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Effect of the Second-Order Potential in the Slow-Drift Oscillation of a Floating Structure in Irregular Waves

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The slow-drift phenomenon is important when the waves are irregular and the sea spectrum has a relatively narrow band. In this paper an expression is derived for the low-frequency force due to the second-order potential. This expression is the leading-order contribution in the wave spectrum bandwidth and can be exactly determined without computing the second-order potential. It is shown that this effect is of importance when the water depth is relatively shallow or the typical wave period relatively long.

1. Introduction

SLOW-DRIFT OSCILLATION of a moored structure in irregular waves may be an important problem whenever the restoring forces are small. This condition often arises for the motions in the horizontal plane of a moored ship or even for the vertical motions in a semisubmersible vessel with small waterplane area.

The origin of this phenomenon is as follows (see, for instance, reference [1][2]): Nonlinear interaction, in irregular waves, induce exciting forces at low frequency. If the floating structure has a small restoring force in one of its six degrees of freedom, the corresponding natural period is relatively long. So the nonlinear low-frequency forces, although of small magnitude, can excite large motions due to the amplification provided by resonance. Former investigations have shown that this nonlinear phenomenon is, in many cases, the dominating one in determining maximum mooring line tensions [2].

We recognize here three conditions for the existence of the slow-drift oscillation, namely:

(*a*) Small restoring force in some of the six degrees of freedom.
(*b*) Irregular (nonharmonic) waves.
(*c*) $\Delta\omega$, the nondimensional bandwidth of the sea spectrum, should be relatively small.

This last condition is quite common in the field of sea waves and its necessity can be easily understood. In fact the forces at low frequencies are spread over a range of frequencies of order $0 \leq \omega/\bar{\omega} \leq 0(\Delta\omega)$, where $\bar{\omega}$ is the average frequency of the sea. To excite resonance, $\omega \approx \omega_n$, where ω_n is the natural frequency of the system. Then $\omega_n/\bar{\omega} \sim 0(\Delta\omega)$ and the following estimates can be used: The vertical motion of a semisubmersible has, typically, a natural period around 30 sec and the horizontal motion of a moored ship has a natural period around 80 sec. If the average period of the sea is around 8 sec, then $\Delta\omega \approx 0.25$ to excite resonance in the semisubmersible or $\Delta\omega \approx 0.1$ for the ship. In both cases the bandwidth is relatively smaller than one.

Once the physical origin of the phenomenon is recognized, it is certainly desirable to derive a theory that allows one to predict the low-frequency oscillation. The most difficult point is to compute the exciting forces, since they are nonlinear, and it is with this task that the literature in the field is mainly concerned. We present next a brief overview of several approaches used by different researchers.

The nonlinear forces at low frequencies are the result of two distinct components: one is related to the second-order effect of the first-order (linear) potential and the other is the effect of the second-order potential.

Newman [3] has argued that the leading-order contribution, in $\Delta\omega$, is due to the first-order potential. More than that, he shows that this contribution can be approximated by the mean drift force in harmonic waves multiplied by the amplitude modulation of the sea spectrum. Newman's result is reviewed in the present paper (see Section 8). Pinkster [4] dedicates most of his paper deriving an exact way to compute the effect of the first-order potential. He analyzes the case where the water depth is arbitrary but deals with the influence of the second-order potential in a crude way. He argues that this effect is important when the waves are long and diffraction is small, and from this he computes only the second-order potential due to the incoming wave. As we are going to see, his assumption is only partially correct. Faltinsen and Løkens analyze the case of an infinitely long horizontal cylinder in a beam sea (strip theory) in water of infinite depth, but they compute exactly the effect of the second-order potential. By comparing their results with the ones derived by Newman, they conclude that the approximation introduced by this author is good enough for practical application.

It is important to keep in mind the amount of numerical work involved in these last two studies. For instance, if we assume that the sea is approximated by a sum of (n) harmonic waves, Pinkster computes $\frac{1}{2}(n^2 + n)$ nonlinear interactions among the linear potentials and Faltinsen and Løkens, besides this, also compute $\frac{1}{2}(n^2 + n)$ second-order potentials associated with combinations of possible low frequencies. This tremendous amount of numerical work, together with the fact that the importance of the second-order potential is weak in deep water, has certainly played a role in the decision, common among all these research efforts, to substantially or even totally disregard this effect.

Ships, however, are usually moored in relatively shallow water and, as it is well known, the second-order potential becomes more important the shallower the water is. The results of the research work quoted above are inconclusive in this case. Newman disregards from the outset the second-order potential. Pinkster, although analyzing relatively shallow water, does so only in a crude way and Faltinsen and Løkens analyze only deep water. The main objective of this paper is to study, in a consistent and relatively easy way, the effect of the second-order potential, irrespective of the water depth.

The starting point is to take note of the fact that $\Delta\omega$, the sea spectrum bandwidth, is "small." A consistent asymptotic theory in this small parameter is derived and an expression for the low-

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[2] Numbers in brackets designate References at end of paper.

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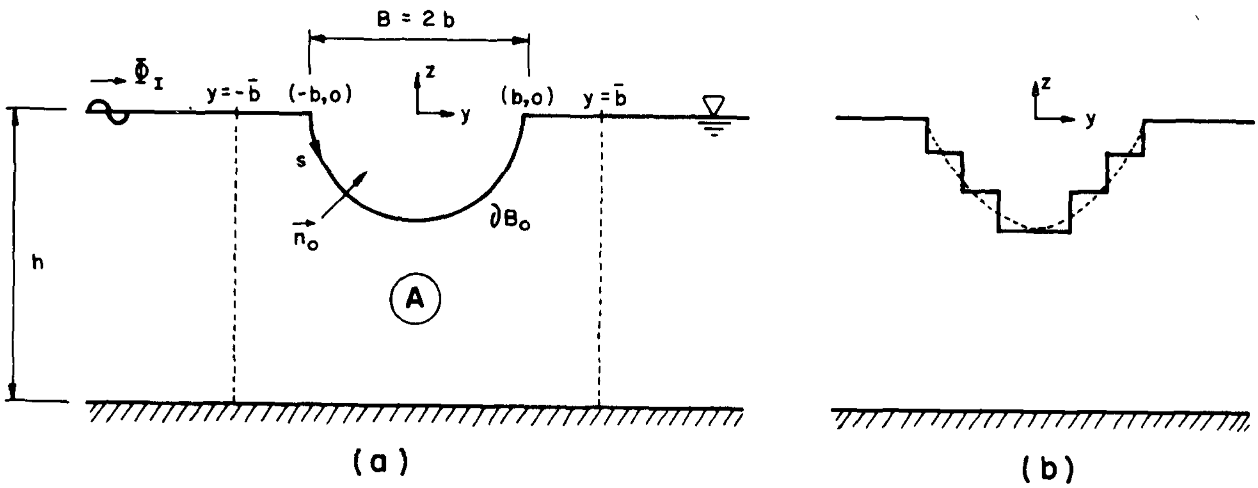

Fig. 1 Geometric definitions

frequency exciting force is then obtained. In this paper, as in Faltinsen and Løkens, only the case of an infinitely long cylinder in a beam sea is analyzed, but the results are rewarding. In fact if $\hat{Q}_{K,LF}(t)$ is the exciting force in the mode K ($K = 1$, sway; $K = 2$, heave; $K = 3$, roll), then its asymptotic expression can be written as

$$\hat{Q}_{K,LF}(t) = \{\rho g \bar{A} B \cdot \delta \cdot [Q^{(1)}_{K,LF}(t) + Q^{(2)}_{K,LF}(t)]\}\,[1 + 0(\Delta\omega^2;\delta)]$$

where ρ = water density, g = acceleration of gravity, $\bar{A}$ = average wave amplitude, B = beam of ship, $\delta = \bar{A}/B$ = small-amplitude parameter, $\Delta\omega$ = nondimensional bandwidth [see equation (79)], and

$Q^{(1)}_{K,LF}(t)$ = effect, in low frequency, of first-order potential

$Q^{(2)}_{K,LF}(t)$ = effect, in low frequency, of second-order potential

The expression for $\hat{Q}_{K,LF}(t)$ is correct to leading order in δ and to second order in $\Delta\omega$. If $\Delta\omega \approx 0.25$, then the error is of order 6 percent, which is generally quite reasonable. Furthermore, the functions $Q^{(1)}_{K,LF}(t)$; $Q^{(2)}_{K,LF}(t)$ can be written as

$$Q^{(1)}_{K,LF}(t) = \bar{Q}_k(\bar{\omega}) \cdot |F(t)|^2 + \left[P_k(\bar{\omega}) \cdot F(t)\,\frac{1}{\omega}\,\frac{dF^*}{dt}(t) + P^*_k(\bar{\omega}) \cdot F^*(t)\,\frac{1}{\omega}\,\frac{dF}{dt}(t)\right]$$

$$Q^{(2)}_{K,LF}(t) = Q^{(k)}_{20}(\bar{\omega})\,\frac{1}{\bar{\omega}}\,\frac{d}{dt}\,(|F(t)|^2)$$

where (*) stands for the complex conjugate, $F(t)$ is the amplitude modulation, and $\{\bar{Q}_k(\bar{\omega}); P_k(\bar{\omega}); Q^{(k)}_{20}(\bar{\omega})\}$, $k = 1,2,3$ are coefficients that can be computed with basic knowledge of the linear harmonic potential in the average frequency $\bar{\omega}$.

Clearly the amount of numerical work is comparable to that for solving the linear problem, and is, by far, much less demanding than the method of either Pinkster or Faltinsen and Løkens. Furthermore the expressions for these coefficients are relatively simple, and they provide insight into the relative importance between the two factors when the water becomes shallow. It is worthwhile, here, to emphasize one important point: The value of $Q^{(k)}_{20}(\bar{\omega})$ can be computed without solving any second-order problem.

2. The nonlinear problem

Let δ be the small parameter associated with the wave amplitude and $\Phi(y,z,t)$ the potential correct to second order in δ.

There are two sources of nonlinearities: the free surface and the body boundary condition. If $v(t)$, $w(t)$, and $\theta(t)$ are the generalized displacement in sway, heave, and roll, the function $\Phi(y,z,t)$ must satisfy the following set of equations:

(i) $$\nabla^2\Phi = 0 \left(\nabla^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

(ii) $$\frac{1}{g}\cdot\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial\Phi}{\partial z}\bigg|_{z=0} = -\frac{1}{g}\left[2\cdot\nabla\Phi\cdot\nabla\left(\frac{\partial\Phi}{\partial t}\right) - \frac{1}{g}\cdot\frac{\partial\Phi}{\partial t}\cdot\frac{\partial}{\partial z}\left(\frac{\partial^2\Phi}{\partial t^2} + g\cdot\frac{\partial\Phi}{\partial z}\right)\right]_{z=0}$$

(iii) $$\frac{\partial\Phi}{\partial z}\bigg|_{z=-h} = 0$$

(iv) $$\begin{aligned}\nabla\Phi\cdot\vec{n}_0|_{\partial B_0} &= n_{y,0}\cdot\left(\frac{dv}{dt} - z\cdot\frac{d\theta}{dt}\right) + n_{z,0}\left(\frac{dw}{dt} + y\cdot\frac{d\theta}{dt}\right)\\ &+\left[\theta(t)\cdot\left(\frac{dw}{dt} + y\cdot\frac{d\theta}{dt}\right) - [w(t) + y\cdot\theta(t)]\cdot\frac{\partial}{\partial z}\left(\frac{\partial\Phi}{\partial y}\right)\right]n_{y,0}\\ &-\left[\theta(t)\cdot\left(\frac{dv}{dt} - z\cdot\frac{d\theta}{dt}\right) + [v(t) - z\cdot\theta(t)]\cdot\frac{\partial}{\partial y}\left(\frac{\partial\Phi}{\partial z}\right)\right]n_{z,0}\\ &-\frac{1}{2}\cdot\frac{d}{dt}[\theta^2(t)]\cdot(y\cdot n_{y,0} + z\cdot n_{z,0})\\ &+\left[-\theta(t)\cdot\frac{\partial\Phi}{\partial z} + [v(t) - \theta(t)\cdot z]\cdot\frac{\partial^2\Phi}{\partial z^2}\right]\cdot n_{y,0}\\ &+\left[\theta(t)\cdot\frac{\partial\Phi}{\partial z} + [w(t) + \theta(t)\cdot y]\cdot\frac{\partial^2\Phi}{\partial y^2}\right]\cdot n_{z,0}\end{aligned}$$

(v) Radiation condition (1)

In equation (1), $\vec{n}_0 = n_{y,0}\cdot\vec{j} + n_{z,0}\cdot\vec{k}$ is the normal to the cross section ∂B_0, as indicated in Fig. 1, where ∂B_0 is the rest position of the body. The radiation condition will be explicitly stated later in the paper, but it is important to keep in mind that Φ is the distortion of a free wave (or train of waves) propagating from left to right.

The nonlinear term at the free surface is given by Newman [5], but the derivation of (1)(iv) will be omitted in this paper. The generalized displacements $\{v(t); w(t); \theta(t)\}$ can be determined from the equation of the body's motion, as shown in Section 4.

Expression (1)(iv) contains second derivatives of Φ, in a way not very convenient for the purposes of this work. It is desirable to transform it to an equivalent expression, whose deduction is indicated below. In fact let (s) be the *curvilinear coordinate*, as shown in Fig. 1(a), and suppose that the cross section is approximated by a stepwise contour line; see Fig. 1(b). The size of the steps can be arbitrarily small and we can easily see, in this case, that

(a) If $|n_{y,0}| = 1$, then $n_{z,0} = 0$ and $n_{y,0} \cdot \frac{\partial}{\partial z} = -\frac{\partial}{\partial s}$

(b) If $|n_{z,0}| = 1$, then $n_{y,0} = 0$ and $n_{z,0} \cdot \frac{\partial}{\partial y} = \frac{\partial}{\partial s}$

Using these relations in (1)(iv), we obtain

$$\nabla\phi \cdot \vec{n}_0|_{\partial B_0} = n_{y,0} \cdot \left(\frac{dv}{dt} - z \cdot \frac{d\theta}{dt}\right) + n_{z,0} \cdot \left(\frac{dw}{dt} + y \cdot \frac{d\theta}{dt}\right) + \frac{d}{dt}\left[\theta(t) \cdot \left(w(t) + \frac{1}{2} \cdot y \cdot \theta(t)\right) \cdot n_{y,0} - \theta(t) \cdot \left(v(t) - \frac{1}{2} \cdot z \cdot \theta(t)\right) \cdot n_{z,0}\right] + \frac{\partial}{\partial s} \cdot \left\{\frac{\partial \Phi}{\partial s}\left[(v(t) - z \cdot \theta(t)) \cdot n_{y,0} + (w(t) + y \cdot \theta(t)) \cdot n_{z,0}\right]\right\} \quad (2)$$

It is not difficult[3] to extend this demonstration for an arbitrary cross section. As is clear, this can be done by approximating ∂B_0 by a stepwise contour line and then letting the size of the steps go to zero. Details will not be given here, but it can be shown that (2) is valid for all ∂B_0 that define a "regular" fluid region. The convenience of (2) will be realized later in this work.

It is important to compare the orders of magnitude of the *nonlinear correction* and the *linear term*. If $\Phi = \Phi_1 + \Phi_2 + \ldots$, then, from the boundary condition at the free surface, we get

$$\partial\Phi_1/\partial z \sim 0\left(\frac{\omega^2}{g} \cdot \Phi_1\right)$$

$$\partial\Phi_2/\partial z \sim 0\left[\frac{\omega}{g} \cdot \left(\frac{\partial\phi_1}{\partial y}\right)^2\right]$$

From mass conservation

$$\frac{\partial\Phi_1}{\partial y} \sim 0\left(\frac{1}{\tanh K_0 h} \cdot \bar{\omega}\bar{A}\right)$$

where $\bar{\omega}^2/g = K_0 \cdot \tanh K_0 h$, $\bar{A}$ = wave amplitude (average), and $\bar{\omega}$ = wave frequency (average). Thus

$$\frac{\Phi_2}{\Phi_1} \sim 0\,(\delta_0)$$

$$\delta_0 = \frac{K_0 \bar{A}}{\tanh K_0 h} \quad (3)$$

and the wave is said to be of small amplitude when $\delta_0 \ll 1$.

3. Nondimensional variables

The problem can be linearized, or else the solution can be written as an asymptotic series, if the wave amplitude and body motion are both small. The first condition implies $\delta_0 \ll 1$ and the second $\delta \ll 1$ where

$$\delta = \frac{A}{B} \quad (4)$$

In (3), B is the typical dimension of the cross section (see Fig. 1) and A is the typical value of the wave amplitude. Notice also that the condition $\delta \ll 1$ is essential for the potential theory, since only then the influence of flow separation and vorticity can be disregarded.

In this work, we shall suppose the cross section to be given, although the wave frequency (ω) and the water depth (h) can change. It is convenient, then, to take δ as the small parameter and the role played by δ_0 will be discussed when needed. In this way we introduce the following scales

- length scale = B
- time scale = $\omega_B^{-1} = \sqrt{B/g}$
- scale for body displacement = A
- scale for angle of rotation = A/B
- scale for potential = gA/ω_B
- pressure scale = $\rho g A$
- force scale = $\rho g A B$ (unit of length)

From here on, the dimensional quantities will be designated by a (̂). Then

$$t = \omega_B \hat{t}$$

$$(y;z) = \frac{1}{B}(\hat{y};\hat{z})$$

$$h = \hat{h}/B$$

$$\Phi(y,z,t) = \frac{\omega_B}{gA} \cdot \hat{\Phi}(\hat{y},\hat{z},\hat{t}) \quad (5)$$

where the quantities on the left are nondimensional. The generalized displacements are written as

$$q_1(t) = \frac{1}{A} \cdot v(\hat{t}) \qquad \text{(sway)}$$

$$q_2(t) = \frac{1}{A} \cdot w(\hat{t}) \qquad \text{(heave)} \quad (6)$$

$$q_3(t) = \frac{B}{A} \cdot \theta(\hat{t}) \qquad \text{(roll)}$$

Correct to second order, we write then

$$\Phi(y,z,t) = \delta \cdot \Phi_1(y,z,t) + \delta^2 \cdot \Phi_2(y,z,t) + \ldots$$

$$q_k(t) = \delta \cdot q_k^{(1)}(t) + \delta^2 \cdot q_k^{(2)}(t) + \ldots \qquad (k = 1,2,3) \quad (7)$$

Placing (6) into (1) and (2), one obtains, after separating terms of like order in δ:

(a) *Linear problem*

(i) $\nabla^2\Phi_1 = 0$

(ii) $\left.\frac{\partial^2\Phi_1}{\partial t^2} + \frac{\partial\Phi_1}{\partial z}\right|_{z=0} = 0$

(iii) $\left.\frac{\partial\Phi_1}{\partial z}\right|_{z=-h}$

(iv) $\nabla\Phi_1 \cdot \vec{n}_0|_{\partial B_0} = \left(\frac{dq_1^{(1)}}{dt} - z \cdot \frac{dq_3^{(1)}}{dt}\right) \cdot n_{y,0} + \left(\frac{dq_2^{(1)}}{dt} + y \cdot \frac{dq_3^{(1)}}{dt}\right) \cdot n_{z,0}$

(v) Radiation condition (8)

Again the radiation condition will be specified later.

[3] This could eventually be done by recalling the Cauchy-Riemann equations and the conjugate stream function, to relate normal and tangential derivatives.

(b) *Nonlinear correction*

(i) $\nabla^2 \Phi_2 = 0$

(ii) $$\left.\frac{\partial^2 \Phi_2}{\partial t^2} + \frac{\partial \Phi_2}{\partial z}\right|_{z=0} = -\left[2\nabla\phi_1 \cdot \nabla\left(\frac{\partial \Phi_1}{\partial t}\right) - \frac{\partial \Phi_1}{\partial t}\cdot\frac{\partial}{\partial z} \cdot \left(\frac{\partial^2 \Phi_1}{\partial t^2} + \frac{\partial \Phi_1}{\partial z}\right)\right]_{z=0}$$

(iii) $$\left.\frac{\partial \Phi_2}{\partial z}\right|_{z=-h} = 0$$

(iv) $$\nabla \Phi_2 \cdot \vec{n}_0|_{\partial B_0} = \left(\frac{dq_1^{(2)}}{dt} - z\cdot\frac{dq_3^{(2)}}{dt}\right)\cdot n_{y,0} + \left(\frac{dq_2^{(2)}}{dt} + y\cdot\frac{dq_3^{(2)}}{dt}\right)\cdot n_{z,0} + \frac{d}{dt}\left[q_3^{(1)}(t)\cdot\left(q_2^{(1)}(t) + \frac{1}{2}yq_3^{(1)}(t)\right)\cdot n_{y,0} - q_3^{(1)}(t)\cdot\left(q_1^{(1)}(t) - \frac{1}{2}zq_3^{(1)}(t)\right)\cdot n_{z,0}\right] + \frac{\partial}{\partial s}\left\{\frac{\partial \Phi_1}{\partial s}\cdot\left[\left(\frac{dq_1^{(1)}}{dt} - z\cdot q_3^{(1)}(t)\right)\cdot n_{y,0} + \left(\frac{dq_2^{(1)}}{dt} + y\cdot q_3^{(1)}(t)\right)\cdot n_{z,0}\right]\right\}$$

(v) Radiation condition (9)

In (8)(iv) we have used (2), and the nonlinear radiation condition deserves special consideration. From Bernoulli's equation the pressure is given by

$$p(y,z,t) = -\frac{\partial \Phi_1}{\partial t} - z + \delta\cdot\left[\frac{\partial \Phi_2}{\partial t} + \frac{1}{2}(\nabla\Phi_1)^2\right] + \ldots \quad (10)$$

and the free-surface displacement $\eta(y,t)$ can be written as

$$\eta(y,t) = \delta\cdot\eta_1(y,t) + \delta^2\cdot\eta_2(y,t) + \ldots$$

$$\eta_1(y,t) = -\frac{\partial \Phi_1}{\partial t} \quad (11)$$

From (3), (4) and (5) we obtain here

$$\frac{\Phi_2}{\Phi_1} \sim 0\left(\frac{K_0}{\tanh K_0 h}\right) \to 0\left(\frac{1}{h}\right) \quad \text{when } K_0 h \to 0 \quad (12)$$

where K_0 is the typical wave number of the sea spectrum ($\bar{\omega}^2 = K_0 \cdot \tanh K_0 h$; $\bar{\omega}$ = average frequency). Expression (12) shows that the effect of the second-order term increases as the water becomes shallow.

4. Linear solution—harmonic waves

For a narrow-band spectrum the leading contribution in $(\Delta\omega)$ is harmonic. It is thus natural to analyze the properties of this solution, which will be done in the following four sections. In Section 8 we will review the problem of nonharmonic excitation.

We intend to discuss briefly here the linear solution for a harmonic wave with frequency (ω). The dispersion relation

$$\omega^2 = K_0 \cdot \tanh K_0 h \quad (13)$$

is correct with an error factor of the form $[1 + 0(\delta_0)^2]$. The (linear) incident wave is given by

$$\Phi_I(y,z,t) = \frac{1}{2}[\phi_I(y,z)\cdot e^{-i\bar{\omega}t} + (*)] \quad (14)$$

$$\phi_I(y,z) = -\frac{i}{\omega}\cdot\frac{\cosh K_0(z+h)}{\cosh K_0 h}\cdot e^{iK_0 y} \quad (14)$$

(cont'd)

where (*) means the complex conjugate of the expression between brackets. If

$$v_k(y,z) = \begin{cases} = n_{y,0} & \text{when } K = 1 \text{ (sway)} \\ = n_{z,0} & \text{when } K = 2 \text{ (heave)} \\ = -z\cdot n_{y,0} + y\cdot n_{z,0} & \text{when } K = 3 \text{ (roll)} \\ = -(\nabla\Phi_I\cdot\vec{n}_0)|_{\partial B_0} & \text{when } K = 4 \text{ (diffraction)} \end{cases} \quad (15)$$

then let $\phi_k(y,z)$, $k = 1,2,3,4$, be the solution of the problem

(i) $\nabla^2\phi_k = 0$

(ii) $$\left.\frac{\partial \phi_k}{\partial z}\right|_{z=0} = \omega^2\phi_k|_{z=0}$$

(iii) $$\left.\frac{\partial \phi_k}{\partial z}\right|_{z=-h} = 0$$

(iv) $\nabla\phi_k\cdot\vec{n}_0|_{\partial B_0} = v_k(y,z)$

(v) $$\phi_k(y,z) \sim -\frac{i}{\omega}\cdot\frac{\cosh K_0(z+h)}{\cosh K_0 h}\cdot\begin{Bmatrix}T_k\\R_k\end{Bmatrix}\cdot e^{iK_0|y|} \text{ when } y \to \pm\infty \quad (16)$$

Equation (16) can be solved by the Hybrid Element Method (see [6]). In the region $|y| \leq \bar{b}$, $\phi_k(y,z)$ is obtained numerically and for $|y| \geq \bar{b}$ it is given by

$$\phi_k(y,z) = -\frac{i}{\omega}\cdot\frac{\cosh K_0(z+h)}{\cosh K_0 h}\cdot\begin{Bmatrix}T_k\\R_k\end{Bmatrix}\cdot e^{iK_0|y|} + \sum_{n=1}^{\infty} A_{n,k}^{\pm}\cdot f_n(z)\cdot e^{-K_n(|y|-\bar{b})}$$

for

$$y \gtrless \pm\bar{b} \quad (17)$$

where

$$f_n(z) = F_n\cdot\cos K_n(z+h) \qquad \omega^2 = -K_n\cdot\tan K_n h$$

$$F_n^2 = \frac{1}{h}\cdot\frac{4K_n h}{2K_n h + \sin 2K_n h}; \qquad \int_{-h}^{0} f_n^2(z)dz = 1 \quad (18)$$

In (17) the coefficients T_k, R_k and $A_{n,k}^{\pm}$ can be determined in the following way: Once $\phi_k(y,z)$ is numerically computed in the region $|y| \leq \bar{b}$, then

$$T_k = -\frac{4\cdot K_0\omega\cdot\cosh K_0 h}{2K_0 h + \sinh 2K_0 h}\cdot\int_{-h}^{0}\phi_k(+\bar{b},z)\cdot\cosh K_0(z+h)\cdot dz$$

$$R_k = -\frac{4\cdot K_0\omega\cdot\cosh K_0 h}{2K_0 h + \sinh 2K_0 h}\cdot\int_{-h}^{0}\phi_k(-\bar{b},z)\cdot\cosh K_0(z+h)\cdot dz$$

$$A_{n,k}^{\pm} = \int_{-h}^{0}\phi_k(\pm\bar{b},z)\cdot f_n(z)\cdot dz \quad (19)$$

Details about the numerical computation of $\phi_k(y,z)$ in the region $|y| \leq \bar{b}$ can be found in [6].

The excitation forces are

$$Q_k^{(e)} = i\omega \int_{\partial B_0} (\phi_I(y,z) + \phi_4(y,z)) \cdot v_k(y,z) \cdot d\partial B_0 \quad (20)$$

and the hydrodynamic coefficients are given by

$$m_{\ell j}^{(a)} = \int_{\partial B_0} (\text{Real } \phi_\ell(y,z)) \cdot v_j(y,z) \cdot d\partial B_0$$

$$d_{\ell j}^{(a)} = \int_{\partial B_0} \omega(\text{Imag } \phi_\ell(y,z)) \cdot v_j(y,z) \cdot d\partial B_0$$

$$(\ell, j = 1,2,3) \quad (21)$$

where $m_{\ell j}^{(a)} = m_{j\ell}^{(a)}$ and $d_{\ell j}^{(a)} = d_{j\ell}^{(a)}$ are elements of the added mass and radiation damping matrices, respectively $[M_a] = [m_{\ell j}^{(a)}]$; $[D_a] = [d_{\ell j}^{(a)}]$. If $[M]$, $[D_v]$, and $[K]$ are the inertial mass, viscous damping, and restoring forces matrices, the harmonic generalized displacement, $\{q_{k,h}\}$, is the solution of the system

$$[-\omega^2([M] + [M_a]) - i\omega([D_v] + [D_a]) + [K]] \cdot \{q_{k,h}\} = \{Q_k^{(e)}\} \quad (22)$$

Once determined $q_{k,h}$, $k = 1,2,3$, we define

$$T = 1 + T_4 - i\omega \sum_{k=1}^{3} q_{k,h} \cdot T_k$$

$$R = R_4 - i\omega \sum_{k=1}^{3} q_{k,h} \cdot R_k \quad (23)$$

$$A_n^\pm = A_{n,k}^\pm - i\omega \sum_{k=1}^{3} q_{k,h} \cdot A_{n,k}^\pm$$

Then

$$\phi_L(y,z) = \phi_I(y,z) + \phi_4(y,z) - i\omega \sum_{j=1}^{3} q_{k,h} \cdot \phi_k(y,z)$$

$$\eta_L(y) = i\omega\phi_L(y,0) \quad (24)$$

where for $y \gtrless \pm\bar{b}$

$$\phi_L(y,z) = -\frac{i}{\omega} \cdot \frac{\cosh K_0(z+h)}{\cosh K_0 h} \cdot \begin{Bmatrix} Te^{iK_0 y} \\ e^{iK_0 y} + R \cdot e^{-iK_0 y} \end{Bmatrix} + \sum_{n=1}^{\infty} A_n^\pm \cdot f_n(z) \cdot e^{-K_n(|y| - \bar{b})}$$

$$\eta_L(y) = \begin{Bmatrix} Te^{iK_0 y} \\ e^{iK_0 y} + R \cdot e^{-iK_0 y} \end{Bmatrix} + i\omega \sum_{n=1}^{\infty} A_n^\pm \cdot f_n(0) \cdot e^{-K_n(|y| - \bar{b})} \quad (25)$$

With (25) the linear harmonic solution can be written as

$$\Phi_1(y,z,t) = \left[\frac{1}{2} \cdot \phi_L(y,z) \cdot e^{-i\omega t} + (*)\right]$$

$$q_k^{(1)}(t) = \left[\frac{1}{2} \cdot q_{k,h} \cdot e^{-i\omega t} + (*)\right] \quad (26)$$

$$\eta_1(y,t) = \left[\frac{1}{2} \cdot \eta_L(y) \cdot e^{-i\omega t} + (*)\right]$$

It is worthwhile to keep in mind that

$$\phi_L(y,z) = \phi_L(y,z;\omega);\ q_{k,h} = q_{k,h}(\omega);\ \eta_L(y) = \eta_L(y;\omega)$$

(ω) being the frequency.

5. Radiation potential in zero frequency

Once the linear solution for harmonic waves is derived, it is natural, in the context of this work, to study the nonlinear correction. It is convenient, however, that we postpone this discussion to the next section and address, here, another sort of problem. The relevance of this will become apparent a bit later.

We start by considering the equation of motion for the low-frequency oscillation, $\{q_{k,\mathrm{LF}}(t)\}$. As is clear from equation (22) we must solve, now, the system of differential equations

$$([M] + [M_a]_{\mathrm{LF}} \cdot \{\ddot{q}_{k,\mathrm{LF}}\} + ([D_v] + [D_a]_{\mathrm{LF}}) \cdot \{\dot{q}_{k,\mathrm{LF}}\} + [K] \cdot \{q_{k,\mathrm{LF}}\} = \delta \cdot \{Q_{k,\mathrm{LF}}(t)\} \quad (27)$$

where $[M]$; $[D_v]$ and $[K]$ are the inertial mass, viscous damping and restoring forces matrices; $[M_a]_{\mathrm{LF}}$ and $[D_a]_{\mathrm{LF}}$ are the added mass and radiation damping matrices in low frequency; $\delta \cdot \{Q_{k,\mathrm{LF}}(t)\}$ is the exciting force in low frequency.

In (27), $\dot{q} = dq/dt$ and the right-hand side show, explicitly, that the exciting forces are of order δ.

The matrices $[M]$ and $[K]$ have already been defined in Section 4, and the effect of viscous damping is small in low frequency. In fact if $F_v(t)$ is the dimensional viscous force, then it has typically the expression

$$F_v(t) = \frac{1}{2}\rho C_D B(\dot{v})^2 = \rho g B A \left(\frac{1}{2}\delta \cdot C_D \cdot (\dot{q}_{\mathrm{LF}})^2\right) = \rho g B A \cdot D_v \cdot \dot{q}_{\mathrm{LF}}$$

So

$$[D_v] \sim 0(\bar{\omega} \cdot \Delta\omega \cdot \delta \cdot A_{\mathrm{LF}}) \quad (28)$$

where A_{LF} is the amplitude of the low-frequency oscillation.

Since $[M] \sim 0(1)$, then $[D_v]$ is pretty small, unless A_{LF} is large. The smallness of the viscous damping effect in low frequency is one of the reasons why the amplitude of the slow drift oscillation can be large, in spite of the fact that the exciting force is of order (δ). This point will be discussed further at the end of this section.

The main objective of this work is to determine the low-frequency exciting force, $\{Q_{k,\mathrm{LF}}(t)\}$, but (27) shows that the matrices $[M_a]_{\mathrm{LF}}$ and $[D_a]_{\mathrm{LF}}$ must also be computed. These are the added mass and the radiation damping matrices in the range of frequencies $0 \le \omega/\bar{\omega} \le 0(\Delta\omega)$ where, again, ($\Delta\omega$) is the nondimensional bandwidth of the sea spectrum.

These matrices can be determined from (21) where now $\phi_k(y,z)$ is the solution of (16) when $\omega = \bar{\omega} \cdot \Delta\omega$. Consistent with the error $[1 + 0(\Delta\omega)^2]$ assumed in this work, the potential $\phi_k(y,z)$ can be approximated by $\phi_k^{(0)}(y,z)$, which is the solution of the equation [see (16) and assume $\omega = 0$]:

(i) $\nabla^2 \phi_k^{(0)} = 0$

(ii) $\left.\dfrac{\partial \phi_k^{(0)}}{\partial z}\right|_{z=0} = 0$

(iii) $\left.\dfrac{\partial \phi_k^{(0)}}{\partial z}\right|_{z=-h} = 0$

(iv) $\nabla \phi_k^{(0)} \cdot \vec{n}_0|_{\partial B_0} = v_k(y,z) \quad (k = 1,2,3)$

(v) $\phi_k^{(0)}(y,z) \sim [B_{0,k}^\pm + D_{0,k}^\pm(|y| - \bar{b})] \cdot \dfrac{1}{\sqrt{h}}$ when $y \to \pm\infty$ (29)

In (v) we have used the most general solution of (i), (ii), (iii) that is not exponentially growing when $|y| \to \infty$.

Equation (29) is the standard flow equation, but the values of the velocity at infinity, $D_{0,k}^\pm$, are unknown. Furthermore, if mass is conserved, the solution $\phi_k^{(0)}(y,z)$ exists but is not unique. In fact, if $\phi_k^{(0)}(y,z)$ is a solution, then $\phi_k^{(0)}(y,z) + C$ also is. In a flow problem the constant C is irrelevant, but here it plays a role. For instance the added mass in heave can now be written as

$$m_{22}^{(0)} = \int_{\partial B_0} \phi_2^{(0)}(y,z) \cdot n_{z,0} \cdot d\partial B_0 + (\text{Real } C) \cdot 2b$$

and so $m_{22}^{(0)}$ depends on the specific value of C.

This problem is well known, but there is a point that is worthwhile discussing: It will be seen later that the second-order potential at zero frequency, $\phi_{20}(y,z)$, is essential in the theory proposed here. It happens that $\phi_{20}(y,z)$ satisfies an equation similar to (29), but with a nonhomogeneous term at the free surface (see Section 6). So the solution for $\phi_{20}(y,z)$ is nonunique and the constant C will be important again. The determination of this value follows a reasoning very much the same as the one to be used here, and this motivates a close analysis of (29).

We start by introducing the linear functionals

$$V_k(\psi) = \int_{\partial B_0} v_k(y,z) \cdot \psi(y,z) \cdot d\partial B_0 \quad (k = 1,2,3) \tag{30}$$

where $\psi(y,z)$ belongs to a sufficiently broad class of functions (see Appendix 1).

If equation (29)(i) is multiplied by $\psi(y,z) = 1$ and integrated in the fluid region we obtain the *mass conservation equation*

$$D_{0,k}^{+} + D_{0,k}^{+} = -\frac{V_k(1)}{\sqrt{h}} \tag{31}$$

where from (15) and (30)

$$V_k(1) = L_w \cdot \delta_{k2}, \quad k = 1,2,3 \tag{32}$$

δ_{kj} being the Kronecker delta function and L_w the waterline of the body ($L_w = B$ in Fig. 1).

Problem (29) has a solution if and only if $D_{0,k}^{\pm}$ satisfy (31), although they can otherwise be arbitrary. Certainly we could invoke the symmetry of the cross section to write

$$D_{0,k}^{+} = D_{0,k}^{-} = -\frac{1}{2\sqrt{h}} \cdot (L_w) \cdot \delta_{k2} \quad (k = 1,2,3) \tag{33}$$

but it is convenient to consider them undetermined as yet.

It can be shown (see Appendix 1) that $\phi_k^{(0)}(y,z)$ can be expressed as

$$\phi_k^{(0)}(y,z) = \bar{\phi}_{k,R}^{(0)}(y,z) + (B_{0,k}^{+} + B_{0,k}^{-}) \cdot \frac{1}{2\sqrt{h}} + (B_{0,k}^{+} - B_{0,k}^{-}) \cdot p_A(y,z)(|y| \le \bar{b})$$

$$\phi_k^{(0)}(y,z) = [B_{0,k}^{\pm} + D_{0,k}^{\pm}(|y| - \bar{b})] \cdot g_0(z) + \sum_{n=1}^{\infty} L_n^{\pm}(\phi_k^{(0)}) \cdot g_n(z) \cdot e^{-\lambda_n(|y|-\bar{b})} \quad (y \gtrless \pm\bar{b}) \tag{34}$$

where $\bar{\phi}_{k,R}^{(0)}(y,z)$; $p_A(y,z)$ are well behaved functions, defined in the region $|y| \le \bar{b}$, and can be numerically computed.

In (34) also

$$g_0(z) = 1/\sqrt{h}$$

$$g_n(z) = (2/h)^{1/2} \cdot \cos \lambda_n(z + h); \ \lambda_n = \frac{n\pi}{h}$$

$$L_n^{\pm}(\psi) = \int_{-h}^{0} \psi(\pm\bar{b},z) \cdot g_n(z) \cdot dz \tag{35}$$

and $B_{0,k}^{\pm}$; $D_{0,k}^{\pm}$ are related by means of

$$\bar{G} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} B_{0,k}^{+} \\ B_{0,k}^{-} \end{bmatrix} = \begin{bmatrix} V_k(p+) \\ V_k(p-) \end{bmatrix} + \begin{bmatrix} D_{0,k}^{+} \\ D_{0,k}^{-} \end{bmatrix} \tag{36}$$

where $\bar{G}$ is a positive number defined in Appendix 1 and

$$p^{\pm}(y,z) = \frac{1}{2\sqrt{h}} \pm p_A(y,z) \quad (|y| \le \bar{b}) \tag{37}$$

The singular equation (36) has a solution if and only if mass is conserved [see (31) and notice that $V_k(p+ + p-) = (1/\sqrt{h})V_k(1)$]. If this is the case we can write

$$B_{0,k}^{\pm} = \frac{1}{2\bar{G}} \cdot V_k(p\pm) + \frac{1}{2\bar{G}} \cdot D_{0,k}^{\pm} + C \tag{38}$$

The values of C; $D_{0,k}^{\pm}$ are unknown and cannot be determined within the context of the mathematical problem. A physical argument is needed and, in this case, we recall that $\phi_k^{(0)}(y,z)$ intends to be the limit of $\phi_k(y,z)$, the solution of (16), when $\omega \to 0$. Since

$$\phi_k(y,z) \sim A_0^{\pm}(\omega) \cdot e^{iK_0(|y|-\bar{b})} \cdot f_0(z) \quad \text{when } y \to \pm\infty \tag{39}$$

where

$$f_0(z) = \left(\frac{1}{h} \cdot \frac{4K_0h}{2K_0h + \sinh 2K_0h}\right)^{1/2} \cdot \cosh K_0(z + h)$$

then, for $\omega \ll 1$, the amplitude $A_0^{\pm}(\omega)$ is determined from the equation (see (36) and reference [6])

$$\begin{bmatrix} \bar{G} - i \cdot K_0 & -\bar{G} \\ -\bar{G} & \bar{G} - iK_0 \end{bmatrix} \cdot \begin{bmatrix} A_0^{+}(\omega) \\ A_0^{-}(\omega) \end{bmatrix} = \begin{bmatrix} V_k(p+) \\ V_k(p-) \end{bmatrix} \quad (\omega = K_0\sqrt{h})$$

and so

$$A_0^{\pm}(\omega) = \frac{i}{2} \cdot \frac{1}{K_0\sqrt{h}} \cdot V_k(1) + \frac{1}{2\bar{G}} \cdot V_k(p\pm) \tag{40}$$

Now $f_0(z) \to g_0(z)$ when $\omega \to 0$ and from (39) it follows

$$\phi_k(y,z) \sim [A_0^{\pm}(\omega) + i \cdot K_0 \cdot A_0^{\pm}(\omega)(|y| - \bar{b})] \cdot g_0(z)$$

when $\omega \to 0$. Using (40) and looking to (34) we obtain

$$D_{0,k}^{\pm} = -\frac{1}{2\sqrt{h}} \cdot V_k(1)$$

$$B_{0,k}^{\pm} = \frac{1}{2\bar{G}} \cdot V_k(p\pm) \tag{41}$$

$$\lim_{\omega \to 0} [\omega \cdot \text{Imag}(\phi_k(y,z))] = \frac{1}{2\sqrt{h}} \cdot V_k(1)$$

Expression (41) for $D_{0,k}^{\pm}$ coincides with (33) [see (32)]. Also $C = -(1/2\bar{G}) \cdot D_{0,k}^{\pm}$ is the proper value of the constant, and from (21)

$$m_{\ell j}^{(0)} = \int_{\partial B_0} \phi_\ell^{(0)}(y,z) \cdot v_j(y,z) \cdot d\partial B_0$$

$$d_{\ell j}^{(0)} = \lim_{\omega \to 0} (d_{\ell j}^{(a)}(\omega)) = \frac{1}{2\sqrt{h}} \cdot V_\ell(1) \cdot V_j(1) \quad (\ell,j = 1,2,3) \tag{42}$$

where $V_k(1)$ is defined in (32).

From (42) we obtain

$$[M_a]_{\text{LF}} = [m_{\ell j}^{(0)}](1 + 0(\Delta\omega)^2)$$

$$[D_a]_{\text{LF}} = [d_{\ell j}^{(0)}](1 + 0(\Delta\omega)^2) \tag{43}$$

The radiation potential at zero frequency, $\phi_k^{(0)}(y,z)$, must be always computed in order to determine the low-frequency oscillations. As we are going to see in Section 8, we will also use these potentials to compute the exciting forces in low frequency, due to the effect of the second-order potential. We close this section with an analysis of the order of magnitude of the slow-drift oscillation. We first make the following observation: For a body floating on the free surface, like a ship, the slow-drift phenomenon is important in sway, since only in this mode is the restoring force small. But then from (43), (42) and (32), $d_{11}^{(0)} = d_{13}^{(0)} = d_{33}^{(0)} = 0$ and so $[D_a]_{LF}$ can be taken as zero in (27). If the cross section is totally submerged, as the cross section of the semisubmersible is, the slow drift can be important in heave and roll, but now $d_{\ell j}^{(0)} = 0$ since L_w, the waterline, is zero. It follows that $[D_a]_{LF}$ can always be taken zero in (27), and if $A_{LF}(\Omega)$ is the amplitude of the harmonic response due to the input $Q_{k,LF}(t) = \exp(i \cdot \Omega \cdot \bar{\omega} \cdot \Delta\omega \cdot t)$, where $\Omega \sim 0(1)$, then from (27) we obtain

$$[(-\bar{\omega}^2 \cdot (\Delta\omega)^2 \cdot \Omega^2 \cdot M + K) + i \cdot \bar{\omega} \cdot \Omega \cdot \Delta\omega \cdot (C_v \cdot \bar{\omega} \cdot \Delta\omega \cdot \delta \cdot A_{LF})] \cdot A_{LF}(\Omega) = \delta$$

In the above expression M is the total mass, of order 1, K is the small restoring coefficient, and $C_v \cdot \Delta\omega \cdot \bar{\omega} \cdot \delta \cdot A_{LF}$, with $C_v \sim 0(1)$, is the viscous damping; see (28).

As has been said before, the sea spectrum will excite resonance if $\bar{\omega} \cdot \Delta\omega$ is such that $K \sim 0(\bar{\omega} \cdot \Delta\omega)^2$. Writing, in this case, $K = \bar{\omega}^2 \cdot (\Delta\omega)^2 \cdot \ddot{K}$, $\ddot{K} \sim 0(1)$, we obtain

$$[(\ddot{K} - \Omega^2 M) + i \cdot \Omega C_v \delta \cdot A_{LF}] \cdot A_{LF}(\Omega) = \frac{\delta}{(\bar{\omega} \cdot \Delta\omega)^2}$$

If $\Omega_R = (\ddot{K}/M)^{1/2}$ is the resonant frequency, then $\Omega_R \sim 0(1)$ and

(*a*) $A_{LF}(\Omega) \sim 0(\delta/(\bar{\omega}\Delta\omega)^2)$, when $|\Omega - \Omega_R| \sim 0(1)$
(*b*) $A_{LF}(\Omega) \sim 0(1/(\bar{\omega}\Delta\omega))$, when $|\Omega - \Omega_R| \ll 1$

if $K \sim 0(\bar{\omega}\Delta\omega)^2$. If K is small but larger than $(\Delta\omega \cdot \bar{\omega})$, the response is quasi-static and $A_{LF} \sim 0(\delta/K)$.

To get an idea about the numbers let us assume that

$B = 20$ m

$\bar{\omega} = \omega_B = (g/B)^{1/2}$ ($\bar{T}$ = average period of sea $\simeq 9$ sec)

$\Delta\omega = 0.20$ (T_n = natural period of ship $\simeq 45$ sec)

The amplitude of the low-frequency oscillation will be given by $\hat{q}_{LF} = \bar{A} \cdot A_{LF}$, where $\bar{A}$ is the average amplitude of the waves and

$A_{LF} \approx 1.25$ if $|\Omega - \Omega_R| \sim 0(1)$ ($\bar{A} \simeq 1$ m; $\delta = 1/20$)
$A_{LF} \approx 5$ if $|\Omega - \Omega_R| \ll 1$

This order-of-magnitude analysis shows, quite clearly, the importance of the low-frequency oscillation.

6. Nonlinear correction—harmonic waves

Once the linear solution $\{\Phi_1(y,z,t;\omega);\ q_k^{(1)}(t;\omega)\}$ [see (25)] is known, one can determine the nonlinear correction $\Phi_2(y,z,t;\omega)$ solution of (8).

The term $\left(\dfrac{dq_1^{(2)}}{dt} - z \cdot \dfrac{dq_3^{(2)}}{dt}\right) \cdot n_{y,0} + \left(\dfrac{dq_2^{(2)}}{dt} + y \cdot \dfrac{dq_3^{(2)}}{dt}\right) \cdot n_{z,0}$

in (8)(iv) is associated with the radiation problems and contributes to the added mass and radiation damping matrices, as has been shown in Section 4. The excitation of Φ_2 is provided by the remaining terms in (8)(iv) and (8)(ii). Since the linear solution is harmonic we obviously have (RHS = right-hand side)

$$\text{RHS of (8)(ii)} = L_{20}(y) + [L_{22}(y) \cdot e^{-2i\omega t} + (*)]$$
$$\text{RHS of (8)(iv)} = B_{20}(y,z) + [B_{22}(y,z) e^{-2i\omega t} + (*)]$$

where RHS of (8)(iv) means the remaining terms of this boundary condition. In the above expression [see (26)]

$$L_{20}(y) = \frac{\omega}{4}\frac{\partial}{\partial y}\left[i \cdot \phi_L(y,0) \cdot \frac{\partial \phi_L^*}{\partial y}\right]$$

$$B_{20}(y,z) = \frac{1}{4}\frac{\partial}{\partial s}\left\{\frac{\partial \phi_L^*}{\partial s}[(q_{1,H} - z \cdot q_{3,H}) \cdot n_{y,0} + (q_{2,H} + y \cdot q_{3,H}) \cdot n_{z,0}] + (*)\right\} \quad (44)$$

The functions $L_{22}(y)$ and $B_{22}(y,z)$ can also be computed but they will not be used in this work. From (44) and (25) we obtain

$$L_{20}(y) \sim 0(e^{-K_1 \cdot |y|}) \quad \text{when } |y| \to \infty \quad (45)$$

The potential $\Phi_2(y,z,t)$ can also be written as

$$\Phi_2(y,z,t) = \Phi_{20}(y,z) + [\Phi_{22}(y,z) \cdot e^{-2i\omega t} + (*)] \quad (46)$$

and since we are interested in low frequency, only $\phi_{20}(y,z)$ will be needed. From (8) it follows

(i) $\nabla^2 \phi_{20} = 0$

(ii) $\left.\dfrac{\partial \phi_{20}}{\partial z}\right|_{z=0} = L_{20}(y)$

(iii) $\left.\dfrac{\partial \phi_{20}}{\partial z}\right|_{z=-h} = 0$

(iv) $\nabla \phi_{20} \cdot \vec{n}_0|_{\partial B_0} = B_{20}(y,z)$

(v) $\phi_{20}(y,z) \sim [A_{20}^{\pm} + U_{20}^{\pm}(|y| - \bar{b})] \cdot \dfrac{1}{\sqrt{h}}$ when $y \to \pm\infty$ (47)

The radiation condition (47)(v) is equivalent to (29)(v), since $L_{20}(y) \to 0$ when $|y| \to \infty$; see (45). The values of $A_{20}^{\pm}$; $U_{20}^{\pm}$ are, as yet, unknown and we postpone the discussion of (47) to the next section. Here we will derive a useful expression and later we will recall some results associated with mean values.

It is important, in order to deduce the mass conservation equation and other equalities, to compute the integral

$$\mathrm{I}(\psi) = -\iint_{A_\infty} \nabla \phi_{20} \cdot \nabla \psi \cdot dA_\infty \quad (48)$$

where A_∞ is the entire fluid region. The function $\psi(y,z)$ will be restricted to the class where $(\nabla\psi)^2$ is Lebesgue integrable and such that

$$\psi(y,z) \sim [B^{\pm} + D^{\pm}(|y| - \bar{b})] \cdot \frac{1}{\sqrt{h}} + 0(e^{-\beta|y|}) \quad \text{when } y \to \pm\infty \quad (49)$$

where $\beta > 0$ and $B^{\pm}$; $D^{\pm}$ can take any value.

From (47) and (49) it follows that

$$\mathrm{I}(\psi) = -\int_F L_{20}(y) \cdot \psi(y,0) \cdot dy - \int_{\partial B_0} B_{20}(y,z) \cdot \psi(y,z) \cdot d\partial B_0 - [U_{20}^{+}(B^{+} - \bar{b} \cdot D^{+}) + U_{20}^{-} \cdot (B^{-} - \bar{b} \cdot D^{-})] - [U_{20}^{+} \cdot D^{+} + U_{20}^{-} \cdot D^{-}] \cdot |y|$$

The integrals over the free surface F and ∂B_0 can be simplified. In fact if we assume, for a while, that $n_{z,0} = 0$ when ($y = \pm b$; $z = 0$) (see Fig. 2), then

$$\frac{\partial \phi_L}{\partial y}(\pm b;0) = i\omega q_{1,H} \quad \text{[see (8)]}$$

$$\frac{\partial \phi_L}{\partial s}(\pm b;0) = \frac{\partial \phi_L}{\partial z}(\pm b;0) = \pm\omega^2 \cdot \phi_L(\pm b;0)$$

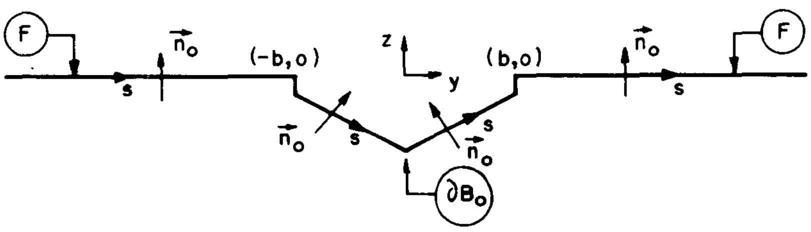

Fig. 2 Contour $C = F \cup \partial B_0$

$$n_{y,0}(\pm b;0) = \mp 1$$

$$\frac{\partial \phi_L}{\partial s}(\pm b;0) \cdot n_{y,0}(\pm b;0) = -\omega^2 \cdot \phi_L(\pm b;0)$$

Placing these relations and (44) in $I(\psi)$ we obtain, after integrating by parts

$$I(\psi) = \frac{K_0}{2\omega}(1 - |R|^2) \cdot [B^- + D^-(|y| - \bar{b})] \cdot \frac{1}{\sqrt{h}}$$
$$- \frac{K_0}{2\omega} \cdot |T|^2 \cdot [B^+ + D^+(|y| - \bar{b})] \cdot \frac{1}{\sqrt{h}}$$
$$+ \frac{1}{2} \cdot \text{Real}\left\{ i\omega \int_F \phi_L(y,0) \cdot \frac{\partial \phi_L^*}{\partial y}(y,0) \cdot \frac{\partial \psi}{\partial y}(y;0) \cdot dy \right.$$
$$+ \int_{\partial B_0} [(q_{1,H} - z \cdot q_{3,H}) \cdot n_{y,0} + (q_{2,H} + y \cdot q_{3,H}) \cdot n_{z,0}]$$
$$\left. \cdot \frac{\partial \phi_L^*}{\partial s} \cdot \frac{\partial \psi}{\partial s} \cdot ds \right\} - [U_{20}^+(B^+ - \bar{b} \cdot D^+) + U_{20}^- \cdot (B^- - \bar{b} \cdot D^-)]$$
$$- [U_{20}^+ \cdot D^+ + U_{20}^- \cdot D^-] \cdot |y|$$

where R and T are the reflection and transmission coefficients; see (23). The above equality can be easily generalized to sections that do not cross the free surface at a right angle, but this will not be pursued here. Notice that the integrands in F and ∂B_0 have a similar expression. In fact

$$i\omega \cdot \phi_L(y,0) = -\frac{1}{i\omega} \cdot (\nabla \phi_L \cdot \vec{n}_0)_{(y,z)\epsilon F}$$

$$(q_{1,H} - z \cdot q_{3,H}) \cdot n_{y,0}$$
$$+ (q_{2,H} + y \cdot q_{3,H}) \cdot n_{z,0} = -\frac{1}{i\omega}(\nabla \phi_L \cdot \vec{n}_0)_{(y,z)\epsilon \partial B_0}$$

So

$$i\omega \int_F [\ldots]\, dy + \int_{\partial B_0} [\ldots]\, d\partial B_0$$
$$= \frac{i}{\omega} \int_C (\nabla \phi_L \cdot \vec{n}_0)_C \cdot \frac{\partial \phi_L^*}{\partial s} \cdot \frac{\partial \psi}{\partial s} \cdot ds$$

where C is the line indicated in Fig. 2.

The integral $I(\psi)$ is, in general, divergent, as the term that increases with $|y|$ indicates. It is important, however, to distinguish the convergent and divergent parts and for this we define

$$J(\psi) = \frac{1}{2} \cdot \text{Real}\left\{ i\omega \int_{F_A} \phi_L(y,0) \cdot \frac{\partial \phi_L^*}{\partial y}(y,0) \cdot \frac{\partial \psi}{\partial y}(y,0) dy \right.$$
$$+ \int_{\partial B_0} [(q_{1,H} - z \cdot q_{3,H}) \cdot n_{y,0} + (q_{2,H} + y \cdot q_{3,H}) \cdot n_{z,0}] \times$$
$$\cdot \frac{\partial \phi_L^*}{\partial s} \cdot \frac{\partial \psi}{\partial s} \cdot ds + i\omega \int_{F^+} \phi_L(y,0)$$
$$\cdot \frac{\partial \phi_L^*}{\partial y}(y,0) \cdot \left[\frac{\partial \psi}{\partial y}(y,0) - \frac{D^+}{\sqrt{h}} \right] \cdot dy$$
$$+ i\omega \int_{F^-} \phi_L(y,0) \cdot \frac{\partial \phi_L^*}{\partial y}(y,0) \cdot \left[\frac{\partial \psi}{\partial y}(y,0) + \frac{D^-}{\sqrt{h}} \right] \cdot dy$$
$$+ i\omega \cdot \frac{D^+}{\sqrt{h}} \cdot \int_{F^+} \left[\phi_L \cdot \frac{\partial \phi_L^*}{\partial y} + \frac{i \cdot K_0}{\omega^2} \cdot |T|^2 \right] dy$$
$$\left. - i\omega \cdot \frac{D^-}{\sqrt{h}} \cdot \int_{F^-} \left[\phi_L \cdot \frac{\partial \phi_L^*}{\partial y} - \frac{iK_0}{\omega^2}(1 - |R|^2) \right] dy \right\} \quad (50)$$

where F_A; $F^\pm$ are the free surfaces in the regions $|y| \leq \bar{b}$; $y \gtrless \pm \bar{b}$, respectively. Notice that $J(\psi)$ is convergent in the class (49) and so $I(\psi)$ can be written as

$$I(\psi) = J(\psi) - (B^+ - \bar{b} \cdot D^+) \cdot \left[U_{20}^+ + \frac{K_0}{2\omega\sqrt{h}} \cdot |T|^2 \right]$$
$$- (B^- - \bar{b} \cdot D^-) \cdot \left[U_{20}^- - \frac{K_0}{2\omega\sqrt{h}}(1 - |R|^2) \right]$$
$$- \left[U_{20}^+ + \frac{K_0}{2\omega\sqrt{h}} \cdot |T|^2 \right] \cdot D^+ \cdot |y|$$
$$- \left[U_{20}^- - \frac{K_0}{2\omega\sqrt{h}}(1 - |R|^2) \right] \cdot D^- \cdot |y|$$
$$- \left[D^- \cdot \frac{K_0}{2\omega\sqrt{h}}(1 - |R|^2) - D^+ \cdot \frac{K_0}{2\omega\sqrt{h}} \cdot |T|^2 \right] (|y| - \bar{b}) \quad (51)$$

Expression (51) will be used in the next section. But first, we shall quote some well-known results related to means values.

(*a*) Mean drift forces

Suppose that $B(t)$ is the wetted surface of the body at time (t), with normal $\vec{n}(t) = n_y(t) \cdot \vec{j} + n_z(t) \cdot \vec{k}$. Let $v_k(y,z;t)$, $k = 1,2,3$, be the expression defined in (15), with $\vec{n}(t)$ in place of $\vec{n}_0$.

The generalized force acting at time (t) is given by

$$Q_k(t) = \left[\int_{\partial B(t)} p(y,z,t) \cdot v_k(y,z,t) \cdot d\partial B(t) \right.$$
$$\left. + \int_{\partial B_0} z \cdot v_k(y,z) \cdot d\partial B_0 \right]$$

where $p(y,z,t)$ is defined in (10) and

$$\left[-\int_{\partial B_0} z \cdot v_k(y,z) \cdot d\partial B_0 \right]$$

is the hydrostatic generalized force.

Since the wave is harmonic, there exists a second-order component of $Q_k(t)$ that is constant in time. All other terms are harmonic, with frequencies ω or 2ω, and their mean values over a period are zero. We can write then, correct to second order in (δ)

$$Q_k(t) = (Q_{k,L} \cdot e^{-i\omega t} + (*)) + \delta \cdot [\bar{Q}_k + (Q_{k,22} \cdot e^{-2i\omega t} + (*))] \tag{52}$$

where $Q_{k,L}$ is the coefficient of the linear force and

$$\bar{Q}_k = \bar{Q}_k(\omega) = \text{coefficient of mean drift force} \tag{53}$$

Notice that

$$\frac{\omega}{2\pi} \int_0^{2\pi/\omega} Q_k(t) \cdot dt = \delta \cdot \bar{Q}_k \cdot (1 + 0(\delta))$$

The values of $\bar{Q}_k(\omega)$ can be computed directly, as shown in Appendix 3. In sway, however, we can use the conservation of linear momentum to derive the simpler relation (see Maruo [7])

$$\bar{Q}_1(\omega) = \frac{1}{4} \cdot \left[1 + \frac{2K_0 h}{\sinh 2K_0 h} \right] \cdot (1 + |R|^2 - |T|^2) \tag{54}$$

(*b*) Mass transport

It is well known that the nonlinearity induces a mass transport. In fact the instantaneous mass flow is given by

$$\begin{aligned} M^{\pm}(t) &= \lim_{y \to \pm\infty} \left[\int_{-h}^{\eta(y,t)} \frac{\partial \phi}{\partial y} \cdot dz \right] \\ &= \delta \cdot (M_1^{\pm} \cdot e^{-i\omega t} + (*)) + \delta^2 \\ &\quad \cdot [\bar{M}^{\pm} + (M_{22}^{\pm} e^{-2i\omega t} + (*))] + \ldots \end{aligned}$$

where $\delta^2 \cdot \bar{M}^{\pm}$ is the mean value of $M^{\pm}(t)$ over a period and

$$\bar{M}^+ = \sqrt{h} \cdot \left[U_{20}^+ + \frac{K_0}{2\sqrt{h}} \cdot |T|^2 \right]$$

$$\bar{M}^- = -\sqrt{h} \cdot \left[U_{20}^- - \frac{K_0}{2\omega\sqrt{h}} (1 - |R|^2) \right] \tag{55}$$

In (55), $U_{20}^{\pm}$ are the asymptotic values of $\pm\partial\phi_{20}/\partial y$ when $y \to \pm\infty$; see (47)(v).

From mass conservation we should certainly have that $\bar{M}^+ + \bar{M}^- = 0$. But then

$$U_{20}^+ + U_{20}^- = \frac{K_0}{2\omega\sqrt{h}} (1 - |R|^2 - |T|^2) \tag{56}$$

Equation (56) stipulates a relation between U_{20}^+ and U_{20}^- and it has been derived by directly invoking mass conservation. We can, however, deduce this relation from the mathematical equation (47). In fact, from the equality $\mathrm{I}(1) = 0$ [see (48)], we obtain [use (50); (51) with $B^{\pm} = \sqrt{h}$; $D^{\pm} = 0$; $\psi(y,z) = 1$]

$$\mathrm{I}(1) = -\sqrt{h} \left[(U_{20}^+ + U_{20}^-) - \frac{K_0}{2\omega\sqrt{h}} (1 - |R|^2 - |T|^2) \right]$$

and so $\mathrm{I}(1) = 0$ if and only if (56) is satisfied.

Relation (56) is similar to (31). Both are related to mass conservation and are necessary conditions for the existence of a solution for equations (29) and (47), respectively.

In Section 5 we invoked symmetry to induce (33), and only later was this identity mathematically demonstrated; see (41). Note that the argument of symmetry was imposed, on physical grounds, on the mathematical problem and could be derived mathematically only when we considered $\phi_k^{(0)}(y,z)$ as the limit of $\phi_k(y,z)$ when $\omega \to 0$.

A similar procedure will be followed now. First a straightforward physical argument will be used to induce the values of $U_{20}^{\pm}$ and only later on, when we consider $\phi_{20}(y,z)$ as the limit of the second-order potential in low frequency, will these relations be mathematically deduced.

The physical argument here is that we expect $\bar{M}^+ = \bar{M}^- = 0$ since, otherwise, we would get a net flow, from right to left, and whose source is nowhere. From this condition we obtain

$$U_{20}^+ = -\frac{K_0}{2\omega\sqrt{h}} \cdot |T|^2$$

$$U_{20}^- = \frac{K_0}{2\omega\sqrt{h}} \cdot (1 - |R|^2) \tag{57}$$

With (57) the expression for $\mathrm{I}(\psi)$ can be written as

$$\mathrm{I}(\psi) = J(\psi) - [D^+ \cdot U_{20}^+ + D^- \cdot U_{20}^-] \cdot (|y| - \bar{b}) \tag{58}$$

7. Force coefficients and radiation condition for $\phi_{20}(y,z)$

We will show in the next section (Section 8) that the numbers

$$Q_{20}^{(k)} = -\omega \int_{\partial B_0} \phi_{20}(y,z) \cdot v_k(y,z) \cdot d\partial B_0; \; k = 1,2,3 \tag{59}$$

represent, asymptotically, the coefficient of the generalized forces in low frequency, due to the effect of the second-order potential.

We will see, next, that $Q_{20}^{(k)}$ can be computed without determining $\phi_{20}(y,z)$. This is a trivial extension of Haskind relations. In fact from (29)(i)

$$\begin{aligned} 0 &= \iint_{A_\infty} \nabla^2 \phi_k^{(0)} \cdot \phi_{20} \cdot dA_\infty \\ &= \iint_{A_\infty} \nabla \cdot (\nabla \phi_k^{(0)} \cdot \phi_{20}) dA_\infty - \iint_{A_\infty} \nabla \phi_k^{(0)} \cdot \nabla \phi_{20} \cdot dA_\infty \end{aligned}$$

From the boundary conditions of (29), (47)(v), and (44) we obtain

$$\begin{aligned} -\iint_{\partial B_0} \phi_{20} \cdot v_k \cdot d\partial B_0 &= \mathrm{I}(\phi_k^{(0)}) + [D_{0,k}^+ \cdot U_{20}^+ + D_{0,k}^- \cdot U_{20}^-] \\ &\quad \cdot (|y| - \bar{b}) + (A_{20}^+ \cdot D_{0,k}^+ + A_{20}^- \cdot D_{0,k}^-) \end{aligned}$$

Using (58) and recalling that $D^{\pm} = D_{0,k}^{\pm}$ when $\psi(y,z) = \phi_k^{(0)}(y,z)$ [see (49) and (29)(v)] we get

$$Q_{20}^{(k)}(\omega) = \omega \cdot J(\phi_k^{(0)}) - \frac{\omega L_w}{2\sqrt{h}} \cdot \delta_{k2} \cdot (A_{20}^+ + A_{20}^-) \tag{60}$$

where δ_{kj} is the Kronecker delta function and we have used (33).

In equation (60), $\phi_k^{(0)}(y,z)$ is the radiation potential in zero frequency; $J(\cdot)$ is the convergent integral (50).

As in Section 5 the values of the constant $A_{20}^{\pm}$ affect the heave coefficient and they can be determined only if we give a more precise physical meaning to $\phi_{20}(y,z)$. For this purpose, suppose we have two waves with frequencies $\omega \pm (\Delta\omega/2)$ where $\Delta\omega \ll 1$. The nonlinear interaction will introduce the frequencies $\{0; \Delta\omega; 2\omega - \Delta\omega; 2\omega + \Delta\omega\}$ and, in the slow-drift phenomenon, we are interested solely in the nonlinear term in the frequency $(\Delta\omega)$. If $\Phi_2(y,z;\Delta\omega t)$ is such a potential, we define

$$\phi_{20}(y,z) = \lim_{\Delta\omega \to 0} \Phi_2(y,z;\Delta\omega t) \tag{61}$$

Condition (61) is similar to one used in Section 5

$$\phi_k^{(0)}(y,z) = \lim_{\omega \to 0} \phi_k(y,z;\omega)$$

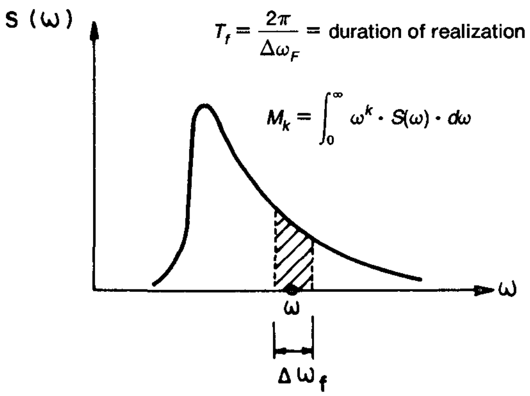

Fig. 3 Amplitude spectrum $S(\omega)$

and it is convenient to consider the pure harmonic wave as if it were the limit of a narrow-band spectrum when the bandwidth goes to zero. From (61) we can compute $A_{20}^{\pm}$; $U_{20}^{\pm}$ (see Appendix 2), and it turns out that $U_{20}^{\pm}$ are given by (57). In other words, the no-net-flow condition has now been deduced from the mathematical equations, once we consider the pure harmonic wave as the limit defined above.

From Appendix 2 we obtain

$$A_{20}^{+} + A_{20}^{-} = \frac{1}{\sqrt{h}} \left[\int_{\bar{b}}^{\infty} d\xi \int_{\xi}^{\infty} L_{20}(y) dy + \int_{-\infty}^{-\bar{b}} d\xi \cdot \int_{-\infty}^{\xi} L_{20}(y) dy \right] \tag{62}$$

With this relation the coefficients $Q_{20}^{(k)}(\omega)$ can be computed at once. In fact the integrals in the region $y \gtrless \pm\bar{b}$ [$F \pm$ in (50)] can be analytically computed, if we use the series expansion (25); (34). The integrals in the region $|y| \leq \bar{b}$ [F_A and ∂B_0 in (50)] must be numerically determined, but this is easy since only the values of the tangent velocity $\{\partial\phi_L/\partial s;\ \partial\phi_k^{(0)}/\partial s\}$ are needed.

In sway the expression for Q_{20} has a simple form. In fact

$$Q_{20}^{(1)}(\omega) = \frac{\omega}{2} \cdot \text{Real} \left\{ i\omega \int_F \phi_L(y,0) \cdot \frac{\partial \phi_L^*}{\partial y}(y,0) \cdot \frac{\partial \Phi_1^{(0)}}{\partial y}(y,0) dy + \int_{\partial B_0} [(q_{1,H} - z \cdot q_{3,H}) \cdot n_{y,0} + (q_{2,H} + y \cdot q_{3,H}) n_{z,0}] \cdot \frac{\partial \phi_L^*}{\partial s} \cdot \frac{\partial \phi_1^{(0)}}{\partial s} \cdot ds \right\} \tag{63}$$

or

$$Q_{20}^{(1)}(\omega) = \frac{1}{2} \cdot \text{Real} \left\{ i \cdot \int_C (\nabla\phi_L \cdot \vec{n}_0) \cdot \frac{\partial \phi_L^*}{\partial s} \cdot \frac{\partial \phi_1^{(0)}}{\partial s} \cdot ds \right\} \tag{64}$$

where C is the line indicated in Fig. 2.

We close this section with an important observation. The values $Q_{20}^{(k)}(\bar{\omega})$ can be explicitly determined in the limit of shallow water or long waves ($K_0h \rightarrow 0$). If we consider, as we have done in this work, that the motion in sway is not restrained when solving the linear problem—which is quite reasonable since the average frequency $\bar{\omega}$ is, in general, much larger than the resonant frequency in sway—then in the limit as $K_0h \rightarrow 0$ we can easily see that the linear solution is given by [see (14), (22), (24)]

$$\begin{aligned} \phi_L(y,z;\bar{\omega}) &\simeq \phi_I(y,z;\bar{\omega}) = -\frac{i}{\omega} \cdot \frac{\cosh K_0(z+h)}{\cosh K_0h} \cdot e^{iK_0y} \\ q_{1,h} &\simeq \frac{i}{\tanh K_0h} \\ q_{2,h} &\simeq 1 \\ q_{3,h} &\simeq K_0 \quad [\text{or } \theta/K_0A = 1, \text{see (6)}] \\ |R|^2 &\simeq 0; |T|^2 \simeq 1 \end{aligned} \tag{65}$$

Physically the expression (65) means that the body follows the fluid particle, and this condition holds if (i) the waves are long or the water is shallow, or (ii) the body's dynamic is unimportant or, in other words, we are below any resonance peak.

Under these conditions the potential $\phi_{20}(y,z)$ must be the counterflux associated with the mass transport of the incoming wave, namely ($\bar{\omega}^2 = \bar{K}_0 \cdot \tanh \bar{K}_0h$):

$$\phi_{20}(y,z) = -\frac{\bar{K}_0}{2\bar{\omega}h} \cdot y \quad \text{when } \bar{\omega} \rightarrow 0 \tag{66}$$

In fact, if we use (65) in (44), (57) and (62), we obtain (see also Appendix 2)

$$\begin{aligned} L_{20}(y) &= 0 & &[(47)(\text{ii})] \\ B_{20}(y,z) &= -\frac{\bar{K}_0}{2\bar{\omega}h} \cdot n_{y,0} & &[(47)(\text{iv})] \\ \phi_{20}(y,z) &\sim -\frac{\bar{K}_0}{2\bar{\omega}h} y \quad \text{when } y \rightarrow \pm\infty & &[(47)(\text{v})] \end{aligned} \tag{67}$$

With these boundary conditions it is easy to check that the only solution of (47) is given by (66). From (59) it follows then that

$$Q_{20}^{(k)}(\bar{\omega}) = \frac{\bar{K}_0}{2h} \int_{\partial B_0} y \cdot v_k(y,z) \cdot d\partial B_0 \quad \text{when } \bar{\omega} \rightarrow 0 \tag{68}$$

The above relation can also be written in an equivalent form if we apply Haskind relation to (47) and use (67). In sway, for instance, we obtain

$$Q_{20}^{(1)}(\bar{\omega}) = -\frac{\bar{K}_0}{2h} \cdot S_0 = \bar{K}_0 \cdot \left[\phi_1^{(0)}(\infty,0) + \frac{1}{2h} \int_{\partial B_0} \phi_1^{(0)}(y,z) \cdot n_{y,0} \cdot d\partial B_0 \right] \tag{69}$$

where S_0 is the cross-sectional area and $\phi_1^{(0)}(y,z)$ is the sway potential in zero frequency. Placing (65) into (63) or (64), we can derive an expression for $Q_{20}^{(1)}(\bar{\omega})$ that coincides with (69).

From (68) it follows that

$$\frac{Q_{20}^{(k)}(\bar{\omega})}{\bar{\omega}} \sim 0(h^{-3/2}) \quad \text{when } K_0h \rightarrow 0 \tag{70}$$

which shows the importance of the second-order potential as the water becomes shallow.

8. Nonlinear diffraction of a narrow-band spectrum

We start this section by considering briefly the properties of the sea spectrum, described by a function $S(\omega)$ as indicated in Fig. 3. It is usual to define

$$\begin{aligned} \bar{\omega} &= \text{average frequency} = (M_2/M_0)^{1/2} \\ \omega_c &= \text{central frequency} = M_1/M_0 \\ \bar{A} &= \text{average amplitude} = (2M_0)^{1/2} = 1 \end{aligned} \tag{71}$$

Notice that $\bar{A} = 1$ in nondimensional variables, where δ is the ratio between the dimensional $\bar{A}$ and the beam B. From the definition (71) we can easily check that

$$\frac{\omega_c}{\bar{\omega}} \le 1 \tag{72}$$

where the equality sign holds if and only if $S(\omega) = S_0 \cdot \delta(\omega - \omega_0)$, that is, for a harmonic wave with frequency ω_0 and amplitude $(2S_0)^{1/2}$.

A realization of this spectrum with duration $T_f = 2\pi/\Delta\omega_f$, where $\Delta\omega_f \ll 1$, can be written as

$$\eta(y,t) = \sum_{j=1}^{\infty}\left[-\frac{i}{2}\cdot A_j \cdot e^{i(\omega_j t - K_j y + \sigma_j)} + (*)\right] \quad (0 \le t \le T_f)$$

$$\omega_j = j \cdot \Delta\omega_f \quad j = 1,2,$$

$$\omega_j^2 = K_j \cdot \tanh K_j h \tag{73}$$

where $\eta(y,t)$ is the displacement of the free surface, due to the incoming wave, the phase σ_j is random, and the amplitudes A_j are given by

$$\frac{1}{2}A_j^2 = S(\omega_j)\cdot\Delta\omega_j \quad (\omega_j = j\cdot\Delta\omega_f) \tag{74}$$

From the definition of $\bar{A}$ and (74) we obtain $(\Delta\omega_f \to 0)$

$$\bar{A} = \left(\sum_{j=1}^{\infty} A_j^2\right)^{1/2} \tag{75}$$

We introduce now the definitions

$$\omega_j = \bar{\omega} + \bar{\omega}\cdot\Omega_j$$

$$\bar{t} = \bar{\omega}\cdot t \tag{76}$$

and the function

$$F(t) = \sum_{j=1}^{\infty}\frac{A_j}{A}\cdot e^{-i(\Omega_j\bar{\omega}t + \sigma_j)} = \sum_{j=1}^{\infty}\frac{A_j}{A}e^{-i(\Omega_j\bar{t}+\sigma_j)} \tag{77}$$

Taking $y = 0$ in (73) we obtain, with the help of (77)

$$\eta(t) = -\frac{i}{2}\cdot\bar{A}\cdot F(t)\cdot e^{-i\bar{\omega}t} + \frac{i}{2}\cdot\bar{A}\cdot F^*(t)\cdot e^{+i\bar{\omega}t} \tag{78}$$

where $\eta(t) = \eta(0;t)$. Note that (rms = root mean square)

$$\text{rms}\, F(t) = \left[\frac{1}{T_f}\int_0^{T_f}|F(t)|^2\cdot dt\right]^{1/2} = 1 \tag{79}$$

$$\text{rms}\,\frac{dF}{d\bar{t}} = \text{rms}\left(\frac{1}{\bar{\omega}}\cdot\frac{dF}{dt}\right) = \Delta\omega = \sqrt{2}\left(1 - \frac{\omega_c}{\bar{\omega}}\right)^{1/2} \tag{80}$$

From (72) it is clear that $\Delta\omega$ is a real quantity such that

$$0 \le \Delta\omega \le \sqrt{2}$$

and $\Delta\omega = 0$ if and only if the wave is harmonic.

The spectrum is said to have a "narrow band" when $\Delta\omega \ll 1$. In this case the wave looks like a harmonic wave whose amplitude modulates slowly in time [see (78) and (79)].

For the semitheoretical Pierson-Moskowitz spectrum (see, for instance, reference [5], page 315) $\Delta\omega \simeq 0.40$, but this is known to represent a fully developed sea with a relatively wide range of frequencies. Other models, like JONSWAP [8], are similar to Pierson-Moskowitz, but with a smaller bandwidth. As has been discussed in the Introduction, the slow-drift phenomenon is of importance when $\Delta\omega\cdot\bar{\omega} \simeq \omega_n$, where ω_n is the natural frequency of the system. Typically this happens in the range $0.1 \le \Delta\omega \le 0.25$, which covers natural periods from 40 to 100 sec for a sea with period around 10 sec. In what follows we will assume $\Delta\omega \ll 1$, and an asymptotic theory, with the error factor $[1 + 0(\Delta\omega)^2]$, will be derived.

Let $\phi_L(y,z;\omega)$ be the linear response due to a harmonic excitation with frequency (ω) and unitary amplitude [see (25)]. We define

$$\phi_{L,j}(y,z) = \phi_L(y,z;\omega)|_{\omega=\omega j}$$

$$\phi_L(y,z) = \phi_L(y,z;\omega)|_{\omega=\bar{\omega}} \tag{81}$$

Since the input is a sum of waves with amplitude A_j and frequency ω_j, the linear response is given by

$$\Phi_1(y,z,t) = \left[\frac{1}{2}\cdot\bar{A}\cdot e^{-i\bar{\omega}t}\cdot\sum_{j=1}^{\infty}\frac{A_j}{\bar{A}}\cdot\phi_{L,j}(y,z)\cdot e^{-i(\bar{\omega}\Omega_j t+\sigma_j)} + (*)\right]$$

$$\eta_1(y,t) = \left[\frac{1}{2}\cdot\bar{A}\cdot e^{-i\bar{\omega}t}\cdot\sum_{j=1}^{\infty}\frac{A_j}{\bar{A}}\cdot\eta_{L,j}(y)\cdot e^{-i(\bar{\omega}\Omega_j t+\sigma_j)} + (*)\right]$$

$$q_k^{(1)}(t) = \left[\frac{1}{2}\cdot\bar{A}\cdot e^{-i\bar{\omega}t}\cdot\sum_{j=1}^{\infty}\frac{A_j}{\bar{A}}\cdot q_{k,H}^{(j)}\cdot e^{-i(\bar{\omega}\Omega_j+\sigma_j)} + (*)\right] \tag{82}$$

For a narrow-band spectrum A_j decreases rapidly if $\Omega_j > 0(\Delta\omega)$. The terms in the series (82) are relevant only when $\Omega_j \sim 0(\Delta\omega)$ and so

$$\phi_{L,j}(y,z) = \phi_L(y,z;\bar{\omega} + \bar{\omega}\Omega_j) = [\phi_L(y,z) + \bar{\omega}\cdot\Omega_j\cdot D_L(y,z)]\cdot(1 + 0(\Delta\omega)^2)$$

$$\eta_{L,j}(y) = \eta_L(y;\bar{\omega} + \bar{\omega}\Omega_j) = [\eta_L(y) + \bar{\omega}\cdot\Omega_j\cdot d_L(y)]\cdot(1 + 0(\Delta\omega)^2)$$

$$q_{k,H}^{(j)} = q_{k,H}(\bar{\omega} + \bar{\omega}\Omega_j) = [q_{k,H} + \bar{\omega}\cdot\Omega_j + \dot{q}_{k,H}]\cdot(1 + 0(\Delta\omega)^2) \tag{83}$$

where

$$D_L(y,z) = \frac{\partial}{\partial\omega}[\Phi_L(y,z;\omega)]_{\omega=\bar{\omega}}$$

$$d_L(y) = \frac{\partial}{\partial\omega}[\eta_L(y;\omega)]_{\omega=\bar{\omega}}$$

$$\dot{q}_{k,H} = \frac{d}{d\omega}[q_{k,H}(\omega)]_{\omega=\bar{\omega}} \tag{84}$$

The functions $\{D_L(y,z); d_L(y)\}$ exist, are well behaved, and can be computed numerically. More is said about them in Appendix 3.

Placing (83) into (82) and using (77), we obtain, with an error factor of the form $[1 + 0(\Delta\omega)^2]$:

$$\Phi_1(y,z,t) = \left[\frac{1}{2}\left(F(t)\cdot\phi_L(y,z) + i\cdot\frac{dF}{dt}(t)\cdot D_L(y,z)\right)e^{-i\bar{\omega}t} + (*)\right]$$

$$\eta_1(y,t) = \left[\frac{1}{2}\left(F(t)\cdot\eta_L(y) + i\cdot\frac{dF}{dt}(t)\cdot d_L(y)\right)e^{-i\bar{\omega}t} + (*)\right]$$

$$q_k^{(1)}(t) = \left[\frac{1}{2}\left(F(t)\cdot q_{k,H} + i\cdot\frac{dF}{dt}(t)\cdot\dot{q}_{k,H}\right)e^{-i\bar{\omega}t} + (*)\right] \tag{85}$$

In (85), $\{\phi_L(y,z); \eta_L(y); q_{k,H}\}$ are referred to the average frequency ω and we use $\bar{A} = 1$; see (71).

We are in fact interested in the excitation force in low frequency. There are two distinct components of this force, namely

$Q^{(1)}_{k,\mathrm{LF}}(t)$ = generalized force, in low frequency, due to the second-order effect of the first-order solution.

$Q^{(2)}_{k,\mathrm{LF}}(t)$ = generalized force, in low frequency, due to the effect of the second-order potential.

and so

$$\hat{Q}_{k,\mathrm{LF}}(t) = \text{excitation force in low frequency (dimensional)}$$

$$= \rho g \bar{A} B \cdot \delta \cdot [Q^{(1)}_{k,\mathrm{LF}}(t) + Q^{(2)}_{k,\mathrm{LF}}(t)] \tag{86}$$

In (86), $\bar{A}$ is the dimensional average amplitude of the spectrum and $\delta = \bar{A}/B$.

It is not difficult to check that the expression for $Q^{(1)}_{k,\mathrm{LF}}(t)$ is given by (see Appendix 3)

$$Q^{(1)}_{k,\mathrm{LF}}(t) = \left\{ \bar{Q}_k(\bar{\omega}) \cdot |F(t)|^2 + \left[P_k(\bar{\omega}) \cdot F(t) \cdot \frac{1}{\bar{\omega}} \cdot \frac{dF^*}{dt}(t) + (*) \right] \right\} \cdot (1 + 0(\Delta\omega^2;\delta)) \tag{87}$$

In (87), $\bar{Q}_k(\bar{\omega})$ is the mean drift coefficient at the frequency $\bar{\omega}$ [see (53) or (54)] and $P_k(\bar{\omega})$ can be computed as indicated in Appendix 3. Again, using conservation of linear momentum, $P_k(\bar{\omega})$ has a relatively simple form in sway. Its expression is given by

$$P_1(\bar{\omega}) = \frac{K_0 \bar{b}}{2}(1 + |T|^2 - |R|^2) - i \cdot \frac{K_0 h}{\sinh 2K_0 h} \cdot \frac{1 - (1 + K_0 h)/(\cosh K_0 h)}{1 + (2K_0 h)/(\sinh 2K_0 h)} (1 + |R|^2 - |T|^2) - \frac{i}{4} \cdot \left(1 + \frac{2K_0 h}{\sinh 2K_0 h}\right) \cdot \bar{\omega} \cdot \left(R \cdot \frac{dR^*}{d\omega} - T \cdot \frac{dT^*}{d\omega}\right) + \mu(\bar{\omega}) \tag{88}$$

where, in (88)

$$K_0 = K_0(\bar{\omega})$$

$$R = R(\bar{\omega}) \qquad \frac{dR^*}{d\omega} = \left(\frac{dR^*}{d\omega}\right)_{\omega=\bar{\omega}} \qquad \frac{dT^*}{d\omega} = \left(\frac{dT^*}{d\omega}\right)_{\omega=\bar{\omega}}$$

$$T = T(\bar{\omega})$$

and $\mu(\bar{\omega})$ is given by

$$\mu(\bar{\omega}) = -\frac{\bar{\omega}}{2} \cdot \mathrm{Real} \left\{ \int_{F_A} \frac{\partial \phi_L^*}{\partial y}(y,0) \cdot \eta_L(y) \cdot dy + \int_{\partial B_0} \frac{\partial \phi_L^*}{\partial y} \cdot [(q_{1,H} - z \cdot q_{3,H}) \cdot n_{y,0} + (q_{2,H} + y \cdot q_{3,H}) \cdot n_{z,0}] \cdot d\partial B_0 \right.$$
$$+ i\bar{\omega} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{K_n}{K_n + K_m} \cdot (A_n^{\pm}) \cdot (A_m^{\pm})^* \cdot f_n(0) \cdot f_m(0)$$
$$+ \sum_{n=1}^{\infty} \left[\frac{e^{iK_0\bar{b}} \cdot K_0 - i \cdot e^{iK_0\bar{b}} \cdot R^* \cdot K_n}{K_n + i \cdot K_0} - \frac{i \cdot e^{-iK_0\bar{b}} + R \cdot e^{-iK_0\bar{b}} \cdot K_0}{K_n - iK_0} \right] \cdot A_n^- \cdot f_n(0)$$
$$\left. + \sum_{n=1}^{\infty} \left[\frac{K_0}{K_n - iK_0} Te^{iK_0\bar{b}} - i \cdot \frac{K_n}{K_n + iK_0} \cdot T^* \cdot e^{-iK_0\bar{b}} \right] \cdot A_n^+ \cdot f_n(0) \right\} \tag{89}$$

In (89), F_A is the free surface of $|y| \leq \bar{b}$; K_0, K_n, $A_n^{\pm}$, T, R, and $f_n(0)$ are defined in (13), (18), and (23) and are related to the linear solution in the frequency $\omega = \bar{\omega}$. Note here that the series converges fast if the water is not too deep and that the integrals over F_A and ∂B_0 must be numerically computed.

It remains now to determine $Q^{(2)}_{k,\mathrm{LF}}(t)$. To leading order in the amplitude parameter δ we can easily see, from (10), that

$$Q^{(2)}_{k,\mathrm{LF}}(t) = -\frac{\partial}{\partial t}\left[\int_{\partial B_0} \Phi_2^{(\mathrm{LF})}(y,z,t) \cdot v_k \cdot d\partial B_0\right] \tag{90}$$

where $\Phi_2^{(\mathrm{LF})}(y,z,t)$ is the component of the second-order potential in low frequency.

Expression (90) indicates that $Q^{(2)}_{k,\mathrm{LF}}(t)$ is already of order $\Delta\omega$. We can then compute this parameter only to leading order in $\Delta\omega$ and for this it suffices to take [see (85)]

$$\Phi_1(y,z,t) = \left[\frac{1}{2} \cdot F(t) \cdot \Phi_L(y,z) \cdot e^{-i\bar{\omega}t} + (*)\right]$$

$$\eta_1(y,t) = \left[\frac{1}{2} \cdot F(t) \cdot \eta_L(y) \cdot e^{-i\bar{\omega}t} + (*)\right]$$

$$q_k^{(1)}(t) = \left[\frac{1}{2} F(t) \cdot q_{k,H} \cdot e^{-i\bar{\omega}t} + (*)\right]$$

Placing these expressions into (8) and separating the low-frequency terms we obtain

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial z}\right)\Phi_2^{(\mathrm{LF})}|_{z=0} = |F(t)|^2 \cdot L_{20}(y) \cdot (1 + 0(\Delta\omega))$$

$$\nabla \Phi_2^{(\mathrm{LF})} \cdot \vec{n}_0|_{\partial B_0} = |F(t)|^2 \cdot B_{20}(y,z) \cdot (1 + 0(\Delta\omega))$$

where $L_{20}(y)$ and $B_{20}(y,z)$ are defined in (44). Since

$$\frac{\partial^2 \Phi_2^{(\mathrm{LF})}}{\partial t^2} \sim 0[(\Delta\omega)^2]$$

it is clear that

$$\phi_2^{(\mathrm{LF})}(y,z,t) = |F(t)|^2 \cdot \phi_{20}(y,z) \cdot [1 + 0(\Delta\omega)] \tag{91}$$

where $\phi_{20}(y,z)$ is solution of (47). Putting (91) into (90) we obtain

$$Q^{(2)}_{k,\mathrm{LF}}(t) = \left[Q^{(k)}_{20}(\bar{\omega}) \cdot \frac{1}{\bar{\omega}} \cdot \frac{d}{dt}(|F(t)|^2)\right] \cdot [1 + 0(\Delta\omega;\delta)] \tag{92}$$

The total slow-drift force can then be written as

$$Q_{k,\mathrm{LF}}(t) = \left\{\bar{Q}_k(\bar{\omega}) \cdot |F(t)|^2 + \left[P_k(\bar{\omega}) \cdot F(t) \cdot \frac{1}{\bar{\omega}} \cdot \frac{dF^*}{dt}(t) + (*)\right] + Q^{(k)}_{20}(\bar{\omega}) \cdot \frac{1}{\bar{\omega}} \cdot \frac{d}{dt}(|F(t)|^2)\right\} [1 + 0(\Delta\omega^2;\delta)] \tag{93}$$

To leading order in $(\Delta\omega)$ we obtain

$$Q_{k,\mathrm{LF}}(t) = \bar{Q}_k(\bar{\omega}) \cdot |F(t)|^2 \cdot [1 + 0(\Delta\omega;\delta)] \tag{94}$$

which agrees with the expression proposed by Newman [3].

Formula (93) has some advantages. First it provides a consistent way to evaluate the slow-drift force, with an error quite acceptable for practical application. Second, it is not necessary to solve any extra diffraction problem to compute the coefficients $\{\bar{Q}_k(\bar{\omega}); P_k(\bar{\omega}); Q^{(k)}_{20}(\bar{\omega})\}$. Only the harmonic linear problem at the frequencies $\bar{\omega}$; $\bar{\omega} \pm \Delta\omega/2$ [to approximate the derivatives $dR^*/d\omega$; $D_L(y,z)$, etc.] and the radiation problem at zero frequency must be solved. With them the coefficients can be determined by direct integration, where in the region $|y| \geq \bar{b}$ the integrals can be done analytically. In the region $|y| \leq \bar{b}$ there are integrals over the free surface and cross section, but they can be trivially computed by a numerical method. The amount of work necessary to evaluate the nonlinear forces $Q_{k,\mathrm{LF}}(t)$ is not much greater than the amount needed to solve the linear problem.

There is also an important consequence of (93), closely related to its simplicity. In fact, the estimates (70) show that the effect of the second-order potential increases with $(h)^{-1/2}$, when the water depth (h) decreases. Furthermore, we can easily assess the relative importance between the effects of the second-order and first-order potentials. If $E_k(\bar{\omega};\Delta\omega)$ is this measure, in the mode $k = 1,2,3$ and for a spectrum with average frequency $\bar{\omega}$ and bandwidth factor $\Delta\omega$, then it can be defined as the ratio

$$\frac{Q_{20}^{(k)}(\bar{\omega}) \cdot \mathrm{rms}\left(\frac{1}{\bar{\omega}} \cdot \frac{d}{dt}|F(t)|^2\right)}{\bar{Q}_k(\bar{\omega}) \cdot \mathrm{rms}\,(|F(t)|^2 - 1)}$$

Since

$$\mathrm{rms}\,(|F(t)|^2 - 1) = \sqrt{2}$$

and

$$\mathrm{rms}\left(\frac{1}{\bar{\omega}} \cdot \frac{d}{dt}|F(t)|^2\right) = \sqrt{2}\,\Delta\omega$$

then

$$E_k(\bar{\omega};\Delta\omega) = \frac{\sqrt{2}\Delta\omega \cdot Q_{20}^{(k)}(\bar{\omega})}{\bar{Q}_k(\bar{\omega})} \tag{95}$$

In sway we obtain

$$E_1(\bar{\omega};\Delta\omega) = \frac{2\sqrt{2}\Delta\omega \cdot Q_{20}^{(1)}(\bar{\omega})}{R_H}$$

$$R_H = \frac{1}{2}\left(1 + \frac{2K_0h}{\sinh 2K_0h}\right) \cdot (1 + |R|^2 - |T|^2) \tag{96}$$

If we disregard, for instance, viscous damping:

$$R_H = \left(1 + \frac{2K_0h}{\sinh 2K_0h}\right) \cdot |R|^2$$

For long waves ($\bar{\omega} \ll 1$), $|R|^2 \sim 0(\bar{\omega}^2)$ and from (69)

$$E_1(\bar{\omega};\Delta\omega) \sim 0\left(\frac{\Delta\omega}{\bar{\omega}} \cdot \frac{S_0}{h\sqrt{h}}\right) \quad \text{when } \bar{\omega} \ll 1$$

This shows that the effect of the second-order potential should be the dominating one when the waves are long and the water depth is not too great. For shorter waves [$\bar{\omega} \sim 0(1)$], $R_H \sim 0(1)$, but this is the region of linear resonance. Since $Q_{20}^{(1)}(\bar{\omega})$ increases quadratically with the amplification factor, the ratio $E_1(\bar{\omega};\Delta\omega)$ should remain of order 1, if the bandwidth $\Delta\omega$ is not too small. So even when diffraction is important, the effect of the second-order potential should be considered, mainly in the case of relatively shallow water.

These qualitative arguments will be checked in the next section, where numerical results are presented, but the simplicity of (93) makes it irrelevant to disregard in any way the effect of the second-order potential.

We close this section with the following observation: Pinkster computes the effect of the second-order potential, taking into account only the incoming wave. His approximation, then, can be physically justified only under conditions where (65) is valid. But we know that in this case the effect of the second-order potential is given explicitly by [see (68) and (92)]

$$Q_{k,\mathrm{LF}}^{(2)}(t) = \left[\left(\frac{\bar{K}_0}{2h} \cdot \int_{\partial B_0} y \cdot v_k(y,z) \cdot d\partial B_0\right) \cdot \frac{d}{dt}|F(t)|^2\right] \cdot [1 + 0(\Delta\omega;\delta)] \tag{97}$$

So in the region where the Pinkster approximation is valid we could use, instead, the simpler formula (97). Also, the discrepancy between (97) and the exact expression (92) delimits the region above which this approximation is not valid anymore. This point is discussed further in the next section.

9. Numerical results

In this section we analyze some few numerical examples for the purpose of discussing the features of the theory presented in this paper. In all cases we consider a rectangular box with beam/draft = 2.0 and radius of gyration equal to $B/4$. We considered two different mass distributions, one with the center of gravity $B/8$ below the free surface and the other with $B/4$. Three different water depths, h/B = 1.0, 2.0, 5.0 have also been analyzed and the viscosity effect in roll has been considered. More detailed numerical results can be found in [9].

Only the coefficients $\bar{Q}_1(\bar{\omega})$ and $Q_{20}^{(1)}(\bar{\omega})$, corresponding to the leading-order contribution of the first- and second-order potential in sway [see (87), (92) and (93)], have been computed. In Figs. 4 and 5 the variation of these coefficients with $\bar{\omega}$ is shown for the case where the center of gravity (CG) is at $z = -B/8$. In Fig. 5 the dotted line indicates the approximation $Q_{20}^{(1)}(\bar{\omega}) = -K_0 \cdot (BD)/2h$ [see (69)] and, for future reference, the plot of $|R|^2$ is also displayed. In Figs. 6 and 7 the same functions are plotted for the case where the CG is "low." In Fig. 7 the dotted lines indicate the approximation (69) for $h/B = 1.0$ and $h/B = 2.0$ and the resonant frequency in roll and heave are shown in Figs. 5 and 7.

In the long-wave (shallow water) regime the effect of the second-order potential is dominant. Furthermore the value of $Q_{20}^{(1)}(\bar{\omega})$ increases as the water becomes shallow and this coefficient depends strongly on the position of the CG.

For $h/B = 1.0$ the approximation (69) is quite good, when the CG is "low," even for waves that are not too long ($\bar{\omega} \cdot \sqrt{B/2g} \approx 0.5$ or $\bar{T} \approx 12$ sec if $B = 20$ m). The same approximation, for the case where the CG is "high," is worse and the reason for this has already been explained in Section 7; in fact, if $z_{\mathrm{CG}} = -B/8$ the roll resonance occurs in low frequency and the dynamics of the body affects the approximation (69) much earlier. The frequency above which $Q_{20}^{(1)}(\bar{\omega})$ and (69) start to diverge is the limit up to which the Pinkster approximation can be used. For $h/B = 2.0$ the range of application of (69) is shorter (see Fig. 7), but this has also been explained in Section 7. In fact the approximation (67) holds good only if the water is shallow ($K_0h \ll 1$) and so, for the same frequency, it is worse the deeper the water is.

There are two features about $Q_{20}^{(1)}(\bar{\omega})$ that must be better explained. One is the fact that for the "high"-CG case, $Q_{20}^{(1)}(\bar{\omega}) = 0$ for a frequency around $\bar{\omega} \cdot (B/2g)^{1/2} \approx 0.8$, irrespective of the water depth; see Fig. 5. The second is the unexpected fact that in the short-wave regime the effect of the second-order potential is higher the higher the water depth is.

To explain the first issue we notice that $\bar{\omega}(B/2g)^{1/2} \approx 0.8$ is just the point where $\bar{Q}_1(\bar{\omega})$ has a maximum. Since this coefficient is proportional to $|R|^2$ we plotted, in Fig. 5, $|R|^2$ as a function of $\bar{\omega}$. We observe, then, that $Q_{20}^{(1)}(\bar{\omega}) = 0$ just at the point where $|R|^2 = 1$. But if $|R|^2 = 1$, then there is no mass transport [see (57)] and so $\phi_{20}(y,z)$ must be zero. This makes even more evident the correspondence between the phenomenon of mass transport and the effect of the second-order potential. Furthermore, if it is clear that the effect of the first-order potential, $\bar{Q}_1(\bar{\omega})$, increases with $|R|^2$ [see (54)], the above observation also makes clear that the effect of the second-order potential, $Q_{20}^{(1)}(\bar{\omega})$, tends to increase with $1 - |R|^2$; see (57). So there is a trend for one to become greater when the other becomes smaller and vice versa. This trend is fully confirmed by the numerical results (see, for instance, Fig. 9). When the CG is "low," $\bar{Q}_1(\bar{\omega})$ has also a maximum at the same frequency $\bar{\omega}(B/2g)^{1/2} \simeq 0.8$. But then this frequency is close to the resonance frequency in "roll" and

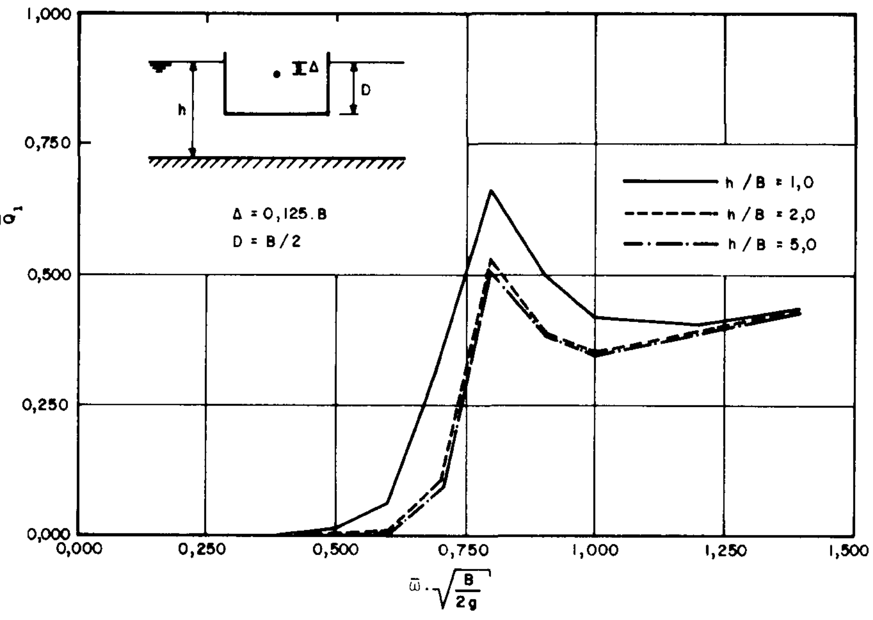

Fig. 4 $\bar{Q}_1(\bar{\omega})$

viscosity plays an important part in the problem. In fact $|R|^2 \simeq 0.85$ at this point, irrespective of the water depth, and so $\phi_{20}(y,z)$ must be different from zero.

Figure 8 displays the contributions to $Q_{20}^{(1)}/\bar{\omega}$ that come from the free surface and the integral around the body [see (63)]. We notice that $(Q_{20}^{(1)}/\bar{\omega})_{\text{Body}}$ is smaller, in absolute value, the deeper the water is. This is in accordance with our *a priori* expectation, but $(Q_{20}^{(1)}/\bar{\omega})$ in the free surface has the opposite behavior when the waves are not too long. We have been unable to explain, in a clear physical way, this sort of unexpected result.

In Fig. 9 the plot of $E_1(\bar{\omega};\Delta\omega)$ as a function of $\bar{\omega}$ is shown, where we took $\Delta\omega = 0.2$. Note that the effect of the second-order potential cannot be disregarded even in the high-frequency regime. Furthermore, we observe that for $h/B = 5.0$, $\Delta\omega = 0.2$, and $z_{CG} = -B/4$:

$$\max\{\text{rms}(\bar{Q}_1(\bar{\omega}) \cdot (|F(t)|^2 - 1))\} \simeq 0.62 \text{ at } \bar{\omega} \cdot (B/2g)^{1/2} \simeq 0.8$$

$$\max\left\{\text{rms}\left(Q_{20}^{(1)}(\bar{\omega}) \cdot \frac{1}{\bar{\omega}} \cdot \frac{d}{dt}|F(t)|^2\right)\right\} \simeq 0.23 \text{ at } \bar{\omega} \cdot (B/2g)^{1/2} \simeq 1.22$$

So the maximum contribution of the second-order potential is more than one third the maximum contribution of the first-order potential. This effect, then, can hardly be disregarded in an actual computation.

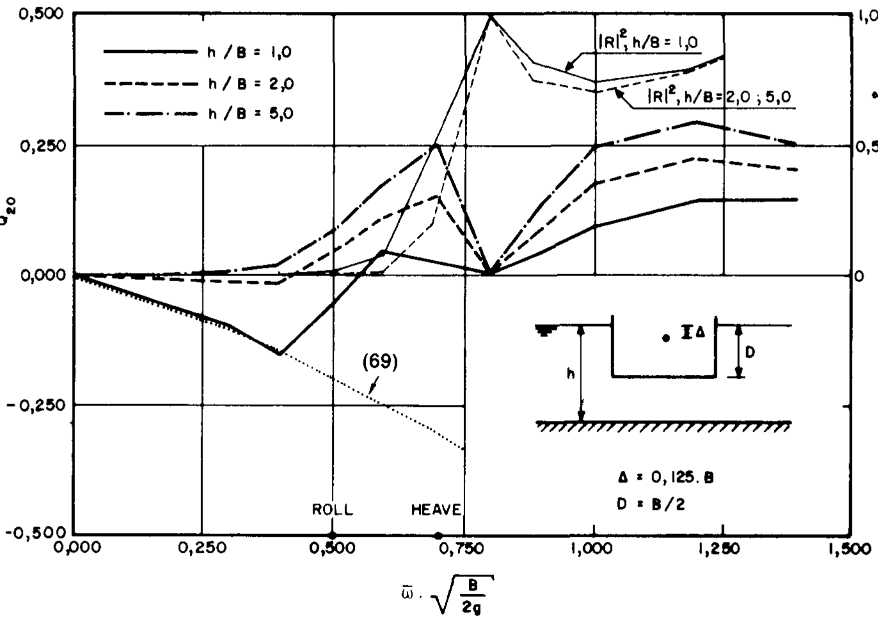

Fig. 5 $Q_{20}^{(1)}(\bar{\omega})$. Also shown: $|R|^2$ and results from equation (69) (······)

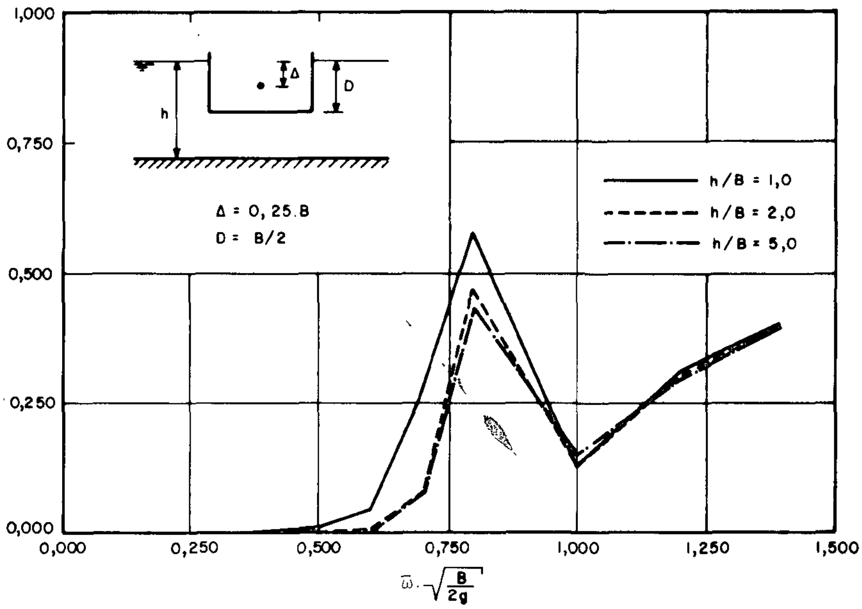

Fig. 6 $\bar{Q}_1(\bar{\omega})$

10. Conclusion

A consistent asymptotic theory, in the small-bandwidth factor $\Delta\omega$, has been derived and an expression for the slow-drift force, correct to an error factor $[1 + 0(\Delta\omega)^2]$, has been obtained. This expression is given by (93), where $F(t)$ is the amplitude modulation of the incoming wave and the coefficients $\{\bar{Q}_k(\bar{\omega});\ P_k(\bar{\omega});\ Q_{20}^{(k)}(\bar{\omega})\}$ can be computed with basically the knowledge of the linear potential at the frequency $\bar{\omega}$. In this way we can easily assess the influence of the second-order potential and this procedure must be confronted with the exact one, as derived by Faltinsen and Løkens [1], where $\frac{1}{2}(n^2 + n)$ nonlinear problems must be solved, n being the number of harmonic components of the irregular wave.

The expression proposed by Newman [3] is the leading-order term of (93) and the approximation proposed by Pinkster, for the influence of the second-order potential, coincides with (93) for the long-wave regime. The new coefficient, $Q_{20}^{(k)}(\bar{\omega})$, is closely related to the phenomenon of mass transport and tends to increase with $1 - |R|^2$, where R is the reflection coefficient. Some features of $Q_{20}^{(k)}(\bar{\omega})$ have been discussed and, in particular, the long-wave approximation (68) has been derived. The behavior of $Q_{20}^{(k)}(\bar{\omega})$ follows closely what could have been anticipated with one important exception: For short waves $Q_{20}^{(k)}(\bar{\omega})$ increases with water depth, which was not expected *a priori.*

An interesting theoretical point has also been discussed in this work: If we consider, as we should, that the harmonic wave is the limit of a narrow-band spectrum when the bandwidth goes to zero, then the standard no-net-flow condition in the mass transport phenomenon arises naturally from the mathematical features of the *nonlinear* diffraction problem.

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Appendix 1

Radiation problem at zero frequency

For $|y| \geq \bar{b}$ the solution of (29) is given by (34), where $B_{0,k}^{\pm} = L_0^{\pm}(\phi_k^{(0)})$; see (35). Then

$$\frac{\partial \phi_k^{(0)}}{\partial y}(\pm\bar{b};z) = \pm D_{0,k}^{\pm} \cdot g_0(z) \mp \sum_{n=1}^{\infty} \lambda_n \cdot L_n^{\pm}(\phi_k^{(0)}) \cdot g_n(z) \tag{98}$$

If we multiply (29)(i) by $\psi(y,z)$ and integrate in A we obtain, after further integrating by parts, using the boundary conditions (29)(i) to (29)(iv) and (98), the weak equation, to determine $\phi_k^{(0)}(y,z)\epsilon W_2^{(1)}(A)$ such that

$$G(\phi_k^{(0)};\psi) - [D_{0,k}^{+} \cdot L_0^{+}(\psi) + D_{0,k}^{-} \cdot L_0^{-}(\psi)] = V_k(\psi) \tag{99}$$

for all $\psi(y,z)\epsilon W_2^{(1)}(A)$.

In (99), $W_2^{(1)}(A)$ is the Hilbert space for all $\psi(y,z)$ such that $(\nabla\psi)^2$ is Lebesgue integrable in the region A, $V_k(\psi)$ is defined in (30), and

$$G(\phi;\psi) = \iint_A \nabla\phi \cdot \nabla\psi \cdot dA + \sum_{n=1}^{\infty} \lambda_n \cdot L_n^{\pm}(\phi) \cdot L_n^{\pm}(\psi) \tag{100}$$

Note that $G(1;1) = 0$ and this bilinear form is singular. However, if

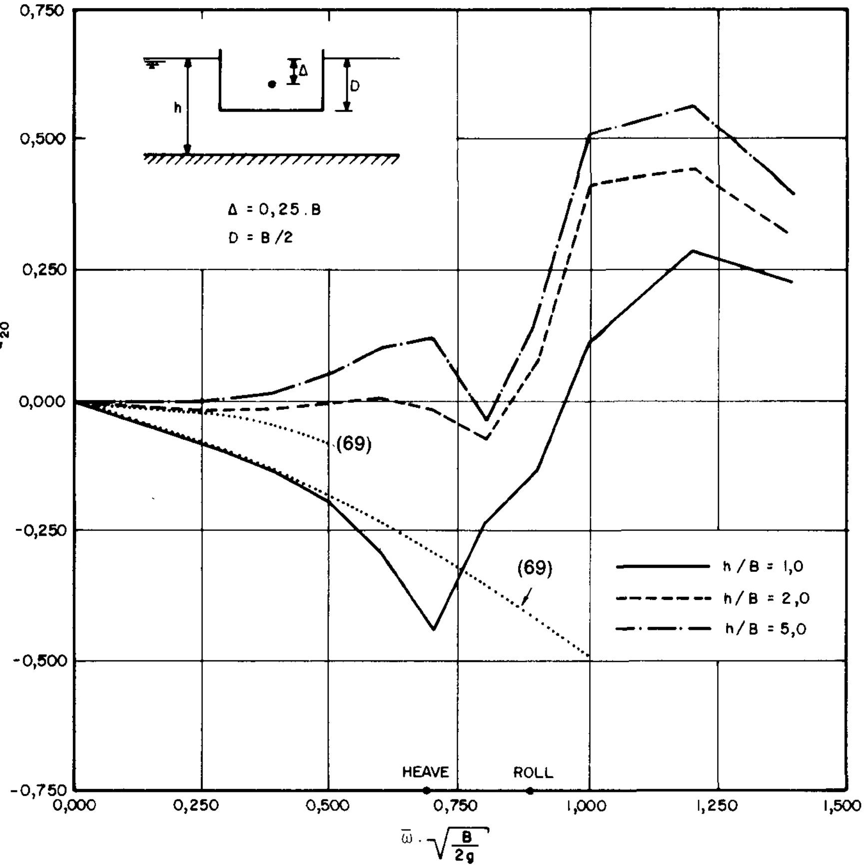

Fig. 7 $\bar{Q}_{20}^{(1)}(\bar{\omega})$. Also shown: results from equation (69) (······)

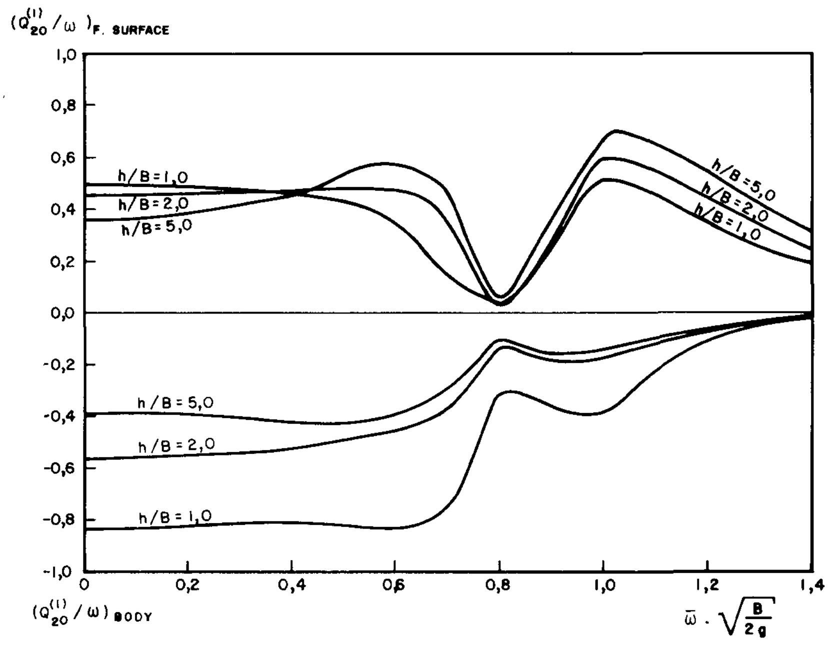

Fig. 8 $(Q_{20}^{(1)}/\bar{\omega})$ on body and free surface ($\Delta = B/8$)

$$\mathring{W}_2^{(1)}(A) = \{\psi_R(y,z)\epsilon W_2^{(1)}(A):\ L_0^+(\psi_R) = L_0^-(\psi_R) = 0\}$$

then $G(\cdot;\cdot)$ is positive definite in $\mathring{W}_2^{(1)}(A)$. If we define now

$$q^{\pm}(y,z) = \frac{1}{2}\left[1 \pm \frac{y}{\bar{b}}\right]\cdot\frac{1}{\sqrt{h}} \quad \left(g_0(z) = \frac{1}{\sqrt{h}}\right) \tag{101}$$

we can write

$$\phi_k^{(0)}(y,z) = B_{0,k}^+\cdot q^+(y,z) + B_{o,k}^-\cdot q^-(y,z) + \phi_{k,R}^{(0)}(y,z);\ \phi_{k,R}^{(0)}(y,z)\epsilon W_2^{(1)}(A)$$
$$\psi(y,z) = L_0^+(\psi)\cdot q^+(y,z) + L_0^-(\psi)\cdot q^-(y,z) + \psi_R(y,z);\ \psi_R(y,z)\epsilon W_2^{(1)}(A)$$

Placing these expressions into (99) we obtain [see (34), (37)]

$$\phi_k^{(0)}(y,z) = \bar{\phi}_{k,R}^{(0)}(y,z) + B_{0,k}^+\cdot p^+(y,z) + B_{0,k}^-\cdot p^-(y,z) \quad (|y| \le \bar{b})$$

where

(*a*) $$\bar{\phi}_{k,R}^{(0)}(y,z)\epsilon\mathring{W}_2^{(1)}(A)$$

is such that

$$G(\bar{\phi}_{k,R}^{(0)};\psi_R) = V_k(\psi_R)$$

for all

$$\psi_R(y,z)\epsilon\mathring{W}_2^{(1)}(A)$$

(*b*) $$p^{\pm}(y,z) = q^{\pm}(y,z) + P_R^{\pm}(y,z)$$

where

$$P_R^{\pm}(y,z)\epsilon\mathring{W}_2^{(1)}(A)$$

and such that

$$G(P_R^{\pm};\psi_R) = -G(q^{\pm};\psi_R) = \mp G\left(\frac{y}{2\bar{b}\sqrt{h}};\psi_R\right) = \mp\frac{1}{2\bar{b}\sqrt{h}}V_k(\psi_R)$$

for all

$$\psi_R(y,z)\epsilon\mathring{W}_2^{(1)}(A)$$

Then

$$P_R^{\pm}(y,z) = \mp\frac{1}{2\bar{b}\sqrt{h}}\cdot\bar{\phi}_{k,R}^{(0)}(y,z)$$

and

$$p^{\pm}(y,z) = \frac{1}{2\sqrt{h}} \pm \frac{1}{2\bar{b}\sqrt{h}}[y + \bar{\phi}_{k,R}^{(0)}(y,z)]$$

From (37)

$$p_A(y,z) = \frac{1}{2\bar{b}\sqrt{h}}\cdot[y + \bar{\phi}_{k,R}^{(0)}(y,z)] \tag{102}$$

(*c*) $$\begin{bmatrix} G(p^+;p^+) & G(p^+;p^-) \\ G(p^+;p^-) & G(p^-;p^-)\end{bmatrix}\cdot\begin{bmatrix} B_{0,k}^+ \\ B_{0,k}^-\end{bmatrix} = \begin{bmatrix} V_k(p^+) \\ V_k(p^-)\end{bmatrix} + \begin{bmatrix} D_{0,k}^+ \\ D_{0,k}^-\end{bmatrix}$$

But

$$G(p^+;p^+) = G(p^-;p^-) = -G(p^+;p^-) = G(p_A;p_A) = \bar{G}$$

See (36).

Appendix 2

Radiation condition for $\phi_{20}(y,z)$

For $|y| \ge \bar{b}$

$$\phi_{20}(y,z) = [\hat{A}_{20}^{\pm} + U_{20}^{\pm}(|y| - \bar{b})]\cdot g_0(z) + \sum_{n=1}^{\infty} A_{20,n}^{\pm}\cdot e^{-\lambda_n(|y|-\bar{b})}\cdot g_n(z) + \phi_{20,P}(y,z) \tag{103}$$

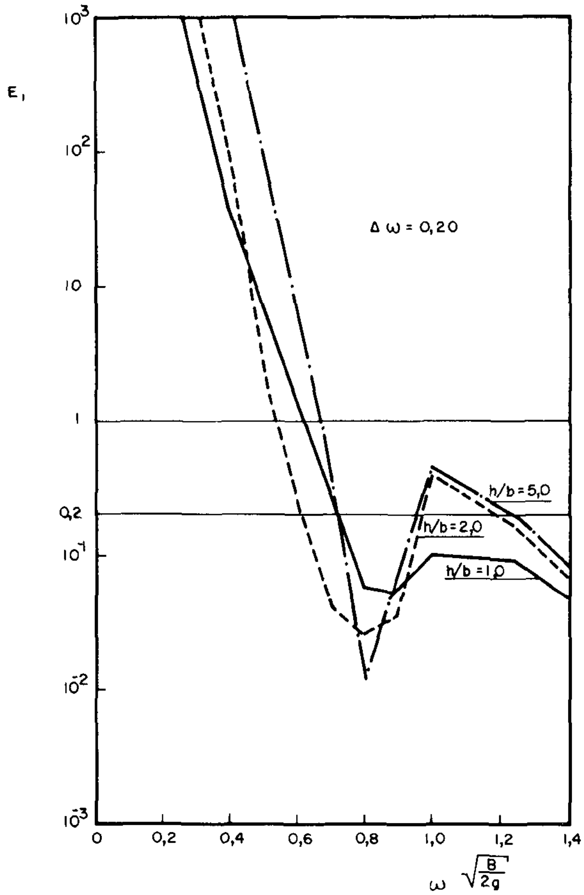

Fig. 9 $\bar{E}_1(\bar{\omega},\Delta\omega)$ for $\Delta\omega = 0.20$

The particular solution $\phi_{20,P}(y,z)$ of (47), ($|y| \ge \bar{b}$), must be such that $[(\partial\phi_{20,P})/\partial y] \to 0$ when $|y| \to \infty$ [see (103) and (47)(v)]. It is given by

$$\phi_{20,P}(y,z) = \sum_{n=0}^{\infty} h_n^{\pm}(y)\cdot e^{-\lambda_n(|y|-\bar{b})}\cdot g_n(z) + L_{20}(y)\cdot p(z) + \frac{1}{h}\cdot\left\{\begin{matrix}\int_{\bar{b}}^{y} d\xi\int_{\xi}^{\infty} L_{20}(\zeta)d\zeta \\ \int_{y}^{-b} d\xi\int_{-\infty}^{\xi} L_{20}(\zeta)d\zeta\end{matrix}\right\} \tag{104}$$

where

$$p(z) = \frac{(z+h)^2}{2h} - \frac{h}{2};\ p_n = \int_{-h}^{0} p(z)\cdot g_n(z)dz$$

and $h_n^{\pm}(y)$ is such that

(i) $$\frac{d^2h_n^{\pm}}{dy^2} \mp 2\lambda_n\frac{dh_n^{\pm}}{dy} = -p_n\cdot e^{-\lambda_n(|y|-\bar{b})}\cdot\frac{d^2L_{20}}{dy^2}$$

(ii) $$h_n^{\pm}(\pm\bar{b}) = -p_n\cdot L_{20}(\pm\bar{b})$$

(iii) $$\frac{dh_n^{\pm}}{dy}(\pm\bar{b}) = -p_n\cdot\frac{dL_{20}}{dy}(\pm\bar{b}) \mp \lambda_n\cdot p_n\cdot L_{20}(\pm\bar{b})$$

Note that

$$\left(\frac{dh_n^{\pm}}{dy} \mp \lambda_n\cdot h_n\right)e^{-\lambda_n(|y|-\bar{b})} \to 0$$

when $|y| \to \infty$ since

$$L_{20}(y) \sim \exp(-K_1|y|)$$

See (45). Then

$$\frac{\partial \phi_{20,P}}{\partial y} \to 0 \text{ as } |y| \to \infty \text{ and } \phi_{20,P}(\pm\bar{b};z) = 0$$

$$\frac{\partial \phi_{20,P}}{\partial y}(\pm\bar{b};z) = \pm\frac{1}{h}\cdot\begin{Bmatrix}\int_{\bar{b}}^{\infty} L_{20}(\xi)\cdot d\xi \\ \int_{-\infty}^{-\bar{b}} L_{20}(\xi)\cdot d\xi\end{Bmatrix}$$

$$\phi_{20,P} \sim \frac{1}{\sqrt{h}}\begin{Bmatrix}\int_{\bar{b}}^{\infty} d\xi \int_{\xi}^{\infty} L_{20}(\zeta)d\zeta \\ \int_{-\infty}^{-\bar{b}} d\xi \int_{-\infty}^{\xi} L_{20}(\zeta)d\zeta\end{Bmatrix} y \to \pm\infty \qquad (105)$$

From (47)(v), (103), (105):

$$A_{20}^{+} = \hat{A}_{20}^{+} + \frac{1}{\sqrt{h}}\cdot\int_{\bar{b}}^{\infty} d\xi\cdot\int_{\xi}^{\infty} L_{20}(\zeta)d\zeta$$

$$A_{20}^{-} = \hat{A}_{20}^{-} + \frac{1}{\sqrt{h}}\cdot\int_{-\infty}^{-\bar{b}} d\xi\cdot\int_{-\infty}^{\xi} L_{20}(\zeta)d\zeta \qquad (106)$$

The problem now is the same as in Appendix 1 where, instead of $V_k(\psi)$, we have

$$V_{20}(\psi) = \int_{\partial B_0} B_{20}(y,z)\cdot\psi(y,z)\cdot d\partial B_0 + \int_{F_A} L_{20}(y)\cdot\psi(y,0)dy$$

$$+ L_0^{+}(\psi)\cdot\left[\frac{1}{\sqrt{h}}\cdot\int_{\bar{b}}^{\infty} L_{20}(\xi)d\xi\right] + L_0^{-}(\psi)\cdot\left[\frac{1}{\sqrt{h}}\cdot\int_{-\infty}^{-\bar{b}} L_{20}(\xi)d\xi\right] \qquad (107)$$

Note that

$$V_{20}(1) = \int_{\partial B_0} B_{20}(y,z)\cdot d\partial B_0 + \int_F L_{20}(y)\cdot dy = -\frac{K_0}{2\omega}(1-|R|^2-|T|^2) \qquad (108)$$

The values $\hat{A}_{20}^{\pm}$ are solutions of [see (36)]

$$\bar{G} = \begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix}\cdot\begin{bmatrix}\hat{A}_{20}^{+} \\ \hat{A}_{20}^{-}\end{bmatrix} = \begin{bmatrix}V_{20}(p^{+}) \\ V_{20}(p^{-})\end{bmatrix} + \begin{bmatrix}U_{20}^{+} \\ U_{20}^{-}\end{bmatrix} \qquad (109)$$

Equation (109) has a solution if and only if

$$U_{20}^{+} + U_{20}^{-} = -V_{20}(p^{+}+p^{-}) = \frac{K_0}{2\omega\sqrt{h}}(1-|R|^2-|T|^2)$$

This is just mass conservation; see (46). So

$$\hat{A}_{20}^{\pm} = \frac{1}{2\bar{G}}[V_{20}(p\pm) + U_{20}^{\pm}] + C \qquad (110)$$

where C and $U_{20}^{\pm}$ can be determined if we consider

$$\phi_{20}(y,z) = \lim_{\Delta\omega\to 0} \Phi_2(y,z;\omega\Delta t)$$

where $\Phi_2(y,z;\Delta\omega t)$ is the low-frequency interaction between two waves with frequencies $\omega \pm \Delta\omega/2$ and wave numbers $K_0 \pm (\Delta k/2)$. If

$$\phi_L^{\pm}(y,z) = \phi_L\left(y,z;\omega\pm\frac{\Delta\omega}{2}\right) \approx -\frac{1}{\omega}\cdot\frac{\cosh K_0(z+h)}{\cosh K_0 h}$$

$$\cdot\begin{Bmatrix}Te^{i\left(K_0\pm\frac{\Delta k}{2}\right)y} \\ e^{i\left(K_0\pm\frac{\Delta k}{2}\right)y} + Re^{-i\left(K_0\pm\frac{\Delta k}{2}\right)y}\end{Bmatrix} + \sum_{n=1}^{\infty} A_n^{\pm}\cdot f_n(z)\cdot e^{-K_n(|y|-\bar{b})}$$

where T, R, and $A_n^{\pm}$ are the coefficients associated with the frequency $\bar{\omega}$, then

$$\Phi_1(y,z,t) = \left[\frac{1}{2}\phi_L^{+}(y,z)e^{-i\left(\omega+\frac{\Delta\omega}{2}\right)t} + \frac{1}{2}\phi_L^{-}(y,z)\cdot e^{-i\left(\omega-\frac{\Delta\omega}{2}\right)t} + (*)\right]$$

Placing this expression into the free-surface term of (8), separating the terms that pulsate with $\Delta\omega$ and disregarding those that tend to zero with $\Delta\omega$, we obtain, at the free surface

$$\left.\frac{\partial\phi_2}{\partial z}\right|_{z=0} = (\Delta\omega)^2\cdot\Phi_2(y,0) + L_{20}(y)$$

$$+\begin{Bmatrix}-\frac{K_0}{2\omega}|T|^2 \\ \frac{K_0}{2\omega}(1-|R|^2)\end{Bmatrix}\cdot\Delta k\cdot\sin(\Delta k|y|)$$

The term $\Delta k\cdot\sin(\Delta k|y|)$ gives a leading-order contribution as we are going to see next. Indeed, if ΔK_0 is the wave number associated with $\Delta\omega$, then with an error $[1+0(\Delta\omega)^2]$ we obtain, for $|y| \geq \bar{b}$

$$\Phi(y,z;\omega\Delta t) = \frac{1}{2}\left[\left(\hat{A}_2^{\pm}\cdot e^{i\Delta K_0(|y|-\bar{b})}\cdot g_0(z) + \sum_{n=1}^{\infty} A_{20,n}^{\pm}\cdot e^{-\lambda_n(|y|-\bar{b})}\right.\right.$$

$$\left.\left.\sum\cdot g_n(z) + \phi_{20,P}(y,z) + \bar{\phi}_{20}^{\pm}(y,z)\right)\cdot e^{-i\Delta\omega t} + (*)\right] \qquad (111)$$

where

$$\nabla^2\bar{\phi}_{20}^{\pm} = 0;\ \partial\bar{\phi}_{20}^{\pm}/\partial z = 0 \text{ at } z = -h$$

and

$$\left.\frac{\partial\bar{\Phi}_{20}}{\partial z}\right|_{z=0} = \frac{K_0}{2\omega}\cdot\begin{Bmatrix}-|T|^2 \\ 1-|R|^2\end{Bmatrix}\cdot\Delta K\cdot\sin\Delta K|y|$$

Imposing

$$\bar{\Phi}_{20}^{\pm}(\pm\bar{b};z) = 0$$

then

$$\bar{\Phi}_{20}^{\pm}(y,z) = \frac{K_0}{2\omega h}\cdot\begin{Bmatrix}-|T|^2 \\ 1-|R|^2\end{Bmatrix}$$

$$\cdot\left[\Delta K\cdot h\cdot\sin(\Delta K|y|)\cdot p(z) + \frac{\sin(\Delta K|y|)-\sin(\Delta K\bar{b})}{\Delta K} + 0((\Delta K)^2)\right]$$

or

$$\bar{\Phi}_{20}^{\pm}(y,z) = \frac{K_0}{2\omega h}\cdot\begin{Bmatrix}-|T|^2 \\ 1-|R|^2\end{Bmatrix}(|y|-\bar{b}) \qquad (112)$$

when $\Delta\omega \to 0$. Now

$$\Phi_2(\pm\bar{b},z;\Delta\omega t) = \left[\frac{1}{2}\left(\hat{A}_2^{\pm}\cdot g_0(z) + \sum_{n=1}^{\infty} A_{n,2}^{\pm}\cdot g_n(z)\right)e^{-i\Delta\omega t} + (*)\right]$$

$$\frac{\partial\Phi_2}{\partial y}(\pm\bar{b},z;\Delta\omega t) = \left[\frac{1}{2}\left(\pm i\cdot\Delta K_0\cdot\hat{A}_2^{\pm} \mp \sum_{n=1}^{\infty}\lambda_n\cdot A_{20,n}^{\pm}\cdot g_n(z)\right.\right.$$

$$\left.\left.\pm\frac{1}{h}\cdot\begin{Bmatrix}\int_{\bar{b}}^{\infty} L_{20}(\xi)d\xi \\ \int_{-\infty}^{-\bar{b}} L_{20}(\xi)d\xi\end{Bmatrix} \pm \frac{K_0}{2\omega h}\begin{Bmatrix}-|T|^2 \\ 1-|R|^2\end{Bmatrix}\right)e^{-i\Delta\omega t} + (*)\right]$$

If

$$\bar{V}_{20}(\psi) = V_{20}(\psi) + L_0^{+}(\psi)\cdot\left[-\frac{K_0}{2\omega\sqrt{h}}\cdot|T|^2\right]$$

$$+ L_0^{-}(\psi)\cdot\left[\frac{K_0}{2\omega\sqrt{h}}(1-|R|^2)\right]$$

then [see Appendix 1 and equation below (39)]:

$$\begin{bmatrix} \bar{G} - i\Delta K_0 & -\bar{G} \\ -\bar{G} & \bar{G} - i\Delta K_0 \end{bmatrix} \cdot \begin{bmatrix} \hat{A}_2^+ \\ \hat{A}_2^- \end{bmatrix} = \begin{bmatrix} \bar{V}_{20}(p+) \\ \bar{V}_{20}(p-) \end{bmatrix}$$

So

$$-i\Delta K_0(\hat{A}_2^+ + \hat{A}_2^-) = \bar{V}(p^+ + p^-) = \bar{V}(1/\sqrt{h}) = 0$$

See equation (108). Then

$$\hat{A}_2^+ = \frac{1}{2\bar{G}}\left(\bar{V}_{20}(p+) - \frac{K_0}{2\omega\sqrt{h}}|T|^2\right)$$
$$\hat{A}_2^- = \frac{1}{2\bar{G}}\left(V_{20}(p-) + \frac{K_0}{2\omega\sqrt{h}}(1-|R|^2)\right) \qquad (113)$$

For $|y| \to \infty$ and $\Delta\omega \to 0$ we obtain

$$\left(\phi_{20} = \lim_{\Delta\omega \to 0} \Phi_2(y,z;\Delta\omega t)\right)$$

$$\phi_{20}(y,z) \sim \left[\hat{A}_2^\pm + (A_{20}^\pm - \hat{A}_{20}^\pm) + \frac{K_0}{2\omega\sqrt{h}} \cdot \begin{Bmatrix} -|T|^2 \\ 1-|R|^2 \end{Bmatrix} \cdot (|y| - \bar{b})\right] \cdot \frac{1}{\sqrt{h}}$$

See (106), (105) and (112). By comparing with (47)(v) we obtain (57) and $\hat{A}_{20}^\pm = \hat{A}_2^\pm$; see (113). Since $\hat{A}_2^+ + \hat{A}_2^- = 0$, then from (106) we obtain (62).

Appendix 3

Force coefficient due to first-order potential

First we consider the coefficient in sway. Using conservation of linear momentum, the total force at time (t) is given by

$$F_y(t) = \int_{\partial B(t)} p(y,z,t) \cdot n_y(t) \cdot d\partial B(t)$$
$$= -\frac{d}{dt}\left[\iint_{A(t)} \frac{\partial \Phi}{\partial y} \cdot dA(t)\right] - \left[\int_0^{\eta(y,t)} \left(\left(\frac{\partial \Phi}{\partial y}\right)^2 + p\right)dz\right]_{y=y^-}^{y=y^+}$$

where $A(t)$ is the fluid region between the vertical lines $y^- \le y \le y^+$ and

$$|f|_{y=y^-}^{y=y^+} = f(y+) - f(y-)$$

In the above expression we will let $|y| \to \infty$.

Using (85) and isolating the quadratic terms in low frequency, due to the first-order potential, we obtain, to leading order in δ:

$$Q_{1,\mathrm{LF}}^{(1)}(t) = -\frac{1}{4}\cdot\left[\int_{-h}^0 \left(\left|\frac{\partial\phi_L}{\partial y}\right|^2 - \left|\frac{\partial\phi_L}{\partial z}\right|^2\right)dz - |\eta_L|^2\right]_{y=y^-}^{y=y^+} \cdot |F(t)|^2$$
$$-\frac{d}{dt}\left[\iint_{\Delta A(t)} \frac{\partial \Phi_1}{\partial y}\cdot d\Delta A(t)\right]_{\mathrm{LF}}$$
$$+\left\{-\frac{i}{4}\cdot\left[\int_{-h}^0\left(\frac{\partial\phi_L^*}{\partial y}\cdot\frac{\partial D_L}{\partial y} - \frac{\partial\phi_L^*}{\partial z}\cdot\frac{\partial D_L}{\partial z}\right)dz + \eta_L^*(y)\cdot d_L(y)\right]_{y=y^-}^{y=y^+}\right.$$
$$\left.\cdot\frac{dF}{dt}(t)\cdot F^*(t) + (*)\right\} \qquad (114)$$

where $A(t) = A_0 + \delta\cdot\Delta A(t) + \ldots$. Note that

$$\bar{Q}_1 = \text{mean drift coefficient}$$
$$= -\frac{1}{4}\left[\int_{-h}^0\left(\left|\frac{\partial\phi_L}{\partial y}\right|^2 - \left|\frac{\partial\phi_L}{\partial z}\right|^2\right)dz - |\eta_L|^2\right]_{y=y^-}^{y=y^+}$$
$$= \frac{1}{4}\cdot\left[1 + \frac{2K_0h}{\sinh 2K_0h}\right]\cdot(1+|R|^2 - |T|^2) \qquad (115)$$

Let $\Delta A^\pm(t)$ be the part of $A(t)$ for $y \gtrless \pm\bar{b}$, and $\Delta\bar{A}(t)$ the part for $|y| \le \bar{b}$. Then

$$\frac{d}{dt}\left[\iint_{\Delta A^\pm(t)} \frac{\partial\Phi}{\partial y}\cdot d\Delta A^\pm(t)\right] = \frac{d}{dt}\cdot\left[\int_{y^-}^{-\bar{b}} \frac{\partial\Phi_1}{\partial y}(y,0)\right.$$
$$\left.\cdot\eta_1(y)dy + \int_{\bar{b}}^{y^+}\frac{\partial\Phi_1}{\partial y}(y,0)\cdot\eta_1(y)\cdot dy\right]_{\mathrm{LF}} = \frac{d}{dt}(|F(t)|^2)$$
$$\cdot\left[\frac{1}{4}\cdot\int_{y^-}^{-\bar{b}} + \int_{\bar{b}}^{y^+}\left(\frac{\partial\phi_L}{\partial y}\cdot\eta_L^* + \frac{\partial\phi_L^*}{\partial y}\cdot\eta_L\right)dy\right]$$

Using (25)

$$\frac{d}{dt}\left[\iint_{\Delta A^\pm(t)}\frac{\partial\Phi_1}{\partial y}\cdot d\Delta A^\pm(t)\right] = \mu_1(\bar{\omega})\frac{1}{\bar{\omega}}\left[F(t)\cdot\frac{dF^*}{dt}(t)\right.$$
$$\left.+F^*(t)\cdot\frac{dF}{dt}(t)\right] - \frac{K_0}{2\bar{\omega}}\left[|T|^2\cdot y^+ + (|R|^2-1)\cdot y^-\right]$$
$$\cdot\frac{d}{dt}(|F(t)|^2) + \frac{K_0\bar{b}}{2\bar{\omega}}\left[1+|T|^2-|R|^2\right]\cdot\frac{d}{dt}(|F(t)|^2) \qquad (116)$$

where $\mu_1(\bar{\omega})$ is the series in expression (89). Also

$$\frac{d}{dt}\left[\iint_{\Delta\bar{A}(t)}\frac{\partial\Phi_1}{\partial y}\cdot d\Delta\bar{A}(t)\right] = \mu_2(\bar{\omega})\cdot\frac{1}{\bar{\omega}}\left[F(t)\cdot\frac{dF^*}{dt}(t) + F^*(t)\cdot\frac{dF}{dt}(t)\right] \qquad (117)$$

where $\mu_2(\bar{\omega})$ are the integrals over F_A and ∂B_0 in (89)

$$(\mu(\bar{\omega}) = \mu_1(\bar{\omega}) + \mu_2(\bar{\omega}))$$

From (84)

$$D_L(y,z) \sim -\frac{d}{d\omega}\left(\frac{1}{\omega}\cdot\frac{\cosh K_0(z+h)}{\cosh K_0h}\right)_{\omega=\bar{\omega}}\cdot\begin{Bmatrix} Te^{iK_0y} \\ e^{iK_0y} + Re^{-iK_0y}\end{Bmatrix}$$
$$-\frac{1}{\bar{\omega}}\cdot\frac{\cosh K_0(z+h)}{\cosh K_0h}\cdot\begin{Bmatrix}(dT/d\omega)\cdot e^{iK_0y} \\ (dR/d\omega)\cdot e^{-iK_0y}\end{Bmatrix}$$
$$-\frac{1}{\bar{\omega}}\cdot\frac{\cosh K_0(z+h)}{\cosh K_0h}\cdot\begin{Bmatrix}Te^{iK_0y} \\ e^{iK_0y} - Re^{-iK_0y}\end{Bmatrix}\cdot\left(\frac{i}{C_g(\bar{\omega})}\cdot y\right), \; y \to \pm\infty$$

$$d_L(y) \sim i\cdot\begin{Bmatrix}(dT/d\omega)\cdot e^{iK_0y} \\ (dR/d\omega)\cdot e^{-iK_0y}\end{Bmatrix}$$
$$-i\cdot\begin{Bmatrix}Te^{iK_0y} \\ e^{iK_0y} - Re^{-iK_0y}\end{Bmatrix}\left(\frac{i}{C_g(\bar{\omega})}\cdot y\right), \; y \to \pm\infty$$

Inserting these expressions into (114) we obtain a convergent term plus a term in y^+ and y^-. These divergent terms cancel the ones that appear in (116) and the convergent one is just

$$-i\cdot\frac{K_0h}{\sinh 2K_0h}\cdot\frac{1-(1+K_0h)/(\cosh K_0h)}{1+(2K_0h)/(\sinh 2K_0h)}(1+|R|^2-|T|^2)$$
$$-\frac{i}{4}\cdot\left(1+\frac{2K_0h}{\sinh 2K_0h}\right)\cdot\bar{\omega}\cdot\left(R\cdot\frac{dR^*}{d\omega} - T\cdot\frac{dT^*}{d\omega}\right)$$

With this expression and (117), (116), and (115), we obtain

$$(\mu(\bar{\omega}) = \mu_1(\bar{\omega}) + \mu_2(\bar{\omega}))$$

See equations (87) and (88).

In heave and roll we must use the formulas

$$Q_{2,\mathrm{LF}}^{(1)}(t) = \left[\int_{\partial B(t)} p_1(y,z,t)\cdot n_z(t)\cdot d\partial B(t) + \int_{\partial B_0} z\cdot n_{z,0}\cdot d\partial B_0\right]_{\mathrm{LF}}$$

$$Q_{3,\mathrm{LF}}^{(1)}(t) = \left[\int_{\partial B(t)} p_1(y,z,t)\cdot(-z\cdot n_y(t) + y\cdot n_z(t))d\partial B(t)\right.$$
$$\left.+\int_{\partial B_0} z(-z\cdot n_{y,0} + y\cdot n_{z,0})d\partial B_0\right]_{\mathrm{LF}}$$

$$p_1(y,z,t) = -z - \frac{\partial \Phi_1}{\partial t} - \frac{1}{2}(\nabla \Phi_1)^2$$

where

$$H_2 = \int_{\partial B_0} z \cdot n_{z,0} d\partial B_0 \text{ and } H_3 = \int_{\partial B_0} z(-z \cdot n_{y,0} + y \cdot n_{z,0}) d\partial B_0$$

are the hydrostatic generalized forces.

The slow-drift forces in heave and roll are important only when the body is submerged. In this case

$$\bar{Q}_2(\bar{\omega}) = \frac{1}{2} \text{Real}\,[Q_{1,H} \cdot q^*_{3,H} - \bar{G}_2(\bar{\omega})]$$

$$\bar{Q}_3(\bar{\omega}) = \frac{1}{2} \text{Real}\,[-Q_{1,H} \cdot q^*_{2,H} + Q_{2,H}(q^*_{1,H} - D_s \cdot q^*_{3,H}) - \bar{G}_3(\bar{\omega})]$$

$$P_2(\bar{\omega}) \doteq -\frac{i\bar{\omega}}{4}\Bigg[Q_{1,H} \cdot \dot{q}^*_{3,H} + (\dot{Q}_{1,H})^* \cdot q_{3,H} - 2\dot{\bar{G}}_2(\bar{\omega})$$
$$- 2 \cdot \text{Real} \int_{\partial B_0} \left(\frac{\partial \phi_L}{\partial y}(q^*_{1,H} - (z + D_s)q^*_{3,H}) + \frac{\partial \phi_L}{\partial z}(q^*_{2,H} + y \cdot q^*_{3,H})\right) n_{z,0} \cdot d\partial B_0\Bigg]$$

$$P_3(\bar{\omega}) = -i\frac{\bar{\omega}}{4}\Bigg[-Q_{1,H} \cdot \dot{q}^*_{2,H} - (\dot{Q}_{1,H})^* \cdot q_{2,H} + Q_{2,H}(\dot{q}^*_{1,H} - D_s \cdot \dot{q}^*_{3,H})$$
$$+ (\dot{Q}_{2,H})^*(q_{1,H} - D_s \cdot q_{3,H}) - 2 \cdot \dot{\bar{G}}_3(\bar{\omega}) - 2 \cdot \text{Real} \int_{\partial B_0} \left(\frac{\partial \phi_L}{\partial y}(q^*_{1,H} - (z + D_s)q^*_{3,H}) + \frac{\partial \phi_L}{\partial z}(q^*_{2,H} + y \cdot q^*_{3,H})\right) \cdot v_3 \cdot d\partial B_0\Bigg]$$

where

$$Z_{CG} = -D_s$$

$$Q_{k,H}(\bar{\omega}) = \text{total linear force in frequency } \bar{\omega}$$

$$G_k(\bar{\omega};\delta\omega) = \int_{\partial B_0} \nabla \phi_L(\bar{\omega}) \cdot \nabla \phi_L(\bar{\omega} + \delta\omega) \cdot v_k \cdot d\partial B_0$$

$$\bar{G}_k(\bar{\omega}) = G_k(\bar{\omega};0) = \int_{\partial B_0} |\nabla \phi_L|^2 \cdot v_k \cdot d\partial B_0$$

$$\dot{\bar{G}}_k(\bar{\omega}) = \left.\frac{dG_k}{d\delta\omega}\right|_{\delta\omega=0} \cong \frac{G_k(\bar{\omega};\delta\omega) - \bar{G}_k(\bar{\omega})}{\delta\omega}$$

This last approximation avoids the computation of $D_L(y,z)$.