mooring lines of the 'Turret', for instance. It is also out of the scope of the present work to discuss the pertinence of this type of hydrodynamic model, although, as we shall show, the stability and dynamic scenarios are strongly dependent on the hydrodynamic coefficients. A number of alternative approachs does exist; see e.g., Faltinsen et al., 1979, Whichers, 1988. However, we are primarily interested in discussing the stability problem, rather than the robustness of the hydrodynamic model, at least at the present moment.

Let, then *Oxyz* be a rigth-handed fixed reference frame, *x* being oriented in the opposite sense of the current velocity vector and *z* pointing upwards. Let *GXY* be a coordinate system attached to the floating unit, where *G* is the center of mass, *X* being oriented towards the bow. We restrict ourselves to the motion in the horizontal plane.Let be, also,

u, *v*: the relative velocity components of G with respect to the water, in the AX e AY directions, respectively;

y : the yaw angle (x*G*X)

$\dot{x}, \dot{y}$: the relative velocity components of G with respect to the fixed frame;

P: the point of mooring;

U: the current velocity intensity;

x_p : the distance *GP*;

l: the distance *OP*, where *O* is the fixed point where the hawser is attached (SPM case);

m: the floating unit mass;

I_Z : the moment of inertia with respect to *GZ*;

T: the tension in the cable (SPM case) or the mooring restoring force ('turret' case).

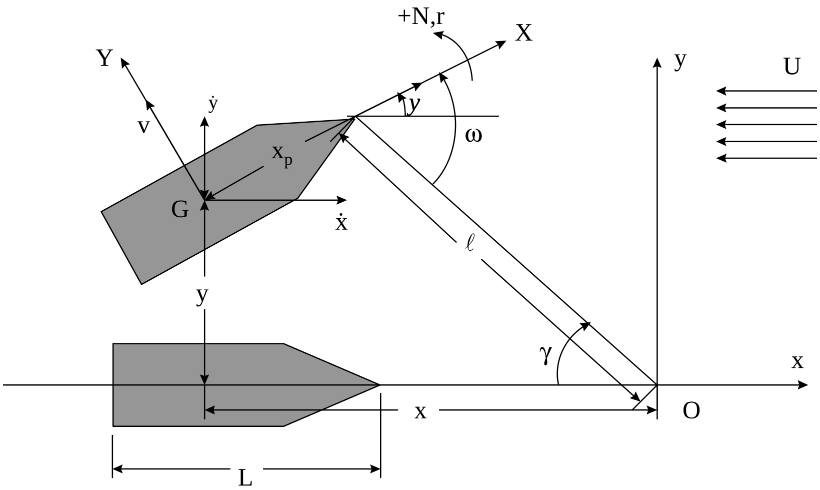

Figure 3 - Reference frames and geometric definitions.

Being **v** the velocity vector of *G* we have:

$$\mathbf{v} = (u\cos y \; - v\,\mathrm{sen}\,y \; - U)\mathbf{i} + (u\,\mathrm{sen}\,y \; + v\cos y\;)\mathbf{j} \qquad . \qquad (1)$$

or else,

$$\begin{aligned} \dot{x} &= u\cos y \; - v\,\mathrm{sen}\,y \; - U \\ \dot{y} &= u\,\mathrm{sen}\,y \; + v\cos y \end{aligned} \qquad (2)$$

For further reference, we also write the inverse form of (3),

$$\begin{aligned} u &= (\dot{x} + U)\cos y \; + \dot{y}\,\mathrm{sen}\,y \\ v &= -(\dot{x} + U)\,\mathrm{sen}\,y \; + \dot{y}\cos y \end{aligned} \qquad . \qquad (4)$$

Let also, by definition,

$$r = \dot{y} \qquad , \qquad (5)$$

and the following geometrical relations (see Figure 2)

$$w = g \; + y \qquad (6)$$

$$l^2 = (y + x_p\,\mathrm{sen}\,y\,)^2 + (x + x_p\cos y\,)^2 \qquad (7)$$

$$\mathrm{sen}\,g \; = \frac{1}{l}(y + x_p\,\mathrm{sen}\,y\,) \qquad (8)$$

$$\cos g \; = -\frac{1}{l}(x + x_p\cos y\,) \qquad (9)$$

noticing that (10), (7) and (8) are strictly valid only for $l \neq 0$, being g not defined for $l = 0$.

The acceleration of the center of mass, with respect to an inertial reference frame, but written in the vessel's frame, reads $\mathbf{a} = (\dot{u} - vr)\mathbf{I} + (\dot{v} + ur)\mathbf{J}$.

The equations of motions are then written,

$$\begin{aligned}
& -A^{11}\dot{u} + A^{22}vr + T\cos w + X(u,v,r) = m(\dot{u} - vr) \\
& -A^{33}\dot{v} - A^{11}ur - A^{23}\dot{r} - T\,\mathrm{sen}\,w + Y(u,v,r) = m(\dot{v} + ur) \\
& -A^{33}\dot{r} - A^{32}(\dot{v} + ur) + N(u,v,r) - Tx_p\,\mathrm{sen}\,w = I_Z\dot{r}
\end{aligned} \tag{11}$$

where *X(u,v,r), Y(u,v,r)* e *N(u,v,r)* are velocity dependent hydrodynamic forces acting on the floating unit and A^{mn} is the added inertia tensor, symmetric by construction (see, Newman, 1978, e.g.).

The SPM case

In the SPM case let *T*, the tension in the cable, be given (Papoulias & Bernitsas, 1988) by

$$T = \begin{cases} S_b\,p\left(\dfrac{l - l_w}{l_w}\right)^q & l \geq l_w \\ 0 & l < l_w \end{cases} \tag{12}$$

where l_w is the original length of the cable (unloaded) and S_b the limit strength, being p and q empirical constants. By defining the state vector

$$\mathbf{x} = (u,v,r,l,g,w)^t \quad , \tag{13}$$

equations **Erro! Vínculo não válido.**, **Erro! Vínculo não válido.** and (14) are transformed into a system of six first-order ordinary differential equations (Papoulias & Bernitsas, 1988),

$$\begin{aligned}
\dot{x}_1 &= \frac{1}{m + A^{11}}\left(F_1(x_1,x_2,x_3) + T\cos x_6 + (m + A^{22})x_2x_3 - A^{23}{x_3}^2\right) \\
\dot{x}_2 &= \frac{(I_Z + A^{33})}{D}\left(F_2(x_1,x_2,x_3) - T\,\mathrm{sen}\,x_6 - (m + A^{11})x_1x_3\right) - \frac{A^{23}}{D}\left(F_3(x_1,x_2,x_3) - A^{23}x_1x_3 - Tx_p\,\mathrm{sen}\,x_6\right) \\
\dot{x}_3 &= -\frac{A^{32}}{D}\left(F_2(x_1,x_2,x_3) + (m + A^{11})x_1x_3 - T\,\mathrm{sen}\,x_6\right) + \frac{(m + A^{22})}{D}\left(F_3(x_1,x_2,x_3) - A^{23}x_1x_3 - Tx_p\,\mathrm{sen}\,x_6\right) \\
\dot{x}_4 &= x_2\,\mathrm{sen}\,x_6 - x_1\cos x_6 + U\cos x_5 + x_px_3\,\mathrm{sen}\,x_6 \\
\dot{x}_5 &= \frac{1}{x_4}\left(x_1\,\mathrm{sen}\,x_6 + x_2\cos x_6 - U\,\mathrm{sen}\,x_5 + x_px_3\cos x_6\right) \\
\dot{x}_6 &= x_3 + \frac{1}{x_4}\left(x_1\,\mathrm{sen}\,x_6 + x_2\cos x_6 - U\,\mathrm{sen}\,x_5 + x_px_3\cos x_6\right)
\end{aligned} \tag{15}$$

where $D = (m + A^{22})(I_Z + A^{33}) - \left(A^{23}\right)^2$, and

$$F_1(x_1,x_2,x_3) = X_u x_1 + \frac{1}{2} X_{uu} x_1^2 + \frac{1}{6} X_{uuu} x_1^3 + m x_2 x_3$$
$$F_2(x_1,x_2,x_3) = Y_v x_2 + \frac{1}{6} Y_{vvv} x_2^3 + Y_r x_3 + \frac{1}{6} Y_{rrr} x_3^3 - m x_1 x_3 \qquad (16)$$
$$F_3(x_1,x_2,x_3) = N_v x_2 + \frac{1}{6} N_{vvv} x_2^3 + N_r x_3 + \frac{1}{6} N_{rrr} x_3^3$$

are the "generalized" hydrodynamic forces given in terms of the well known hydrodynamic derivatives, up to third order, where we take the subscripts for partial derivatives. We notice that (12) is a nonlinear dissipative and autonomous dynamic system.

The 'Turret' case

Another set of state variables, apropriate for the 'Turret' case, could be used instead.[5] Let

$$\mathbf{x} = (u, v, r, x, y, y)^t \quad . \qquad (17)$$

From (2), (3), (4), we get,

$$\dot{x}_1 = \frac{1}{m + A^{11}} \left(F_1(x_1,x_2,x_3) + T\cos w + (m + A^{22}) x_2 x_3 - A^{23} {x_3}^2 \right)$$
$$\dot{x}_2 = \frac{(I_Z + A^{33})}{D} \left(F_2(x_1,x_2,x_3) - T \operatorname{sen} w - (m + A^{11}) x_1 x_3 \right) - \frac{A^{23}}{D} \left(F_3(x_1,x_2,x_3) - A^{23} x_1 x_3 - T x_p \operatorname{sen} w \right)$$
$$\dot{x}_3 = -\frac{A^{32}}{D} \left(F_2(x_1,x_2,x_3) + (m + A^{11}) x_1 x_3 - T \operatorname{sen} w \right) + \frac{(m + A^{22})}{D} \left(F_3(x_1,x_2,x_3) - A^{23} x_1 x_3 - T x_p \operatorname{sen} w \right)$$
$$\dot{x}_4 = x_1 \cos x_6 - x_2 \operatorname{sen} x_6 - U$$
$$\dot{x}_5 = x_1 \operatorname{sen} x_6 + x_2 \cos x_6$$
$$\dot{x}_6 = x_3$$
$$(18)$$

with, again, $D = (m + A^{22})(I_Z + A^{33}) - \left(A^{23}\right)^2$, and

$$\begin{cases} \text{when} \quad l > 0, \quad w = \begin{cases} x_6 + \operatorname{arcsen}\left(\dfrac{x_5 + x_p \operatorname{sen} x_6}{l} \right); & \text{if } \left(x_4 + x_p \cos x_6 \right) < 0 \\ x_6 - \operatorname{arcsen}\left(\dfrac{x_5 + x_p \operatorname{sen} x_6}{l} \right) + p; & \text{if } \left(x_4 + x_p \cos(x_6) \right) \geq 0 \end{cases} \\ \\ \text{when} \quad l = 0, \qquad w(t_k) = w(t_{k-1}) \end{cases} \qquad (19)$$

$$l = \sqrt{(x_5 + x_p \operatorname{sen} x_6)^2 + (x_4 + x_p \cos x_6)^2} \quad . \qquad (20)$$

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[5] Notice that, in the 'turret' case we expect to have *l*=0.

STABILITY ANALYSIS

Standard Linear Analysis

Let the governing equations be written

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad (21)$$

If $\bar{\mathbf{x}}$ is a fixed point then,

$$\mathbf{f}(\bar{\mathbf{x}}) = \mathbf{0} \qquad . \qquad (22)$$

Let $x(t) = \mathbf{x}(t) - \bar{\mathbf{x}}$ be a perturbation around the fixed point**.** Then

$$\dot{x} = \mathbf{A}x \quad , \quad x \in \Re^{\mathbf{6}} \quad \mathbf{A} \in \Re^{\mathbf{6}} \qquad (23)$$

with

$$\mathbf{A} = \mathbf{Df}(\bar{\mathbf{x}}) \qquad (24)$$

being the Jacobian of **f** at $\bar{\mathbf{x}}$. From linear systems theory, asymptotic stability exists if all the eigenvalues of **A** are in the complex left plane. We could use the Routh Hurwitz criterion, for instance, in order to determine the conditions under which this necessary and sufficient condition is fully satisfied; see, e.g., Fernandes & Aratanha, 1996.

For a given floating unit, being known the hydrodynamic coefficients, the control parameters for the SPM case are (U,x_p,l_w). For the 'turret' case (U,x_p,b) are the parameters, where b indicates a subset of parameters characterizing the mooring restoring function. Besides changes in $\bar{\mathbf{x}}$, these parameters are responsible for qualitative variations in the stability and in the dynamic behavior of the system. As we have noticed before, we treat a nonlinear dissipative and autonomous dynamic system, in a hexa-dimensional space. Therefore equilibrium bifurcations, Hopf bifurcations, and even chaotic regimes can be expected. We should remember that nonlinear dissipative terms can lead to Hopf bifurcations, making the system structurally unstable, as well as nonlinear restoring forces to equilibrium bifurcations. Different scenarios where a number of atractors compete with each other can lead to chaos.We also notice that, in the SPM case the desirable equilibrium is obviously given by

$$\bar{\mathbf{x}} = (U,0,0,l_0,0,0)^t \qquad , \qquad (25)$$

where

$$l_0 = l_w \left\{ 1 + \left(\frac{X(U,0,0)}{S_b p} \right)^{\frac{1}{q}} \right\} = l_w \left\{ 1 + \left(\frac{T_0}{S_b p} \right)^{\frac{1}{q}} \right\} \qquad , \qquad (26)$$

is the stretched length of the cable in this position, whereas, in the 'Turret' case the desirable fixed point is (see (27)),

$$\bar{\mathbf{x}} = (U,0,0,0,0,0) \qquad . \qquad (28)$$

Equilibrium Bifurcations

We take *U* as invariant. The array of control parameters will be denoted by

$$m = (x_p, l_w)^t \quad , \qquad (29)$$

in the SPM case and by

$$m = (x_p, b)^t \qquad (30)$$

in the 'Turret' case. The dynamical system will be writen in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, m) \quad , \quad \mathbf{x} \in \Re^{\mathbf{6}} \quad m \in \Re^{\mathbf{2}} \qquad (31)$$

Supercritical Pitchfork Bifurcation

There is an obvious plane of symmetry, condition under which pitchfork bifurcations usually arise. According to Papoulias and Bernitsas, 1988, the equilibrium equation can be reduced to (see appendix A)

$$\frac{1}{6}(N_{vvv} - x_p Y_{vvv})\,\bar{v}^3 + (N_v - x_p Y_v)\bar{v} = 0 \quad . \qquad (32)$$

This equation can be rewritten in the form of a cubic equation in $\bar{v}$

$$f(\bar{v}, l) = \bar{v}^3 - l\bar{v} = 0 \qquad (33)$$

where, as defined in Papoulias & Bernitsas, 1988,

$$l = -6\frac{(N_v - x_p Y_v)}{(N_{vvv} - x_p Y_{vvv})} \qquad (34)$$

We should notice that this control parameter depends on a ratio between two diferences. Thus, errors in evaluating the hydrodynamic coefficients can be highly amplified. At the bifurcation point $(\bar{v}_0, l_0)$ both

$$\begin{aligned} &f(\bar{v}, l) = 0 \\ &f_{\bar{v}}(\bar{v}, l) = \frac{¶f}{¶\bar{v}} = 0 \end{aligned} \qquad (35)$$

must be satisfied. Then, from (31)

$$\bar{v} = \begin{cases} 0 \\ \pm\sqrt{l} \end{cases} \qquad (36)$$

and, from (31), l=0. Therefore

$$(\bar{v}_0, \bar{l}_0) = (0,0) \qquad (37)$$

and hence, from (30)

$$x_{p_0} = \frac{N_v}{Y_v} \qquad (38)$$

that is the critical value below which supercritical pitchfork bifurcation occurrs.

On the other hand, taking $(\dot{x}\ ,\ \dot{y}, \dot{y}\ ) = (0,0,0)$ in (39) we have the equilibrium values for the relative velocity components given by

$$\begin{aligned} u &= \bar{u} = U \cos \overline{y} \\ v &= \bar{v} = -U \operatorname{sen} \overline{y} \end{aligned} \quad . \qquad (40)$$

After equilibrium bifurcation occurs, $\bar{v} = \pm\sqrt{l}$, and so,

$$\operatorname{sen} \overline{y} = \pm \frac{\sqrt{l}}{U} \qquad (41)$$

gives the new equilibrium positions. If l is negative there is no real stable bifurcated solutions. Notice, also, that $l \le U^2$ must be satisfied, as well. In other words, at $\overline{y} = \pm p/2$ structural stability is broken. We shall return to this point later on.

Structural Stability and Subcritical Pitchfork Bifurcation

The previous analysis, recovered from Papoulias & Bernitsas, 1988, takes the hydrodynamic derivatives given in terms of the relative velocity component *v*, defined by equations (2) and (3), as usually done in marine hydrodynamics. We can think of reconducting such a reasoning, but now in terms of the heading angle ψ , perhaps enhancing our physical understanding. As a direct result, a condition for stability loss concerning supercritical pitchfork equilibrium bifurcations is discussed, and a new type, the subcritical one (or cathastrophic) appears vividly. For, taking (42), solved in terms of (u, v) we get,

$$\begin{aligned} \frac{\P v}{\P y} &= -u \\ \frac{\P u}{\P y} &= v \end{aligned} \quad , \qquad (43)$$

valid for all states. If we assume $(\dot{x}, \dot{y}) = (0,0)$ - as in a *captive model experiment*, where a *restrained* small-scale model of the vessel is driven by a constant current velocity *U*, being measured the hydrodynamic forces and moment $(X^{(c)}, Y^{(c)}, N^{(c)})$, acting upon the model, - from equation (3) we get[6]

$$\begin{aligned} u &= U \cos y^{(c)} \\ v &= -U \operatorname{sen} y^{(c)} \end{aligned} \quad . \qquad (44)$$

We can think of $(X^{(c)}, Y^{(c)}, N^{(c)})$ as functions of $y = y^{(c)}(u, v) = \arctan(-v/u)$ and *U*, such that

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[6] The subscript *c* stands for *captive.*

$$
\begin{aligned}
X^{(c)} &= X^{(c)}(y\,;U)\\
Y^{(c)} &= Y^{(c)}(y\,;U)\\
N^{(c)} &= N^{(c)}(y\,;U)
\end{aligned} \qquad . \tag{45}
$$

Accordingly, with the use of (46), we get

$$
Y_v^{(c)} = \frac{\P Y^{(c)}}{\P v} = \frac{\P Y^{(c)}}{\P y}\frac{\P y}{\P v} = -\frac{1}{u} Y_y^{(c)}
$$

that after (47) is used, reads

$$
Y_v^{(c)} = -\frac{1}{U \cos y^{\,(c)}} Y_y^{(c)} \qquad , \tag{48}
$$

leading, as can be easily verified, to

$$
Y_{vvv}^{(c)} = -\frac{1}{U^3 \cos^3 y^{\,(c)}} Y_{yyy}^{(c)} \qquad . \tag{49}
$$

Analogously,

$$
N_v^{(c)} = -\frac{1}{U \cos y^{\,(c)}} N_y^{(c)} \tag{50}
$$

and also,

$$
N_{vvv}^{(c)} = -\frac{1}{U^3 \cos^3 y^{\,(c)}} N_{yyy}^{(c)} \tag{51}
$$

Equations (52)-(53) are valid only under $\left(\dot{x},\dot{y}\right) = (0{,}0)$, i.e. if (54) holds. Substituting (55)-(56) in (57) it follows at equilibrium ($y = y^{\,(c)}(\bar{u},\bar{v}) = \bar{y}$), that[7]

$$
x_{\mathrm{p}}\ \{\ Y_y\ \tan\bar{y}\ +\ \frac{1}{6}Y_{yyy}\ \tan^3\bar{y}\}\ =\ N_y\ \tan\bar{y}\ +\ \frac{1}{6}N_{yyy}\ \tan^3\bar{y} \qquad . \tag{58}
$$

Notice that (44) is satisfied for any $\bar{y}$, particularly at $\bar{y} = 0$ and $\bar{y} = p$, fixed points. For these two fixed points we may have super- (as previously presented) or sub-critical pitchfork bifurcations. But, as we shall see, confirming results presented in the last section, if super-critical pitchfork bifurcation scenario exists, $\bar{y} = \pm p/2$ will be an unstable solution, leading to a former (locally) sub-critical bifurcation, when the dynamic system looses structural stability.

Firstly we shall restrict ourselves to a local analysis around $y = 0$. Obviously, the same reasoning can be applied around $y = p$ as well. Equation (44) can then be written,

[7] From now on, $Y_y = Y_y^{(c)}\Big|_{(y=\bar{y})}$, etc., are implied.

$$x_p \{ Y_y \bar{y} + \frac{1}{6} Y_{yyy} \bar{y}^3 \} = N_y \bar{y} + \frac{1}{6} N_{yyy} \bar{y}^3 \quad (59)$$

or else,

$$\bar{y}^3 - l_c \bar{y} = 0 \quad (60)$$

where

$$l_c = -6 \frac{(N_y - x_p Y_y)}{(N_{yyy} - x_p Y_{yyy})} \quad (61)$$

plays the role of the l parameter, previously defined. As before,equation (62) admits up to three roots

$$\begin{aligned} \bar{y} &= 0 \\ \bar{y} &= \pm\sqrt{l_c} \end{aligned} \quad . \quad (63)$$

Let, now, $M(y)$ be a cubic restoring moment, such that

$$M(y) = \frac{1}{6} y^3 (x_p Y_{yyy} - N_{yyy}) + y (x_p Y_y - N_y) \quad (64)$$

and take a non-dissipative one-dimensional dynamic system (one degree of freedom)

$$I\ddot{y} + M(y) = 0 \quad (65)$$

as representing our system around the considered fixed point. Equation (66) can be put in the form

$$\ddot{y} + a(y^3 - l_c y) = 0 \quad (67)$$

where

$$a = \frac{x_p Y_{yyy} - N_{yyy}}{6I} \quad . \quad (68)$$

Two possibilities do exist: **(i)** $a > 0$; **(ii)** $a < 0$. Let us analyse the alternatives that appear:

(i) If $a > 0$; or, equivalently, $\begin{cases} x_p > N_{yyy} / Y_{yyy} ; \text{ if } Y_{yyy} > 0 \\ x_p < N_{yyy} / Y_{yyy} ; \text{ if } Y_{yyy} < 0 \end{cases}$; two different situations arise:

(ia) if $l_c < 0$:

In this case $\bar{y} = 0$ is the only fixed point, stable, a *center*.

As $a > 0$, and $Y_v < 0$ always, so that, from(69), $Y_y > 0$ if $U > 0$ and $\cos y > 0$, we then have the following condition,

$$x_p > \frac{N_y}{Y_y} = \frac{N_v}{Y_v} \quad ; \quad (70)$$

(ib) if $l_c > 0$, conversely, we then have,

$$x_p < \frac{N_y}{Y_y} = \frac{N_v}{Y_v} \tag{71}$$

In this case three are the fixed points,

$\bar{y} = 0$: a saddle point (unstable, therefore)

$\bar{y} = \pm\sqrt{l_c}$: nodes (stable)

This situation corresponds to the super-critical bifurcation, previously studied.

(ii) If $a < 0$; or, equivalently, $\begin{cases} x_p < N_{yyy} / Y_{yyy} \text{ ; if } Y_{yyy} > 0 \\ x_p > N_{yyy} / Y_{yyy} \text{ ; if } Y_{yyy} < 0 \end{cases}$; again, two different situations arise:

(iia) If $l_c < 0$:

$\bar{y} = 0$ is the only fixed point, unstable, a *saddle point.*

As $a < 0$, and, as mentioned before, $Y_v < 0$ always, so that, from (72), $Y_y > 0$ if $U > 0$ and $\cos y > 0$, we then have valid the following condition,

$$x_p < \frac{N_y}{Y_y} = \frac{N_v}{Y_v} \quad ; \tag{73}$$

(iib) If $l_c > 0$, conversely, we then have,

$$x_p > \frac{N_y}{Y_y} = \frac{N_v}{Y_v} \tag{74}$$

In this case three are the fixed points,

$\bar{y} = 0$: a *center* (stable)

$\bar{y} = \pm\sqrt{l_c}$: *saddle point*s (unstable).

Notice that $l_c = 0$ is a bifurcation point, therefore. The *stability condition* $x_p > N_y / Y_y$ is a necessary condition, either for $a > 0$ or $a < 0$. This condition (53), is the same one that has been previously achieved, (34). It is also the same necessary condition that emerges if the Routh-Hurwitz criterion is applied to the linearised system, (see Fernandes, 1995).

If $x_p < N_y / Y_y$ and if $a > 0$, the pitchfork bifurcation is of the *supercritical* (smooth) type, whereas, if $a < 0$ it is of the *subcritical* (cathastrophic) type. Notice that the *sign of the third-order derivative* Y_{vvv}, *that under equilibrium hypotheses is related to* Y_{yyy} *as given by* (75), *controls the bifurcation scenario, if super- or sub-critical.*

Summarizing, stability around $\bar{y} = 0$ exists if the necessary condition (53) holds. Otherwise ($x_p < N_y / Y_y$), two bifurcation scenarios arise:

(i) $a > 0$: *supercritical bifurcation* and then a new equilibrium point is reached: $\bar{y} = \pm\sqrt{l_c}$ (local analysis); $\text{sen}\,\bar{y} = \pm\frac{\sqrt{l}}{U}$ (global analysis);

(ii) $a < 0$: *subcritical bifurcation.* The system would not reach, (smoothly) another point of equilibrium. As a matter of fact, looking at the original equation, $\bar{y} = p$ would be reached as the new stable fixed point, of the center type.

Notice that, although somewhat simplified, the present analysis do consider terms up to the third-order, retaining all the features of the system regarding equilibrium point stability analysis. This kind of features is unacessible under the Routh-Hurwitz criterion applied to the linearised equations. Notice also that dissipation terms, not considered in this simplified analysis, will transform *centers* in *stable foci.*

Another important point is that, even when a supercritical (smooth) bifurcation occurs, there is a limit value for a stable bifurcated equilibrium, namely $\bar{y} = \pm p/2$. This corresponds to the condition $l = U^2$, as already mentioned. In fact, (76) can be written in the form (77), with the restoring moment given as

$$M(y) = \frac{1}{6}\tan^3 y\,(x_p Y_{yyy} - N_{yyy}) + \tan y\,(x_p Y_y - N_y) \quad . \qquad (78)$$

Whenever $y \to \pm p/2$ the *restoring moment* changes sign suddenly, tranforming itself into a *repulsive moment*. Thus stability is broken at $y \to \pm p/2$ and, usually $\bar{y} = p$ will be the new atractor point. This is in fact a loss in the structural stability for the dynamic system, considering up to the third-order hydrodynamic coefficients. Notice also that, from, (79) - (80) in (81), and comparing it to (82), that,

$$l = U^2 \cos^2 \bar{y}\, l_c(\bar{y}) \quad , \qquad (83)$$

where the parameter λ_c is determined for $\psi = \overline{\psi}$ from (84). The bifurcated equilibrium equation (85) can then be transformed into

$$\tan \bar{y} = \pm\sqrt{l_c(\bar{y})} \quad . \qquad (86)$$

This equation, together with the definition (87), shows clearly that, for each $y = \bar{y}$, there is a “turning point”, $x_{p1} = x_{p1}(y) = N_{yyy} / Y_{yyy}$, concerning the loss of stability for supercritical pitchfork bifurcations. Additionaly, this latter form has the advantage, if compared to (88), of being (at least explicitly) independent of *U.* The above discussion will be clearly exemplified in the next section.

The figures below ilustrate the present local analysis, where $M(y)$ is the restoring moment.

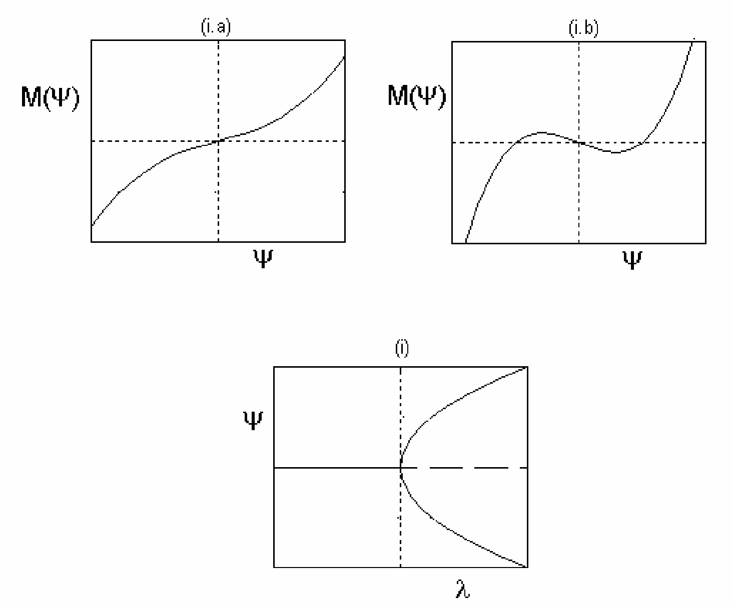

Figure 4 - Pitchfork Bifurcation Diagram. Supercritical Case: $a > 0$**;** $x_p > N_{yyy} / Y_{yyy}$; $Y_{yyy} > 0$**.**

Local analysis around $\overline{\psi} = 0$**.**

(i.a) $l_c < 0$**;** $x_p > N_y / Y_y = N_v / Y_v$ **. (i.b)** $l_c > 0$**;** $x_p < N_y / Y_y = N_v / Y_v$ **.**

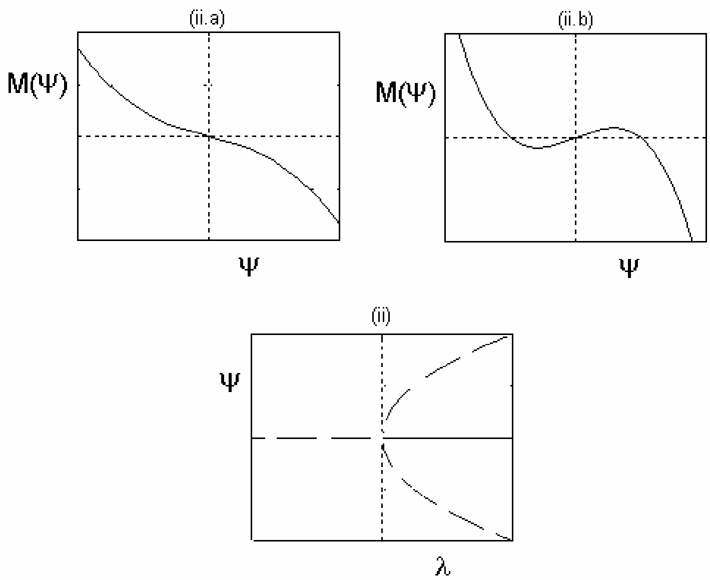

Figure 5 - Pitchfork Bifurcation Diagram. Subcritical Case: $a < 0$**;** $x_p < N_{yyy} / Y_{yyy}$; $Y_{yyy} > 0$ **.**

Local analysis around $\overline{y} = 0$**.**

(ii.a) $l_c < 0$**;** $x_p < N_y / Y_y = N_v / Y_v$ **. (ii.b)** $l_c > 0$**;** $x_p > N_y / Y_y = N_v / Y_v$ **.**

EXAMPLES ON STABILITY ANALYSIS AND BIFURCATIONS

The studied cases and their hydrodynamic derivatives

Two different tankers moored at either a SPM or a 'Turret' type systems have been taken. Table 1 shows the particulars of these vessels. The characteristcs of both the hawser cable, for the SPM case, and the restoring mooring force function, for the "Turret' case, are shown in Table 2. Table 3 presents the nondimensional hydrodynamic derivatives, extracted, respectively from Takashina, 1986 and Bernitsas & Kekridis, 1985. It should be noticed that the Takashina coefficients were obtatined by means of captive model experiments, in either static drift tests at Froude number 0.066, or yaw rotating tests (for four different constant yaw rates $\dot{\psi} = r = \text{constant}$). It should also be pointed out that Takashina's nondimensionalized parameters differ somewhat from Bernitsas & Kekridis, as shown in Table B1, appendix B. For comparison purposes Table 3 shows, for the Takashina tanker, the nondimensional hydrodynamic derivatives in both forms. Notice, however, that equations of motions are invariant in form, under both nondimensionalization procedures, the velocity being always equal to unity. Both have been considered as linear.

Table 1. Tanker's particulars

	Tanker (1)	Tanker (2)
Length (Lbpp) (m)	270	325
Breadth (m)	44.8	-
Draft (m)	10.8	-
Displacement (m^3)	90699	310669
Mass (t)	92967	318436

Table 2. Hawser Breaking Strength for the SPM and Restoring Force Charactheristics for the 'Turret' System.

	Hawser Breaking Strength (*) S_B	**Turret System Restoring Coefficient** K
	12200 kN	33.0 kN/m
Dimensionless (A)	8.0	6.0
Dimensionless (B)	0.2252	0.194

(*) $(q = 1.0; p = 5.0.;)$; i.e., linear and 20% stretching at rupture.

Table 3. Tankers' Dimensionless Parameters and Hydrodynamic Derivatives[8]**.**

	Tanker 1		Tanker 2
	(A)[9]	**(B)**[10]	**(B)**
m	2.258	0.0903	0.0181
I_z	0.14112	0.00564	-
$I_z + A^{rr}$	-	-	0.00222
A^{uu}	0.050xm	0.004516	0.0009
A^{vv}	0.650xm	0.05871	0.0171
A^{rr}	0.043xm	0.003884	-
A^{vr}	-	-	~0.0
A^{rv}	-	-	~0.0
X_u	-0.01	-0.0004	-0.003
Y_v	-0.285	-0.0114	-0.0261
Y_r	-	-	0.00365
N_r	-0.030	-0.0012	-0.0048
N_v	-0.0028	-0.00011	-0.0105
Y_{vvv}	-0.894	-0.21456	-0.045
N_{vvv}	-0.0093	-0.002232	0.00611
N_{rrr}	-	-	0.00611

[8] Only shown hydrodynamic derivatives actually used in the present time domain simulations.

[9] Values according to Takashina, 1986, and table 2, column (A). Notice that in that paper the Taylor expansion coefficients (n!) that appear in the definition of the hydrodynamic derivatives are implicitly considered in the equation of motion. In other words, these values should be corrected by the corresponding factor if the hydrodynamic forces are calculated as in equation (**Erro! Apenas o documento principal.**).

[10] Values calculated from column (A) but already corrected according to table 2, column (B).

Stability, bifurcations and dynamic behaviours

Consider the fixed point corresponding to $\overline{y} = 0$. The necessary condition for stability is given by equation (89). If applied to ***tanker (1)***, irrespective if SPM or 'Turret' cases are considered, we get $x_p/L > N_v/Y_v = 0.0028/0.285 = 0.009824$. But, in the present case, $N_{vvv}/Y_{vvv} = 0.0093/0.894 = 0.0104$, and so $N_v/Y_v < N_{vvv}/Y_{vvv}$. Notice, also that, in the present case, $Y_{vvv} < 0$, such that, for $\cos y > 0$, we have $Y_{yyy} > 0$. This implies that the stability condition (ia) $(a > 0; l_c < 0)$, can be written as $(x_p/L > N_{vvv}/Y_{vvv}; x_p/L > N_v/Y_v)$, i.e., $x_p/L > 0.0104$. Condition (ib) $(a > 0; l_c > 0)$, for its turn, would be written $(x_p/L > N_{vvv}/Y_{vvv}; x_p/L < N_v/Y_v)$, i.e., this would never occur, in this case. In words, only the subcritical pitchfork scenario may appear, and supercritical bifurcation will never occur. In fact, condition (iib) $(a < 0; l_c > 0)$ or $(x_p/L < N_{vvv}/Y_{vvv}; x_p/L > N_v/Y_v)$, Figure 6 , corresponding to a 'marginal stability' around $\overline{y} = 0$, holds whenever $0.009824 < x_p/L < 0.0104$, and condition (iia), corresponding to $(a < 0; l_c < 0)$, or $(x_p/L < N_{vvv}/Y_{vvv}; x_p/L < N_v/Y_v)$ (corresponding to be $\overline{y} = 0$ an unstable fixed point) applies if $x_p/L < 0.009824$. *Table 5 summarizes the stability analysis*. Figure 7 , shows a number of time domain simulations, starting from the desired equilibrium position by means of a small perturbation in the tranversal relative velocity *v*. The time domain simulations have been performed by means of MATLAB/SIMULINK code, using a fifth-order Runge-Kutta integration scheme. We see that $y = 0$ is a stable equilibrium point for $x_p/L = 0.011$, in accordance to Table 4 (i.a). We see also that $y = p$ is the atractor for the two other conditions exemplified. It should be notice that 100 units of dimensionless time corresponds to about 7.5 hours in real scale, for a current speed of 1 m/s.

As a conclusion *tanker (1) can be considered 'stable'*, since for practical reasons $x_p/L \geq 0.25$, in general. If the connection point is moved aft, only subcritical bifurcation shall appear, therefore.

Table 4. Stability Scenarios for the Tanker (1) concerning $\overline{\psi} = 0$

	stability parameters	**bifurcation scenario**	**situation**
$x_p/L < 0.00982$	$a < 0; l_c < 0$	(ii.a)	unstable
$0.00982 < x_p/L < 0.0104$	$a < 0; l_c > 0$	(ii.b)	subcritical bifurcation[11]
$x_p/L > 0.0104$	$a > 0; l_c < 0$	(i.a)	stable

[11] 'marginally stable'

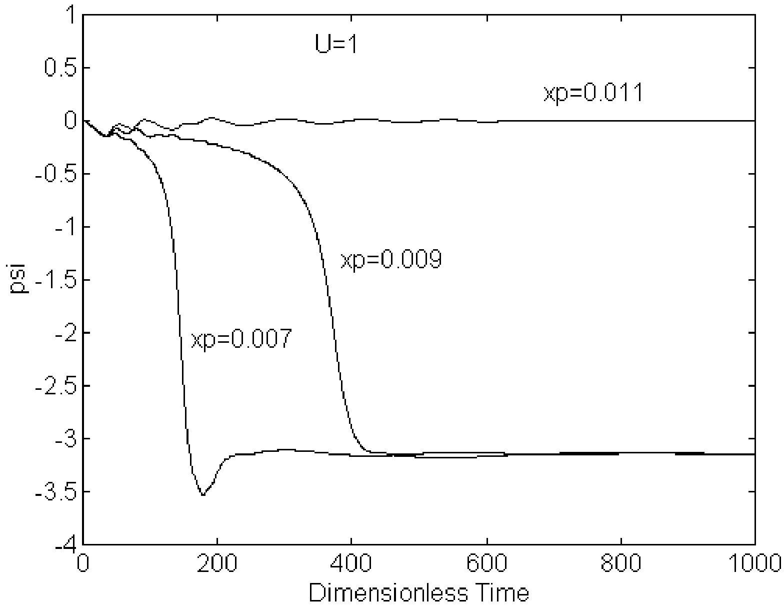

Figure 8 - Subcritical Pitchfork Bifurcations for tanker (1) in SPM; x_p/L as a parameter. Dimensionless time $t' = tU/L$. Hawser length: $l_w/L = 2$. Initial conditions:

$$x_1 = u/U = 1; x_2 = v/U = 0.1; x_3 = r\,L/U = 0; x_4 = l/L = 2; x_5 = g = 0; x_6 = w = 0.$$

For the ***tanker (2)***, however, we get $N_v/Y_v = 0.0105/0.0261 = 0.4023$ and $N_{vvv}/Y_{vvv} = -0.0061/0.045 = -0.1358$. Hence, we have $N_{vvv}/Y_{vvv} < N_v/Y_v$. Notice that, even in the present case, $Y_{vvv} < 0$, such that, for $\cos y > 0$, we have $Y_{yyy} > 0$. The stability condition (ia) $(a > 0; l_c < 0)$, or $(x_p/L > N_{vvv}/Y_{vvv}; x_p/L > N_v/Y_v)$, leads to, $x_p/L > 0.4023$. Condition (ib) $(a > 0; l_c > 0)$, or $(x_p/L > N_{vvv}/Y_{vvv}; x_p/L < N_v/Y_v)$, gives $-0.1358 < x_p/L < 0.4023$, in this case. In words, supercritical bifurcation scenario is now present. On the other hand, condition (iib) $(a < 0; l_c > 0)$ or $(x_p/L < N_{vvv}/Y_{vvv}; x_p/L > N_v/Y_v)$, that would correspond to a 'marginal stability' around $\bar{y} = 0$, cannot hold, and condition (iia) $(a < 0; l_c < 0)$, or $(x_p/L < N_{vvv}/Y_{vvv}; x_p/L < N_v/Y_v)$ (corresponding to be $\bar{\psi} = 0$ an unstable fixed point) applies if $x_p/L < -0.1358$. Table 5 summarizes the stability analysis.

Table 5. Stability Scenarios for the Tanker (2) concerning $\bar{y} = 0$

	stability parameters	**bifurcation scenario**	**situation**
$x_p/L < -0.1358$	$a < 0; l_c < 0$	(ii.a)	unstable
$-0.1358(*) < x_p/L < 0.4023$	$a > 0; l_c > 0$	(i.b)	supercritical bifurcation
$x_p/L > 0.4023$	$a > 0; l_c < 0$	(i.a)	stable

(*) see discussion below)

Condition (iib) cannot hold and so, subcritical bifurcation will never occur in this case.

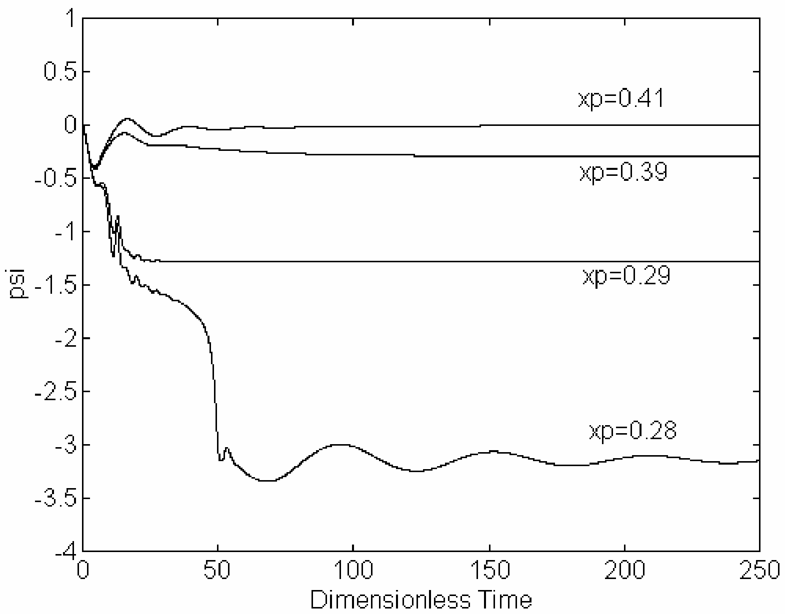

Figure 9 - Supercritical Pitchfork Bifurcations (showing some cases with subsequent loss of structural stability) for tanker (2) in SPM; x_p/L as a parameter. Dimensionless time $t' = tU/L$. Hawser length: $l_w/L = 2.5$. Initial conditions:

$$x_1 = u/U = 1; x_2 = v/U = 0.1; x_3 = r\,L/U = 0; x_4 = l = 2.5; x_5 = g = 0; x_6 = w = 0.$$

Figure 10 shows a number of time domain simulations, also starting from the desired equilibrium position by means of a small perturbation in the tranversal relative velocity *v*.

Notice, however, that if supercritical bifurcation occurs, i.e. condition (ia) transforms into (ib), stability around the bifurcated equilibrium position breaks down whenever $l > U^2$, i.e. if $\overline{y} \to \pm p/2$ (otherwise equation (90) would have no real roots). This fact can be also interpreted as a loss of structural stability, regarding the system equations. Looking at Figure 11 we see that this situation happens for $\left(x_p/L\right)^* \cong 0.28$, as could be predicted, and now $\overline{y} = p$ is the new atractor.

In fact, from (91) and (92), under the condition $l = U^2$, we get from table 3,

$$\left(\frac{x_p}{L}\right)^* = \frac{(N_{vvv} + 6N_v)}{\left(Y_{vvv} + 6Y_v\right)} = \frac{(0.00611) + 6\times(-0.0105)}{(-0.045) + 6\times(-0.0261)} = 0.2822\,,$$

confirming the numerical simulations (see Figure 12). Table 5 must then be corrected, in order to account for the loss of stability of supercritical pitchfork bifurcations. This is shown in Table 6. Figure 13 refers to equation (93), showing, as in Papoulias & Bernitsas, 1988, $\overline{y}$ as a function of x_p/L.

As mentioned before the present stability analysis, concerning the desirable equilibrium position $\overline{y} = 0$**,** does not depend on the type of mooring system used, if SPM or 'Turret'. Figures 14 and 15 show the time domain simulations for the corresponding 'Turret' cases. Conditions are the same. Stability behavior is unchanged, as predicted but, the time histories, although similar, are not. Particularly, for tanker 2 a limit-cycle (Hopf bifurcation), around the atractor $\overline{y} = p$, is got for $x_p/L = 0.28$.

Table 6. Stability Scenarios for the Tanker (2) concerning $\overline{y} = 0$. Considering 'unstable' supercritical bifurcations

	stability parameters	**bifurcation scenario**	**situation**
$x_p/L < -0.1358$	$a > 0; l_c > 0$	(ii.a)	unstable
$-0.1358 < x_p/L < 0.2822$	$a > 0; l_c > 0$ $l > U^2$	(i.b) unstable	'unstable' supercritical bifurcation
$0.2822 < x_p/L < 0.4023$	$a > 0; l_c > 0$ $l < U^2$	(i.b)stable	'stable' supercritical bifurcation
$x_p/L > 0.4023$	$a > 0; l_c < 0$	(i.a)	stable

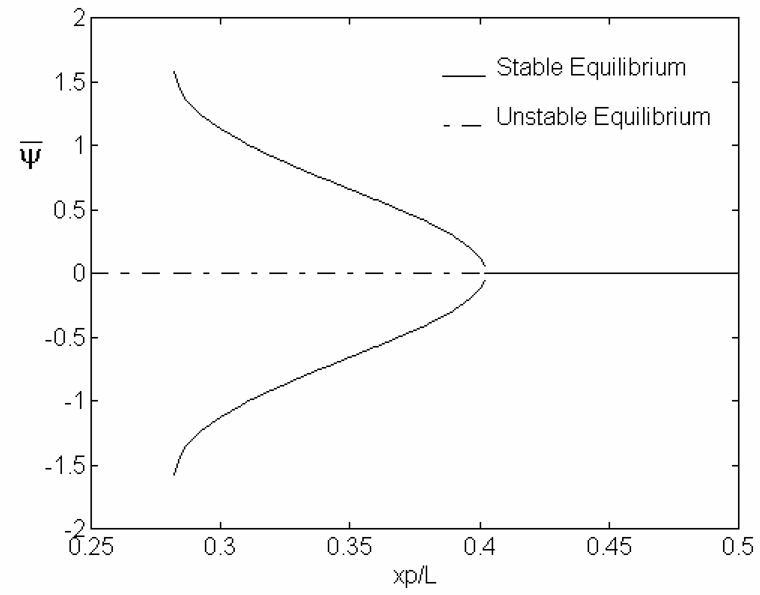

Figure 16 - Diagram of Supercritical Pitchfork Bifurcation for tanker (2)

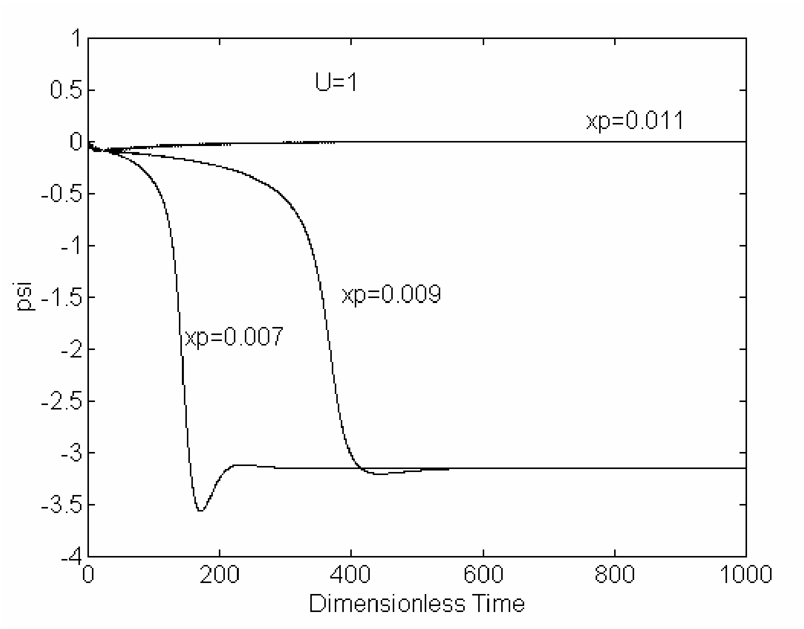

Figure 17 - Subcritical Pitchfork Bifurcations for tanker (1) in 'Turret'; x_p/L as a parameter. Dimensionless time $t' = tU/L$. Initial conditions:

$x_1 = u/U = 1; x_2 = v/U = 0.1; x_3 = r\,L/U = 0; x_4 = x/L = -x_p/L; x_5 = y/L = 0; x_6 = y = 0.$

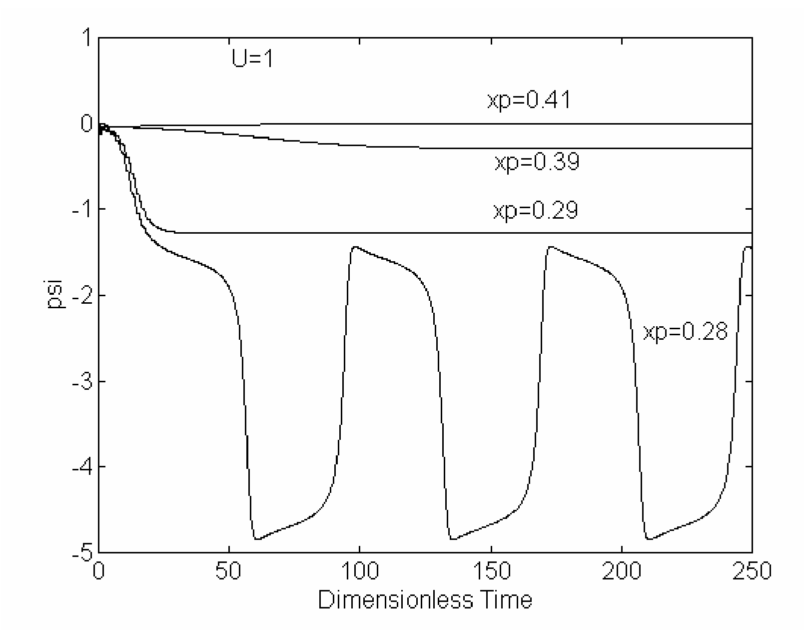

Figure 18 - Supercritical Pitchfork Bifurcations for tanker (2) in 'Turret'; x_p/L as a parameter. Dimensionless time $t' = tU/L$. Initial conditions:

$x_1 = u/U = 1; x_2 = v/U = 0.1; x_3 = r\,L/U = 0; x_4 = x/L = -x_p/L; x_5 = y/L = 0; x_6 = y = 0.$

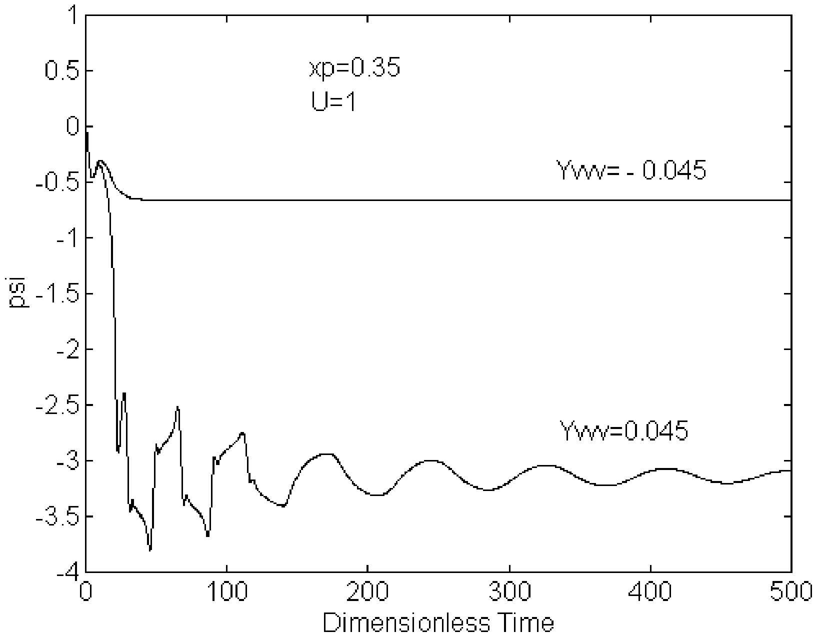

Figure 19 - Effect of Y_{vvv} sign in the Pitchfork Bifurcations Scenario for tanker (2) in SPM; $x_p/L = 0.35$. Dimensionless time $t' = tU/L$. Initial conditions:
$x_1 = u/U = 1; x_2 = v/U = 0.1; x_3 = r\,L/U = 0; x_4 = l = 2.5; x_5 = g = 0; x_6 = w = 0.$

Figure 20 exemplifies as the sign of the third-order derivative Y_{vvv} may change the whole picture. If sign is reversed such that $Y_{vvv} = 0.045$, *the supercritical scenario*, for which condition (i.a) $(a > 0; l_c < 0)$ is now written $(N_v/Y_v < x_p/L < N_{vvv}/Y_{vvv})$, *will not be possible anymore*. In fact, now $N_v/Y_v = 0.0105/0.0261 = 0.4023$ and $N_{vvv}/Y_{vvv} = 0.0061/0.045 = 0.1358$. Hence, we have $N_{vvv}/Y_{vvv} < N_v/Y_v$, such that (i.a) cannot hold. Instead, subcritical pitchfork bifurcation scenario will appear, conversely. Explicitly, condition (iib) $(a < 0; l_c > 0)$ or $(x_p/L > N_{vvv}/Y_{vvv}; x_p/L > N_v/Y_v)$, would correspond to a 'marginal stability' around $\bar{y} = 0$, and condition (iia) $(a < 0; l_c < 0)$, or $(N_{vvv}/Y_{vvv} < x_p/L < N_v/Y_v)$ (corresponding to be $\bar{\psi} = 0$ an unstable fixed point) applies if $0.1358 < x_p/L < 0.4023$. Notice, however, that this is a local analysis and gives no information concerning the existence or not of other atractors and if structural stability is preserved.

Finally, figure 21 shows time-histories corresponding to tanker (2), for different values of hawser cable length, taking $x_p/L = 0.7$ (the attatchement point on a bridge at the bow). According to table 7, $y = 0$ is a stable fixed point. Neverthless, a Hopf Bifurcation may appear, leading to a steady limit cycle. This behavior can depend also on hawser cable length. A "dynamic behaviour map", as shown, for example in Papoulias & Bernitsas, 1988, could be constructed, describing as a function of x_p/L and l_w/L, the kind of dynamic behavior that migth be expected.

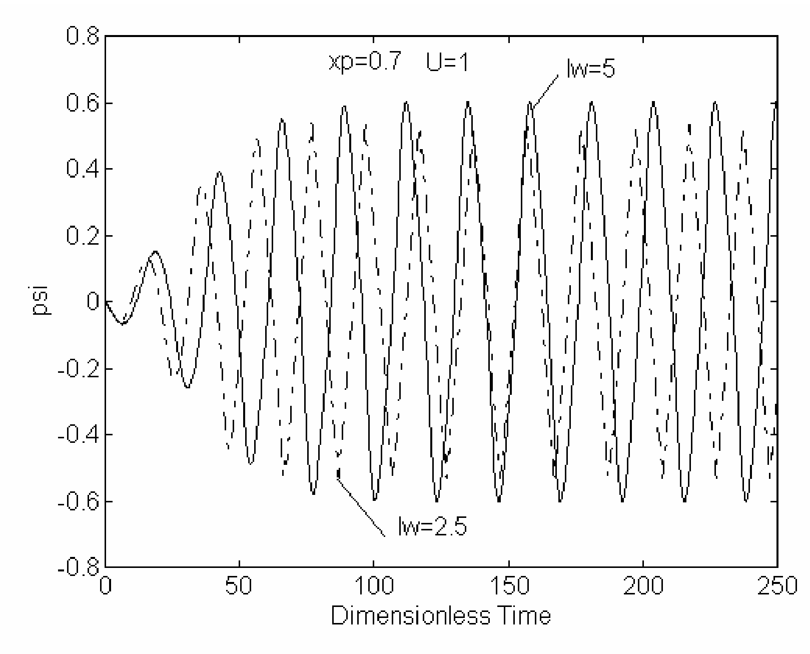

Figure 22 - Hopf Bifurcations (limit cycles) for tanker (2) in SPM; l_w/L as a parameter. Dimensionless time $t' = tU/L$. $X_u = -0.001$; $x_p/L = 0.7$. Initial conditions:

$x_1 = u/U = 1; x_2 = v/U = 0.1; x_3 = r\,L/U = 0; x_4 = l_w/L; x_5 = g = 0; x_6 = w = 0.$

CONCLUSIONS

Stability analysis of Single Point (SPM) and 'Turret' Mooring Systems for Floating Production Storage and Offloading Systems, under the action of steady ocean currents, has been performed by means of bifurcation theory. Theoretical results previously published by Papoulias & Bernitsas, 1988, have been recovered and discussed. The most important result is the necessary condition for stability, governed by the longitudinal position of the attachement point. Stability analysis has been enlarged somewhat, showing how bifurcation theory applied to a third-order model, based on the standard "hydrodynamic derivatives" type, is able to qualify two distinct equilibria bifurcation scenarios of the super- and sub-critical pitchfork type. *It has been shown that the sign of the third-order hydrodynamic derivative of lateral force with respect to the lateral component of relative velocity governs the type of bifurcation scenario*. Usually third-order hydrodynamic coefficientes are small, and experimental errors can easily lead to changes in algebraic sign, conducting to a totally different bifurcation scenario.

Additionally, when super-critical pitchfork bifurcation scenario is present a condition for structural stability loss has been stabilished and discussed as well. Finally, some practical examples, taking two tankers of different size, moored either in SPM or in 'Turret' configurations have been presented, ilustrating the features predicted by bifurcation analysis, through a number of time-domain simulations, performed with a fifth-order Runge-Kutta integration scheme. Some examples, concerning the appearance of Hopf bifurcations have also been presented and discussed.

Acknowledgements

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Appendix A

In fact, using **Erro! Vínculo não válido.**, **Erro! Vínculo não válido.** and (9) under equilibrium assumption, equations (19) can be written,

$$\begin{aligned} X(\overline{u},\overline{v},\overline{r}) + m\overline{r}\overline{v} + \overline{T}\cos\overline{w} = 0 \\ Y(\overline{u},\overline{v},\overline{r}) - m\overline{r}\overline{u} - \overline{T}\,\mathrm{sen}\,\overline{w} = 0 \\ N(\overline{u},\overline{v},\overline{r}) - Tx_p\,\mathrm{sen}\,\overline{w} = 0 \end{aligned} \quad . \qquad (94)$$

The second and third equations are then combined, leading to

$$x_p \; \{ \; Y(\overline{u},\overline{v},\overline{r}) \; - \; m\,\overline{r}\,\overline{u} \; \} \; = \; N(\overline{u},\overline{v},\overline{r}) \quad . \qquad (95)$$

As equilibrium is assumed, the angular velocity is null $(\overline{r} = 0)$. Taking the hydrodynamic derivatives up to third order, we then have, at equilibrium,

$$x_p \; \{ \; Y_v\overline{v} \; + \; \frac{1}{6}Y_{vvv}\overline{v}^3 \} \; = \; N_v\overline{v} \; + \; \frac{1}{6}N_{vvv}\overline{v}^3 \quad . \qquad (96)$$

Appendix B

Table B1. Nondimensionalizers Parameters.

Type of Variable	**(A) According to Takashina, 1986**	**(B) According to Bernitsas & Kekridis, 1985**
velocity	U	U
length	L	L
mass (*)	0.5 ρ L^2 T	0.5 ρ L^3
inertia of mass	0.5 ρ L^4 T	0.5 ρ L^5
force	0.5 ρ L T U^2	0.5 ρ L^2 U^2

(*) T is the draft.