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Simple formulas for the natural frequencies of non-uniform cables and beams

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ABSTRACT

The asymptotic development method is used to obtain approximate analytical expressions for the natural frequencies of non-uniform cables and beams. By manipulating the first-order terms, we obtain the mechanical properties (mass, stiffness, etc.) of the equivalent uniform cables and beams having the same (up to the first order) frequencies of the non-uniform one. The second order terms provide an error estimation for the previous expressions. Some examples are reported to illustrate the effectiveness and simplicity of the proposed formulas.

1. Introduction

Beams of varying cross-section are largely used in different branches of mechanical engineering; major examples are helicopter rotor blades, airplane wings, blades in different types of turbines, wind turbine blades, etc. but many others can be found. To a minor extent, tapered beams are also encountered in civil engineering, for example in bridges where pillars and beams may be non-prismatic, and in architecture, where many important architects enjoyed designing non-uniform beams in their realizations.

The determination of natural frequencies of non-uniform beams is an old issue, which dates back at least to Kirchhoff [17], who obtained the analytical solutions for wedge-shaped and cone-shaped beams in terms of Bessel's functions. Other pioneering works are [41,28,43]. A long series of investigations have followed these papers [12,5,21], especially during the 70s [25,18] and the 80s [42]. Sato [33] considered the effect of the axial force of linearly tapered beams, while Filipich et al. [8] studied the effects of a Winkler soil. Goel [11] and Lee and Ke [21,20] introduced elastic boundary conditions. Li [23] considered the presence of several cracks and concentrated masses, while the non-uniform beam with several attached oscillators is investigated in [29]. Moving loads are considered in [7], while curved non-uniform beams have been investigated in [22,39,34]. Rotating non-uniform beams were studied in [3], and composite beams in [38]. In parallel to engineering papers, mathematical oriented works can also be found [10].

A great impetus to this research came from the field of structural optimization, where tapering was properly designed to optimize the dynamical performances of the beam [15,24]. In [14] the Young's modulus is varied, and the optimal control theory is applied.

Recently, the investigation of the natural frequencies of non-uniform beams is undergoing a sort of revival [7,30,36,34,3], also in connection with the functionally graded material/beam [35,31].

In the literature we can basically find two different approaches. In the first, exact solutions have been found for some specific cases of tapering [6], using Bessel's functions [11,17,2] or hypergeometric functions [40,30]. Abrate [1] found the very special class of tapering for which the governing equation can be transformed in that of the uniform media. The drawback of exact solution is that they apply only to specific cases, and cannot be extended.

The second approach consists in looking for approximate solutions [13]. In [2] the Rayleigh quotient is used, while in [33] the Ritz method is used. Sakiyama [32] transformed the differential equation in an integral one, and then solved it numerically. Purely numerical solutions have also been proposed [4,19]. In [37] a finite element formulation considering shear deformation and

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rotatory inertia is proposed. The nonlinear regime has been investigated numerically, too [16].

To the best of the authors' knowledge, no attempt has been made to obtain solutions which are (i) simple, (ii) analytical, (iii) accurate and (iv) general enough to be applied to any situations. This constitutes the goal of this paper, where the asymptotic development method, in the Poincaré–Lindstedt version [27], is applied to obtain accurate analytical approximations of the natural frequencies. Nayfeh [26] used the multiple-time-scale method, but he considered the wave propagation in cables, and not with free vibration of finite-length beams and cables. An asymptotic analysis is developed in [39], but the smallness parameter is the thickness of the beam and not, as in the present paper, the difference with respect to a uniform continuum. The WKB and other approximate methods are used in [36] for the rod (i.e. the cable) problem, while the WKB method is used for beams in [9]; both papers considered a different framework as compared to that of this work.

The paper is organized as follows. In Section 2 the governing equations for the undamped free vibrations of taut inextensible cables and beams are summarized and written in a form convenient for the successive developments. An approximate analytical solution of the governing equation is obtained by the asymptotic development method, by considering terms up to the second order. Simple formulas, valid up to the second order, are obtained, and an error estimation is proposed. The general theory is applied to the case of cables and beams in Sections 3 and 4, respectively, and it is illustrated with some examples; in particular, the homogeneous vertical cable and beam of uniform strength are considered. The paper ends with some conclusions (Section 5).

2. Problem formulation

2.1. The cable

The free vibration of a taut inextensible cable are governed by the well-known equation:

$$[N(\hat{z})u'(\hat{z},\hat{t})]' - \rho A(\hat{z})\ddot{u}(\hat{z},\hat{t}) = 0, \tag{1}$$

where $u(\hat{z},\hat{t})$ is the transversal displacement of the cable, $N=N(\hat{z})$ is the axial force (positive for traction) and $\rho A = \rho A(\hat{z})$ is the mass per unit length. Prime denotes derivative with respect to the physical space variable $\hat{z}$ and dot derivative with respect to the physical time $\hat{t}$. Note that $N'(\hat{z}) = -q(\hat{z})$, $q(\hat{z})$ being the distributed load per unit length along the axis of the cable (e.g. the weight for a vertical cable), so that when $q \neq 0$ the axial load is not constant, which is the case we investigate in this work.

Instead of dealing with (1), it is useful to work with a dimensionless equation. Let N_0 and ρA_0 be the reference axial force and mass per unit length; they will be chosen in due course. Let $N(\hat{z}) = N_0 + \tilde{N}(\hat{z})$ and $\rho A(\hat{z}) = \rho A_0 + \tilde{\rho A}(\hat{z})$, and let L be the length of the cable. By introducing the dimensionless space variable, defined by

$$\hat{z} = zL, \tag{2}$$

and dimensionless time, defined by

$$\hat{t} = tL\sqrt{\frac{\rho A_0}{N_0}}. \tag{3}$$

Eq. (1) can be rewritten in the form (note that u is not dimensionless)

$$\{[1+g_1(z)]u'(z,t)\}' - [1+g_2(z)]\ddot{u}(z,t) = 0, \tag{4}$$

where

$$g_1(z) = \frac{\tilde{N}(z)}{N_0} \quad \text{and} \quad g_2(z) = \frac{\tilde{\rho A}(z)}{\rho A_0} \tag{5}$$

are the dimensionless varying parts of the axial force and of the mass per unit length. Prime and dot now mean derivative with respect to z and t, respectively.

Looking for classical linear oscillations of (4) entails assuming $u(z,t) = \sin(\omega t)v(z)$, obtaining

$$\{[1+g_1(z)]v'(z)\}' + \omega^2[1+g_2(z)]v(z) = 0, \tag{6}$$

and solving the associated eigenvalue problem.

Associated to the governing equation (6) there are the boundary conditions at $z=0$ and $z=1$:

$$v = 0. \tag{7}$$

Remark. The same Eq. (6) holds also for the axial vibration of a beam. In this case we have $g_1(z) = \tilde{EA}(z)/EA_0$, EA being the axial stiffness, and boundary conditions different from (7) can be considered.

2.2. The beam

We consider the Euler–Bernoulli beam with space dependent bending stiffness $EJ = EJ(\hat{z})$, normal force $N = N(\hat{z})$ and mass per unit length $\rho A = \rho A(\hat{z})$. The governing equation of the free linear vibrations is classical:

$$[EJ(\hat{z})u''(\hat{z},\hat{t})]'' - [N(\hat{z})u'(\hat{z},\hat{t})]' + \rho A(\hat{z})\ddot{u}(\hat{z},\hat{t}) = 0. \tag{8}$$

As it has been done for the cable, it is useful to work with a dimensionless equation. Let EJ_0, N_0 and ρA_0 be the reference bending stiffness, normal force and mass per unit length. Let $EJ(\hat{z}) = EJ_0 + \tilde{EJ}(\hat{z})$, $N(\hat{z}) = N_0 + \tilde{N}(\hat{z})$ and $\rho A(\hat{z}) = \rho A_0 + \tilde{\rho A}(\hat{z})$. By introducing the dimensionless space variable, defined by $\hat{z} = zL$, and the dimensionless time, defined by $\hat{t} = tL^2\sqrt{\rho A_0/EJ_0}$, Eq. (8) can be rewritten in the form

$$\{[1+f_1(z)]u''(z,t)\}'' - \{[\alpha+f_2(z)]u'(z,t)\}' + [1+f_3(z)]\ddot{u}(z,t) = 0, \tag{9}$$

where

$$\alpha = \frac{N_0 L^2}{EJ_0} \tag{10}$$

is the dimensionless reference axial force, and where

$$f_1(z) = \frac{\tilde{EJ}(z)}{EJ_0}, \quad f_2(z) = \frac{\tilde{N}(z)L^2}{EJ_0}, \quad f_3(z) = \frac{\tilde{\rho A}(z)}{\rho A_0}, \tag{11}$$

are the dimensionless varying parts of the bending stiffness, of the axial force and of the mass per unit length.

Looking for classical linear oscillations of (9) entails assuming $u(z,t) = \sin(\omega t)v(z)$, obtaining

$$\{[1+f_1(z)]v''(z)\}'' - \{[\alpha+f_2(z)]v'(z)\}' - \omega^2[1+f_3(z)]v(z) = 0. \tag{12}$$

As opposed to what happens for the cable, here different boundary conditions should be considered. Each boundary can be hinged $(v = v'' = 0)$, fixed $(v = v' = 0)$ or free $(v'' = v''' - (\alpha + f_2)v' = 0)$, although more complex cases (e.g. elastic boundaries) can be analyzed.

2.3. Asymptotic development

Looking for a perturbative solution around the reference case of a homogeneous cable and beam, we assume (with an abuse of notation)

$$g_1(z) = \varepsilon g_1(z), \quad g_2(z) = \varepsilon g_2(z),$$
$$f_1(z) = \varepsilon f_1(z), \quad f_2(z) = \varepsilon f_2(z), \quad f_3(z) = \varepsilon f_3(z), \tag{13}$$

and, following the Poincaré–Lindstedt method [27], we look for a solution in the form

$$\omega=\omega_0+\varepsilon\omega_1+\varepsilon^2\omega_2+\ldots,$$
$$v(z)=v_0(z)+\varepsilon v_1(z)+\varepsilon^2 v_2(z)+\ldots\ . \qquad (14)$$

These expressions are inserted in the governing equations (6) and (12) and in the boundary conditions, and then expanded in a ε-series. Equating to zero the coefficients of these series provides a sequence of problems for the unknowns ω_i and $v_i(z)$.

3. The cable

3.1. Zero-order solution

Equating to zero the leading order coefficient of the series expansion of the cable provides the equation:

$$v_0''(z)+\omega_0^2 v_0(z)=0 \qquad (15)$$

and the boundary conditions $v_0=0$ for $z=0$ and $z=1$. The solution of this problem, which clearly corresponds to an uniform cable having reference traction and unit mass, is simple and it is given by

$$\omega_0=n\pi,$$
$$v_0(z)=c\ \sin(n\pi z), \qquad (16)$$

where $n\in\mathbb{N}$ is the modal number and c is an arbitrary amplitude.

3.2. First-order solution

Equating to zero the first-order coefficient of the series expansion provides the equation:

$$v_1''(z)+\omega_0^2 v_1(z)+h_1(z)+2\omega_1\omega_0 v_0(z)=0, \qquad (17)$$

where

$$h_1(z)=g_1'(z)v_0'(z)+\omega_0^2 v_0(z)[g_2(z)-g_1(z)]. \qquad (18)$$

The boundary conditions are $v_1=0$ for $z=0$ and $z=1$.

The solution of (17) exists if and only if the solvability condition

$$\int_0^1 [h_1(z)+2\omega_1\omega_0 v_0(z)]v_0(z)\ dz=0 \qquad (19)$$

is satisfied. This gives, after integration by parts

$$\omega_1=-\frac{\int_0^1 h_1(z)v_0(z)\ dz}{2\omega_0\int_0^1 v_0^2(z)\ dz}=\frac{\int_0^1 g_1(z)[v_0'(z)]^2\ dz-\omega_0^2\int_0^1 g_2(z)v_0^2(z)\ dz}{2\omega_0\int_0^1 v_0^2(z)\ dz}. \qquad (20)$$

Eq. (20) shows that the arbitrary amplitude c of $v_0(z)$ does not influence ω_1, which is the first-order correction of the natural frequency ω_0 corresponding to a uniform cable.

Once (19) is satisfied, $v_1(z)$ can be computed by solving (17). An explicit formula obtained by the method of variation of constants is

$$\frac{v_1(z)}{c}=-\frac{\sin(n\pi z)}{2}\int_0^z\{g_1'(\zeta)[1+\cos(2n\pi\zeta)]+n\pi[g_2(\zeta)-g_1(\zeta)]\ \sin(2n\pi\zeta)\}\ d\zeta+\frac{\cos(n\pi z)}{2}\int_0^z\{g_1'(\zeta)\ \sin(2n\pi\zeta)+n\pi[g_2(\zeta)-g_1(\zeta)][1-\cos(2n\pi\zeta)]\}\ d\zeta+\omega_1\left[z\ \cos(n\pi z)-\frac{\sin(n\pi z)}{n\pi}\right]. \qquad (21)$$

As usually done, the part of $v_1(z)$ proportional to $v_0(z)$ is neglected without loss of generality.

By definition we have that

$$g_1(z)=\frac{\tilde{N}(z)}{N_0}=\frac{N(z)-N_0}{N_0}, \qquad (22)$$

so that

$$\int_0^1 g_1(z)[v_0'(z)]^2\ dz=\frac{1}{N_0}\int_0^1 N(z)[v_0'(z)]^2\ dz-\int_0^1 [v_0'(z)]^2\ dz. \qquad (23)$$

Choosing the arbitrary reference axial force equal to

$$N_0=\frac{\int_0^1 N(z)[v_0'(z)]^2\ dz}{\int_0^1 [v_0'(z)]^2\ dz} \qquad (24)$$

we have that the integral on the left hand side of (23) vanishes. Analogously, choosing

$$\rho A_0=\frac{\int_0^1 \rho A(z)v_0^2(z)\ dz}{\int_0^1 v_0^2(z)\ dz} \qquad (25)$$

gives $\int_0^1 g_2(z)v_0^2(z)\ dz=0$.

The conclusion is that with the choices (24) and (25), we have $\omega_1=0$. These values are the properties of the *equivalent uniform* cable having the same frequency (at least up to the first order) of the given non-uniform cable.

Hence, the physical natural frequency of the cable is

$$\hat{f}=\frac{1}{L}\sqrt{\frac{N_0}{\rho A_0}}f=\frac{1}{L}\sqrt{\frac{N_0}{\rho A_0}}\frac{\omega_0}{2\pi}=\frac{n}{2L}\sqrt{\frac{N_0}{\rho A_0}}. \qquad (26)$$

From (24) and (25) we have that

$$\frac{N_0}{\rho A_0}=\frac{\int_0^1 [v_0(z)]^2\ dz}{\int_0^1 \rho A(z)[v_0(z)]^2\ dz}\frac{\int_0^1 N(z)[v_0'(z)]^2\ dz}{\int_0^1 [v_0'(z)]^2\ dz}=\frac{1}{n^2\pi^2}\frac{\int_0^1 N(z)[v_0'(z)]^2\ dz}{\int_0^1 \rho A(z)[v_0(z)]^2\ dz}. \qquad (27)$$

Thus,

$$\hat{f}=\frac{1}{2\pi L}\sqrt{\frac{\int_0^1 N(z)[v_0'(z)]^2\ dz}{\int_0^1 \rho A(z)[v_0(z)]^2\ dz}}, \qquad (28)$$

which is a simple formula providing the natural frequencies of the non-uniform cable. It is worth to underline that, since $\omega_1=0$, this formula is valid *up to the first order*.

An equivalent expression of (28) is

$$\hat{f}=\frac{n}{2L}\sqrt{\frac{\int_0^1 N(z)[1+\cos(2n\pi z)]\ dz}{\int_0^1 \rho A(z)[1-\cos(2n\pi z)]\ dz}}, \qquad (29)$$

which gives the classical expression $\hat{f}=(n/2L)\sqrt{N/\rho A}$ for constant values of the mechanical properties.

Remark. It is not difficult to recognize that (28) is the Rayleigh quotient. The advantage of the proposed method is that of suggesting a given, and easy indeed, trial function in the quotient (i.e. $v_0(z)$) which guarantees that the expression is correct up to the first order. Furthermore, in the following section we will be able to provide a more detailed error estimation.

3.3. Second-order solution

With the double aim of improving the approximation of ω and of estimating the error of the expression (28), we compute the second order term.

Equating to zero the second-order coefficient of the series expansion provides the equation

$$v_2''(z)+\omega_0^2 v_2(z)+h_2(z)+2\omega_2\omega_0 v_0(z)=0, \qquad (30)$$

where

$$h_2(z)=\omega_1^2 v_0(z)+2\omega_1\omega_0\{[g_2(z)-g_1(z)]v_0(z)+v_1(z)\}+\{g_1'(z)v_1'(z)+\omega_0^2 v_1(z)[g_2(z)-g_1(z)]\}+\{\omega_0^2 g_1(z)[g_1(z)-g_2(z)]v_0(z)-g_1(z)g_1'(z)v_0'(z)\}. \qquad (31)$$

The boundary conditions are $v_2=0$ for $z=0$ and $z=1$.

The solution of (30) exists if and only if the solvability condition

$$\int_0^1 [h_2(z)+2\omega_2\omega_0 v_0(z)]v_0(z)\,dz=0 \tag{32}$$

is satisfied. This gives

$$\omega_2=-\frac{\int_0^1 h_2(z)v_0(z)\,dz}{2\omega_0\int_0^1 v_0^2(z)\,dz}. \tag{33}$$

Integrating by parts we have that

$$\int_0^1 h_2(z)v_0(z)\,dz=\omega_1^2\int_0^1 v_0^2(z)\,dz+2\omega_1\omega_0\int_0^1 [g_2(z)v_0^2(z) \\ +v_1(z)v_0(z)]\,dz+(g_1v_1'v_0)|_0^1-\int_0^1 [g_1(z)v_1'(z)v_0'(z) \\ -\omega_0^2 g_2(z)v_1(z)v_0(z)]\,dz. \tag{34}$$

The previous expression is general. In the case $\omega_1=0$, and by taking into account the boundary conditions for v_0, it provides

$$\omega_2=\frac{\int_0^1 g_1(z)v_1'(z)v_0'(z)\,dz-\omega_0^2\int_0^1 g_2(z)v_1(z)v_0(z)\,dz}{2\omega_0\int_0^1 v_0^2(z)\,dz}. \tag{35}$$

We have that

$$\hat{f}^{real}=\hat{f}+\varepsilon^2\frac{\omega_2}{\omega_0}\hat{f}+\cdots \tag{36}$$

where $\hat{f}$ is given by (28). From the previous expression the error can be easily estimated:

$$\frac{\hat{f}^{real}-\hat{f}}{\hat{f}}=\varepsilon^2\frac{\omega_2}{\omega_0}+\cdots \tag{37}$$

3.4. An example

As an illustrative example, we consider a case in which the exact solution is known, so that we can assess the reliability of the proposed method.

We consider the homogeneous heavy vertical cable of uniform strength, i.e. that having a constant stress $N(z)/A(z)=\sigma$. It is possible to show that $A(z)=\overline{A}e^{-xz}$, where

$$x=\frac{\rho g L}{\sigma}, \tag{38}$$

and where ρ is the constant density (mass per unit volume), g the gravity acceleration and $\overline{A}$ is the area of the cable for $z=0$ (z positive downwards), so that $\sigma\overline{A}$ is the traction in the upper end of the cable and $N(z)=\sigma\overline{A}e^{-xz}$.

From (24) and (25) we have

$$N_0=\sigma\overline{A}\left(\frac{1-e^{-x}}{x}\right)\left(\frac{2x^2+4n^2\pi^2}{x^2+4n^2\pi^2}\right), \tag{39}$$

$$\rho A_0=\rho\overline{A}\left(\frac{1-e^{-x}}{x}\right)\left(\frac{4n^2\pi^2}{x^2+4n^2\pi^2}\right), \tag{40}$$

so that from (26) we have

$$\hat{f}=\frac{n}{2L}\sqrt{\frac{\sigma}{\rho}}\sqrt{1+\frac{1}{2}\left(\frac{x}{n\pi}\right)^2}, \tag{41}$$

where the second square root is the correction due to the non-uniformity of the cable. Note that for large mode number n this correction becomes negligible, according to the fact that in this case the wavelength is small and it finds the cable as "piecewise constant".

Furthermore we have

$$\frac{\omega_2}{\omega_0}=-\left[\frac{\tanh(x/2)}{64}\right]\left[\frac{x^3(x^2+4\pi^2)^2}{(x^2+\pi^2)(x^2+2\pi^2)^2}\right], \tag{42}$$

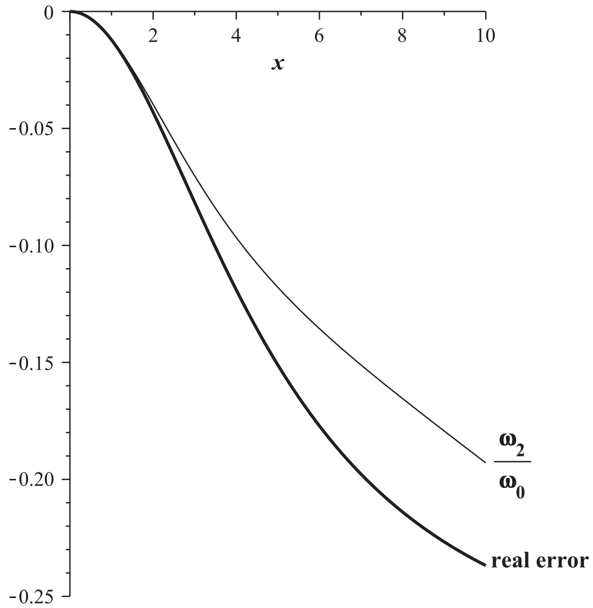

Fig. 1. The real error function and the approximate error function ω_2/ω_0 for the cable.

which provides an error estimation for (41). It is illustrated in Fig. 1. Note that for $x\to 0$ we have $\omega_2/\omega_0\simeq -x^2/(8\pi^2)$.

The dimensional equation of motion in the present case is

$$[\sigma\overline{A}e^{-\hat{x}\hat{z}}u'(\hat{z},\hat{t})]'-\rho\overline{A}e^{-\hat{x}\hat{z}}\ddot{u}(\hat{z},\hat{t})=0, \tag{43}$$

where $\hat{x}=x/L$, so that $\hat{x}\hat{z}=xz$. The general solution of (43) is

$$u(\hat{z},\hat{t})=\sin(\hat{\omega},\hat{t})e^{xz/2}\left[c_1\sin\left(\frac{1}{2}\sqrt{-x^2+\frac{4\rho\hat{\omega}^2L^2}{\sigma}}\right) \\ +c_2\cos\left(\frac{1}{2}\sqrt{-x^2+\frac{4\rho\hat{\omega}^2L^2}{\sigma}}\right)\right], \tag{44}$$

from which, imposing the boundary conditions, we find that

$$\hat{f}^{real}=\frac{n}{2L}\sqrt{\frac{\sigma}{\rho}}\sqrt{1+\frac{1}{4}\left(\frac{x}{n\pi}\right)^2}. \tag{45}$$

Note that in [30] much more complex exact solutions are obtained using the Kummer's hypergeometric function.

Comparing (45) with (41) we find that the approximate solution has a 2 instead of a 4 in the second square root. With (41) and (45) it is possible to compute the real error function $(\hat{f}^{real}-\hat{f})/\hat{f}$, which is also reported in Fig. 1. Note that both curves have exactly the same asymptotic development for $x\to 0$.

4. The beam

In this section we apply to the beam the same approach developed for the cable in the previous section.

4.1. Zero-order solution

Equating to zero the zero-order coefficient of the series expansion of the beam provides the equation

$$v_0''''(z)-\alpha v_0''(z)-\omega_0^2 v_0(z)=0. \tag{46}$$

The general solution of (46) is

$$v_0(z)=c_1\sin(az)+c_2\cos(az)+c_3\sinh(bz)+c_4\cosh(bz), \tag{47}$$

where

$$a=\tfrac{1}{2}\sqrt{-2\alpha+2\sqrt{\alpha^2+4\omega_0^2}},\quad b=\tfrac{1}{2}\sqrt{2\alpha+2\sqrt{\alpha^2+4\omega_0^2}}. \tag{48}$$

Note that $\omega_0=a\sqrt{a^2+\alpha}$ and $b=\sqrt{a^2+\alpha}=\omega_0/a$.

4.1.1. Hinged–hinged beam

For the hinged–hinged beam the boundary conditions are $v(0)=v''(0)=v(1)=v''(1)=0$. They provide

$$a=n\pi,$$
$$\omega_0=n\pi\sqrt{n^2\pi^2+\alpha},$$
$$v_0(z)=c_1\ \sin(n\pi z), \tag{49}$$

where $n\in\mathbb{N}$ is the modal number and c_1 is an arbitrary amplitude.

4.1.2. Fixed–fixed beam

For the fixed–fixed beam the boundary conditions are $v(0)=v'(0)=v(1)=v'(1)=0$. In this case the first-order natural circular frequencies ω_0 are the solutions of the transcendental equation:

$$\alpha\ \sin(a)\sinh(b)+2\omega_0[1-\cos(a)\cosh(b)]=0. \tag{50}$$

In this case $c_2=-c_4=-c_1(b\ \sin(a)-a\sinh(b))/(b[\cos(a)-\cosh(b)])$ and $c_3=-c_1a/b$.

4.1.3. Fixed-free beam

For the fixed-free beam the boundary conditions are $v(0)=v'(0)=v''(1)=v'''(1)-\alpha v'=0$. In this case the first-order natural circular frequencies ω_0 are the solutions of the transcendental equation:

$$2\omega_0^2+\alpha\omega_0\ \sin(a)\sinh(b)+(\alpha^2+2\omega_0^2)\ \cos(a)\cosh(b)=0, \tag{51}$$

In this case $c_2=-c_4=-c_1a(a\ \sin(a)+b\sinh(b))/(a^2\ \cos(a)+b^2\cosh(b))$ and $c_3=(-c_1)a/b$.

The functions $\omega_0(\alpha)$ for different boundary conditions are reported in Fig. 2 for the first natural frequency. Similar curves can be obtained for higher-order frequencies.

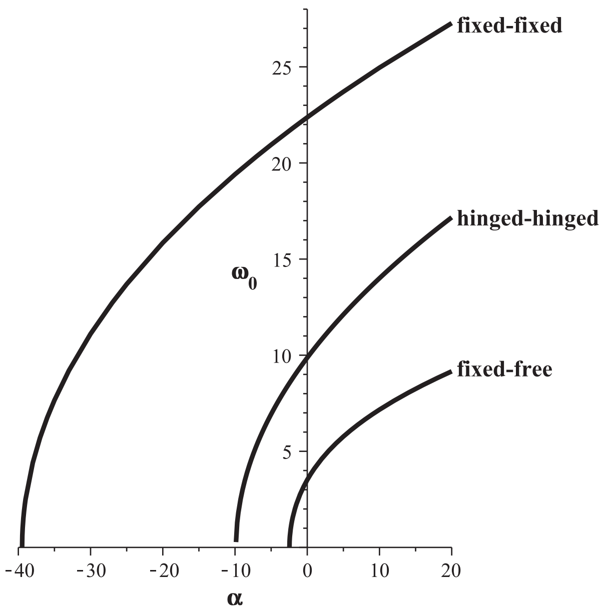

Fig. 2. The functions $\omega_0(\alpha)$ for the different boundary conditions and for $n=1$.

4.2. First-order solution

Equating to zero the first-order coefficient of the series expansion provides the equation:

$$v_1''''(z)-\alpha v_1''(z)-\omega_0^2v_1(z)+h_1(z)-2\omega_1\omega_0v_0(z)=0. \tag{52}$$

where now

$$h_1(z)=2f_1'(z)v_0'''(z)+[f_1''(z)+\alpha f_1(z)-f_2(z)]v_0''(z)$$
$$-f_2'(z)v_0'(z)+\omega_0^2[f_1(z)-f_3(z)]v_0(z). \tag{53}$$

We note that both $h_1(z)$ and $v_1(z)$ are linear with respect to $v_0(z)$.

The solution of (53) exists if and only if the solvability condition

$$\int_0^1[h_1(z)-2\omega_1\omega_0v_0(z)]v_0(z)\,dz=0 \tag{54}$$

is satisfied. This gives

$$\omega_1=\frac{\int_0^1h_1(z)v_0(z)\,dz}{2\omega_0\int_0^1v_0^2(z)\,dz}. \tag{55}$$

Since $h_1(z)$ is linear with respect to $v_0(z)$, the previous equations show that the arbitrary amplitude of $v_0(z)$ does not influence ω_1. It gives the first order correction of the natural frequency ω_0 corresponding to a uniform beam.

Integrating by parts we get

$$\int_0^1h_1(z)v_0(z)\,dz=(f_1'v_0''v_0+f_1v_0'''v_0-f_1v_0''v_0'-f_2v_0'v_0)|_0^1$$
$$+\int_0^1\{f_1(z)[v_0''(z)]^2+f_2(z)[v_0'(z)]^2-\omega_0^2f_3(z)v_0^2(z)\}\,dz. \tag{56}$$

Note that for a fixed or a hinged constraint the boundary terms vanish in (56). This does *not* occur for a free constraint (unless $\alpha f_1=f_2$ at that boundary).

Proceeding as done for the cable, we see that if we choose

$$EJ_0=\frac{(EJ'v_0''v_0+EJv_0'''v_0-EJv_0''v_0')|_0^1+\int_0^1EJ(z)[v_0''(z)]^2\,dz}{(v_0'''v_0-v_0''v_0')|_0^1+\int_0^1[v_0''(z)]^2\,dz},$$
$$N_0=\frac{(-Nv_0'v_0)|_0^1+\int_0^1N(z)[v_0'(z)]^2\,dz}{(-v_0'v_0)|_0^1+\int_0^1[v_0'(z)]^2\,dz},$$
$$\rho A_0=\frac{\int_0^1\rho A(z)v_0^2(z)\,dz}{\int_0^1v_0^2(z)\,dz}, \tag{57}$$

we have that $\omega_1=0$. These values are the properties of the *equivalent uniform* beam having the same frequency (at least up to the first order) of the given non-uniform beam.

We have that $\omega_0(\alpha)$ (see Fig. 2). Furthermore, apart from the hinged–hinged beam, we also have that $v_0(z)$ depends on α, namely $v_0(z;\alpha)$. This means that Eqs. (57) provide $EJ_0(\alpha)$, $N_0(\alpha)$ and $\rho A(\alpha)$. From (10) we then have

$$\alpha=\frac{N_0(\alpha)L^2}{EJ_0(\alpha)}, \tag{58}$$

which is a transcendental equation permitting to compute α, and then all the other quantities of interest.

The physical natural frequency of the beam is

$$\hat{f}=\frac{1}{L^2}\sqrt{\frac{EJ_0}{\rho A_0}}f=\frac{1}{L^2}\sqrt{\frac{EJ_0}{\rho A_0}}\frac{\omega_0}{2\pi}, \tag{59}$$

where all the required expressions have been computed above. This is a simple formula providing the natural frequencies of the non-uniform beam. It is worth underlining that, since $\omega_1=0$, this formula is valid *up to the first order*.

Remark. Contrarily to what happens for the case of the cable, now (59) does *not* come from a Rayleigh quotient (unless $N(z)=0$).

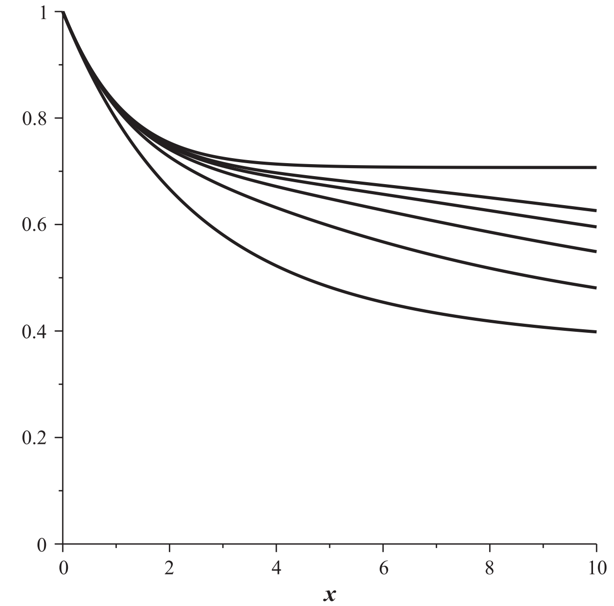

Fig. 3. The correction term for the natural frequency. From the lower to the upper curve: $n=1,2,3,4,5,\infty$.

Choosing $N(z)=\beta\overline{N}(z)$ and noting that N_0 is linear with respect to $N(z)$, we have that $N_0=\beta\overline{N}_0$, so that from (58) we have $\alpha(\beta)$, and then $\omega_0[\alpha(\beta)]=\omega_0(\beta)$. Solving $\omega_0(\beta)=0$ with respect to β provides the buckling load of the non-uniform beam.

4.3. Second-order solution

Equating to zero the second-order coefficient of the series expansion provides the equation:

$$v_2''''(z)-\alpha v_2''(z)-\omega_0^2 v_2(z)+h_2(z)-2\omega_2\omega_0 v_0(z)=0, \tag{60}$$

where

$$\begin{aligned}h_2(z)=&-\omega_1^2 v_0(z)-2\omega_1\omega_0\{[f_3(z)-f_1(z)]v_0(z)+v_1(z)\}\\&+\{2f_1'(z)v_1'''(z)+[f_1''(z)+\alpha f_1(z)-f_2(z)]v_1''(z)-f_2'(z)v_1'(z)\\&+\omega_0^2[f_1(z)-f_3(z)]v_1(z)\}\\&+\{-2f_1'(z)f_1(z)v_0'''(z)-f_1(z)[f_1''(z)+\alpha f_1(z)-f_2(z)]v_0''(z)\\&+f_1(z)f_2'(z)v_0'(z)-\omega_0^2 f_1(z)[f_1(z)-f_3(z)]v_0(z)\}.\end{aligned} \tag{61}$$

We note that both $h_2(z)$ and $v_2(z)$ are linear with respect to $v_0(z)$.

The solution of (61) exists if and only if the solvability condition

$$\int_0^1 [h_2(z)-2\omega_2\omega_0 v_0(z)]v_0(z)\,dz=0 \tag{62}$$

is satisfied. This gives

$$\omega_2=\frac{\int_0^1 h_2(z)v_0(z)\,dz}{2\omega_0\int_0^1 v_0^2(z)\,dz}. \tag{63}$$

Since $h_2(z)$ is linear with respect to $v_0(z)$, the previous equations show that the arbitrary amplitude of $v_0(z)$ does not influence ω_2. It gives the second order correction of the natural frequency ω_0 corresponding to a uniform beam.

Integrating by parts we have that the numerator of (63) becomes

$$\begin{aligned}&\int_0^1 h_2(z)v_0(z)\,dz\\&=-\omega_1^2\int_0^1 v_0^2(z)\,dz-2\omega_1\omega_0\int_0^1[f_3v_0^2(z)+v_1(z)v_0(z)]\,dz\\&\quad+(f_1'v_1''v_0+f_1v_1'''v_0-f_1v_1''v_0'-f_2v_1'v_0)|_0^1\\&\quad+\int_0^1[f_1(z)v_1''(z)v_0''(z)+f_2(z)v_1'(z)v_0'(z)-\omega_0^2 f_3(z)v_1(z)v_0(z)]\,dz.\end{aligned} \tag{64}$$

In the case $\omega_1=0$ we have that

$$\hat{f}^{real}=\hat{f}+\varepsilon^2\frac{\omega_2}{\omega_0}\hat{f}+\ldots, \tag{65}$$

where $\hat{f}$ is given by (59), so that the error is again given by

$$\frac{\hat{f}^{real}-\hat{f}}{\hat{f}}=\varepsilon^2\frac{\omega_2}{\omega_0}+\cdots \tag{66}$$

It is worth noting that in correspondence to the buckling load we have $\omega_0\to 0$, and so the error becomes unbounded.

4.4. An example

We consider the same example used for the cable (see Section 3.4). In particular, we consider the *hinged–hinged* homogeneous heavy vertical beam of uniform strength. We have $A(z)=\overline{A}e^{-xz}$ (x is given by (38); note that now x can be also negative, since a beam can carry compressive axial load), so that $EJ(z)=E\overline{J}e^{-2xz}$, $N(z)=\sigma\overline{A}e^{-xz}$ and $\rho A(z)=\rho\overline{A}e^{-xz}$. E is the Young modulus, $\overline{J}$ is the moment of inertia for $z=0$. It is worth remarking that the bending stiffness is assumed to be proportional to the square of the area.

From (57) we have

$$EJ_0=E\overline{J}\left(\frac{1-e^{-2x}}{2x}\right)\left(\frac{n^2\pi^2}{x^2+n^2\pi^2}\right), \tag{67}$$

$$N_0=\sigma\overline{A}\left(\frac{1-e^{-x}}{x}\right)\left(\frac{2x^2+4n^2\pi^2}{x^2+4n^2\pi^2}\right), \tag{68}$$

$$\rho A_0=\rho\overline{A}\left(\frac{1-e^{-x}}{x}\right)\left(\frac{4n^2\pi^2}{x^2+4n^2\pi^2}\right), \tag{69}$$

so that from (58) we have

$$\alpha=\overline{\alpha}\left(\frac{4}{1+e^{-x}}\right)\left[\frac{(x^2+n^2\pi^2)(x^2+2n^2\pi^2)}{n^2\pi^2(x^2+4n^2\pi^2)}\right],\quad \overline{\alpha}=\frac{\sigma\overline{A}L^2}{E\overline{J}}, \tag{70}$$

while from (59) we have

$$\hat{f}=\frac{n\sqrt{n^2\pi^2+\alpha}}{2L^2}\sqrt{\frac{E\overline{J}}{\rho\overline{A}}}\sqrt{\left(\frac{1+e^{-x}}{8}\right)\left(\frac{x^2+4n^2\pi^2}{x^2+n^2\pi^2}\right)}. \tag{71}$$

The last square root, which is illustrated in Fig. 3, is the correction due to the non-uniformity of the beam. Note that for large mode numbers, $n\to\infty$, it gets closer to $\sqrt{(1+e^{-x})/2}$.

From Eq. (71) we see that $\hat{f}=0$ if and only if $\alpha=-n^2\pi^2$, i.e., if and only if

$$\overline{\alpha}=\overline{\alpha}_{cr}=-n^2\pi^2\left[\frac{n^2\pi^2(x^2+4n^2\pi^2)}{(x^2+n^2\pi^2)(x^2+2n^2\pi^2)}\right]\left(\frac{1+e^{-x}}{4}\right), \tag{72}$$

which corresponds to the buckling load for the considered non-uniform beam. Note that for $x=0$ we recover the Euler critical value $\overline{\alpha}_{cr}=-n^2\pi^2$ for uniform beams. The product of the terms between brackets in (72) is thus the correction term due to the non-uniformity of the beam. Note that for large values of the mode number n, it becomes equal to $(1+e^{-x})/2$.

The ratio ω_2/ω_0, which permits to detect the error, can be computed in closed form. However, its expression is so involved that cannot be reported. It is depicted in Fig. 4 for different values of α_0. Its asymptotic behavior for $x\to 0$ is $\gamma(\alpha_0)x^2$, where the function $\gamma(\alpha_0)$ is depicted in Fig. 5.

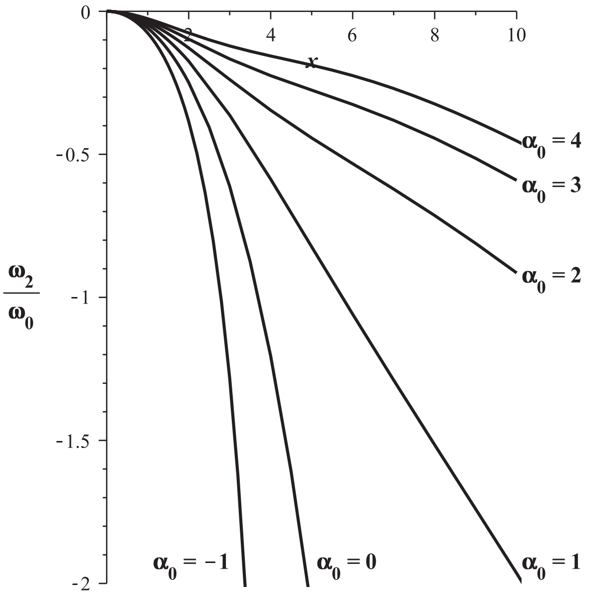

Fig. 4. The error function ω_2/ω_0 for the hinged–hinged beam.

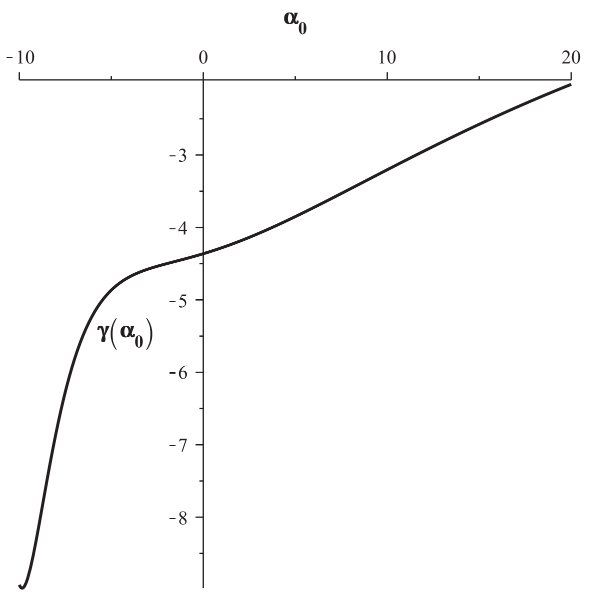

Fig. 5. The function $\gamma(\alpha_0)$ for $n=1$.

4.5. Comparison with an exact solution

To check the effectiveness of the proposed approximate solution, we compare it with an exact one.

We consider the beam with width and thickness varying linearly along the axis. In the absence of axial load, when the mass per unit length is constant, and when both the taper ratios are equal, the governing equation of the free harmonic vibrations is

$$[(1+\varepsilon z)^4 v''(z)]'' - \omega^2 v(z) = 0, \tag{73}$$

where $\varepsilon = (h_L - h_0)/h_0 = (b_L - b_0)/b_0$ is the (unique) taper ratio, h is the thickness and b the width of the beam. This is mainly a theoretical problem, however it has been largely studied in the literature (see for example [11,25]) because it has an exact solution, which is

$$v(z) = c_1\left(z+\frac{1}{\varepsilon}\right)^{-(1/2)\left(1+\sqrt{5-4\sqrt{1+\omega^2/\varepsilon^4}}\right)} + c_2\left(z+\frac{1}{\varepsilon}\right)^{-(1/2)\left(1-\sqrt{5-4\sqrt{1+\omega^2/\varepsilon^4}}\right)}$$
$$c_3\left(z+\frac{1}{\varepsilon}\right)^{-(1/2)\left(1+\sqrt{5+4\sqrt{1+\omega^2/\varepsilon^4}}\right)} + c_4\left(z+\frac{1}{\varepsilon}\right)^{-(1/2)\left(1-\sqrt{5+4\sqrt{1+\omega^2/\varepsilon^4}}\right)}. \tag{74}$$

It is a simpler expression of the same solution obtained in [11,25] by Bessel functions.

By imposing the four appropriate boundary conditions (see Section 4.1) we obtain linear homogeneous systems in the four unknowns c_1, c_2, c_3 and c_4. Setting equal to zero the determinant of the associate matrix, and solving it with respect to ω, permits to obtain the *exact* frequencies $\omega(\varepsilon)$. The first and the second natural frequencies are reported by thick lines in Fig. 6 for different boundary conditions.

Noting that $(1+\varepsilon z)^4 = 1+\varepsilon(4z)+\ldots$ we applied the asymptotic development method developed early by considering $f_1(z)=4z$, $\alpha = f_2(z) = f_3(z) = 0$. Thus, we neglect terms higher that the first, and accordingly we compute only the first-order correction term ω_1. It is exactly (55), which however is not set to zero now to facilitate the comparison with the exact solution.

The results are reported in Table 1 for different boundary conditions. The exact and the approximate solutions are compared in Fig. 6, from which we see a very good agreement, well beyond small values of ε, which confirms the reliability of the proposed solution, especially if we recall that the second-order asymptotic terms are not considered.

5. Conclusions and further developments

The free-vibration problem of non-uniform cables and beam has been addressed by means of the asymptotic development method in order to have simple formula to detect the natural frequencies. The stiffness (for the beam), the normal force and the mass per unit length are allowed to vary along the axis.

After writing the equations in dimensionless form, the Poincaré–Lindstedt method is applied by considering a small variation of the mechanical parameters. The first-order term corresponds to the classical solution for uniform media. The first-order term provides the mechanical properties of the *equivalent* uniform cable/beam, from which simple formulas for the natural frequencies are obtained. Since they come from the vanishing of the first-order frequency correction, they provided an approximation valid up to the first-order. The error was detected by computing the second-order frequency correction of the asymptotic development.

The theoretical results have been illustrated by means of some examples. The vertical heavy cable and beam of constant strength have been considered. They have an exponential variation of the area and of the bending stiffness. Their natural frequencies have been determined, and an error estimation has been provided.

The case of linearly varying width and thickness has also been considered for the beam, since in this case the exact solution is known. Comparison of the exact with our approximate results shows the effectiveness of the proposed method, which stands up to large values of the smallness parameters.

The present work proposes a methodological approach which, at least in principle, can be extended to other cases. For example, different beam models (Timoshenko beam, rotatory inertia, etc.) as well as 2D or 3D continuum models can be considered. It can also

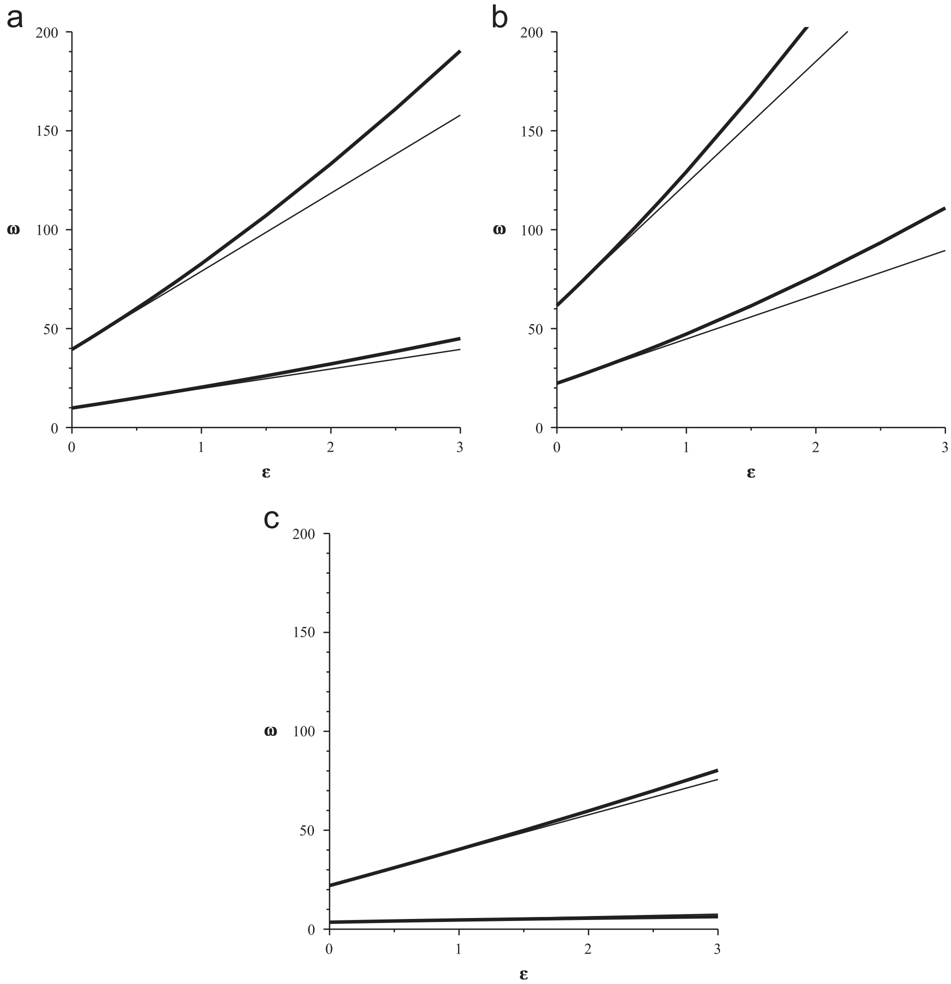

Fig. 6. The functions $\omega(\varepsilon)$ for (a) hinged–hinged, (b) fixed–fixed and (c) fixed-free beams. The thick lines are the exact solutions, the thin lines are the approximate solutions $\omega=\omega_0+\varepsilon\omega_1$ (see Table 1). The first and the second natural frequencies are reported.

Table 1
The zero and first order frequencies for different boundary conditions.

Boundary conditions	Frequency order	ω_0	ω_1
Hinged–hinged	1st	π^2	π^2
	2nd	$4\pi^2$	$4\pi^2$
Fixed–fixed	1st	22.3733	22.3733
	2nd	61.6728	61.6728
Fixed-free	1st	3.5160	1.3604
	2nd	22.0345	17.8854

been extended to the nonlinear case. Another possible development consists in assuming also a time variation of the mechanical properties, thus extending, for example, the problem of a time-varying mass oscillator addressed in [44] or time-varying stiffness oscillators, such as those subjected to parametric excitation of the Mathieu type. It is also expected that some applications in the field of optimization can be found.

Acknowledgments

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