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The linear theory of free vibrations of a suspended cable

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[Plates 10 and 11]

A linear theory is developed for the free vibrations of a uniform suspended cable in which the ratio of sag to span is about 1:8, or less. Both in-plane and out-of-plane motion is considered. It is shown that the analysis of the symmetric in-plane modes is heavily dependent on a parameter which allows for the effects of cable geometry and elasticity. The results of simple experiments are reported which establish the validity of the theory.

Introduction

The theory of the vibrations of suspended cables, which are supported at one or both ends, has been increasingly refined during its development. There is, however, an inadequacy in part of the theory which has apparently gone unnoticed or unsolved until now.

Stated briefly, the inadequacy has arisen because practically all previous theories which are often valid for cables with ratios of sag to span of about 1:8 (and one of which tends to the theory for the vibrations of a vertical cable as the ratio of sag to span becomes very great), cannot be reconciled with the theory of vibrations of a taut string when the ratio of sag to span becomes very small.

When cables, which are fixed at each end, are used to support transverse loads, structural efficiency requires that the profiles of the cables be relatively flat. It appears that a correct linear theory of vibration is missing for cables of this sort, where the ratio of sag to span is about 1:8, or less.

Consequently, attention is confined in this paper to uniform cables, supported at each end, for which the sags are sufficiently small for parabolic profiles to describe accurately their static geometry. The theories to be presented, for which experimental confirmation has been obtained, are of considerable practical importance.

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Historical background†

During the first half of the eighteenth century elements of the theory of vibration of a taut string, which was fixed at each end, were presented by Brook Taylor, D'Alembert, Euler and Daniel Bernoulli. In 1732 Daniel Bernoulli investigated the transverse oscillations of a uniform cable, supported at one end and hanging under gravity. The same problem was also discussed by Euler nearly fifty years later, in 1781. Both Bernoulli and Euler gave the solution for the natural frequencies in the form of an infinite series. This series is now represented by a zero order Bessel function of the first kind, and so their work on this mechanical problem was a forerunner of the theory of Bessel functions.

At this time, however, the theory of partial differential equations was still in its infancy and considerable work had centred around the analysis of discrete, rather than continuous, systems. For example, by 1788 Lagrange and others before him, had obtained solutions of varying degrees of completeness for the vibrations of an inextensible, massless string, fixed at each end, from which numerous weights were hung. The general equations of motion of discrete systems were first given by Lagrange in 1760 and appeared later in Mécanique Analytique in 1788.

The most important contribution to the theory of cable vibrations came in 1820 when Poisson published a paper which gave the general Cartesian partial differential equations of the motion of a cable element under the action of a general force system. These equations were the dynamic analogue of the static equations given by Fuss in 1796. Poisson used these equations to improve the solutions previously obtained for the vertical cable and the taut string.

Thus, by 1820 correct solutions had been given for the linear, free vibrations of uniform cables the geometries of which were the limiting forms of the catenary. Apart from Lagrange's work on the equivalent discrete system, no results had been given for the free vibrations of cables where the sag to span ratio was not either zero or infinite.

In collaboration with Stokes, Rohrs (1851) obtained an approximate solution for the symmetric vertical vibrations of a uniform suspended cable where the ratio of sag to span was small, although appreciable. He arrived at his solution by using a form of Poisson's general equations, correct to the first order and, in addition, used another equation which he termed, 'the equation of continuity of the chain'. He assumed the chain to be inextensible so this continuity equation related only to geometric compatibility.

Routh (1868) gave an exact solution for the symmetric vertical vibrations (and associated longitudinal motion) of a heterogeneous cable which hung in a cycloid. Like Rohrs, he assumed that the cable was inextensible. He showed that the result for the cycloidal cable reduced to Rohr's solution for the uniform cable when the ratio of sag to span was small. Routh also obtained an exact solution for the

† References to the early work on this problem may be found in Routh (1868), Routh (1891), Watson (1966), and Whittaker (1944).

antisymmetric, vertical vibrations (and associated longitudinal motion) of the cycloidal cable.

At this point the subject appears to have been laid to rest until Rannie & von Kármán (1941) and W. D. Rannie (1973, personal communication) independently derived results for both the symmetric and antisymmetric vertical vibrations of an inextensible, three-span cable. In work done in 1945, Vincent (1965) extended Rannie & von Kármán's analysis to allow for the effects of cable elasticity in the calculation of the symmetric vertical motion of the three-span cable. However, he did not explore the nature of the solution so obtained, and, therefore, he appears to have been unaware of the substantial effect which the inclusion of cable elasticity can have on the analysis. A comprehensive study of the theory of vibration in suspension bridges was presented by Bleich *et al.* (1950) in which, among other things, consideration was given to the effects of cable elasticity. Since Bleich *et al.* were concerned with long span cables with ratios of sag to span of about 1:10, the effects of cable elasticity were, in general, not appreciable. All these works were prompted by the aerodynamic failure of the Tacoma Narrows suspension bridge in 1940.

Pugsley (1949) put forward a semi-empirical theory for the natural frequencies of the first three in-plane modes of a uniform suspended cable. He demonstrated the applicability of the results by conducting experiments on cables in which the ratio of sag to span ranged from 1:10 up to approximately 1:4.

By assuming again that the uniform cable was inextensible, Saxon & Cahn (1953) made a major contribution to the theory of the in-plane vibrations. They obtained solutions which effectively reduced to the previously known results for inextensible cables of small sag to span ratios, and for which asymptotic solutions gave extremely good results for large ratios of sag to span. The accuracy of their theory was demonstrated by comparing it with the experiments of Rudnick, Leonard & Saxon, Cahn & Saxon, and Pugsley. In all these experiments the ratio of sag to span was 1:10, or greater.

One of the most interesting aspects of the latter development of the theory of symmetric in-plane vibrations of a suspended cable is that there have been neither theories nor experiments which have sought to explain a discrepancy that arises as the ratio of sag to span reduces to zero. For small sag to span ratios previous theories, which have been derived assuming the cable to inextensible, show that the first symmetric in-plane mode, primarily involving vertical motion, occurs at a frequency which is contained in the first non-zero root of (as will be shown)

$$\tan\left(\tfrac{1}{2}\beta l\right) = \tfrac{1}{2}\beta l$$

namely,

$$(\beta l)_1 \simeq 2.86\pi,$$

where $\beta = (m\omega^2/H)^{\frac{1}{2}}$, and m is the mass per unit length of the cable, H is the horizontal component of cable tension (static), ω is the natural circular frequency of vibration and l is the span of the cable.

However, it has long been known that the frequency of the first symmetric mode of transverse vibration of a taut string is contained in the first root of

$$\cos \tfrac{1}{2}\beta l = 0,$$

namely

$$(\beta l)_1 = \pi.$$

This discrepancy, which amounts to almost 300%, cannot be resolved by the previous analyses of inextensible cables.

Inextensibility is a concept which needs to be used with great care. No real cables are ever inextensible. Clearly, a taut string must stretch when vibrating in a symmetric mode, although standard analyses often overlook this point. Likewise a cable which has a very small sag to span ratio must stretch when vibrating with symmetric vertical motion. However, if the concept of inextensibility is adhered to, it must be concluded from the previous analyses that the classical, first symmetric vertical mode does not exist if even the slightest sag is present. This is at odds with reality and the matter is resolved in the following paragraphs.

The linear theory

Consider a uniform cable which hangs in static equilibrium in a vertical plane through supports, which are located at the same level. The equations of static equilibrium of an element of the cable are

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\left(T\frac{\mathrm{d}x}{\mathrm{d}s}\right) &= 0, \\ \frac{\mathrm{d}}{\mathrm{d}s}\left(T\frac{\mathrm{d}y}{\mathrm{d}s}\right) &= -mg, \end{aligned}\right\} \tag{1}$$

where m is the mass of the cable per unit length, g is the acceleration due to gravity, T is the cable tension, $\mathrm{d}x/\mathrm{d}s$ is the cosine of the angle of inclination and $\mathrm{d}y/\mathrm{d}s$ is the sine of the angle of inclination. The solution of these equations gives the catenary. However, if the slope of the cable is everywhere small, the profile adopted by the cable is accurately given by the parabola.

$$y = \frac{mgl^2}{2H}\left\{\frac{x}{l} - \left(\frac{x}{l}\right)^2\right\}, \tag{2}$$

where l is the span, H is the horizontal component of cable tension specified by

$$H = mgl^2/8d,$$

and d is the sag of the cable. It is here assumed that the static effects of cable elasticity have been accounted for in the determination of H and d (see appendix).

If the cable is given a small, arbitrary displacement from its position of static equilibrium then, subsequently, the equilibrium of an element requires that

$$\left.\begin{aligned}\frac{\partial}{\partial s}\left\{(T+\tau)\left(\frac{\mathrm{d}x}{\mathrm{d}s}+\frac{\partial u}{\partial s}\right)\right\} &= m\frac{\partial^2 u}{\partial t^2},\\ \frac{\partial}{\partial s}\left\{(T+\tau)\left(\frac{\mathrm{d}y}{\mathrm{d}s}+\frac{\partial v}{\partial s}\right)\right\} &= m\frac{\partial^2 v}{\partial t^2}-mg,\\ \frac{\partial}{\partial s}\left\{(T+\tau)\frac{\partial w}{\partial s}\right\} &= m\frac{\partial^2 w}{\partial t^2},\end{aligned}\right\} \tag{3}$$

where u and v are the longitudinal and vertical components of the in-plane motion, respectively, w is the transverse horizontal component of motion (perpendicular to the vertical plane through the supports) and τ is the additional cable tension caused by the motion (see figure 1). The components of motion and the additional tension are functions of both position and time.

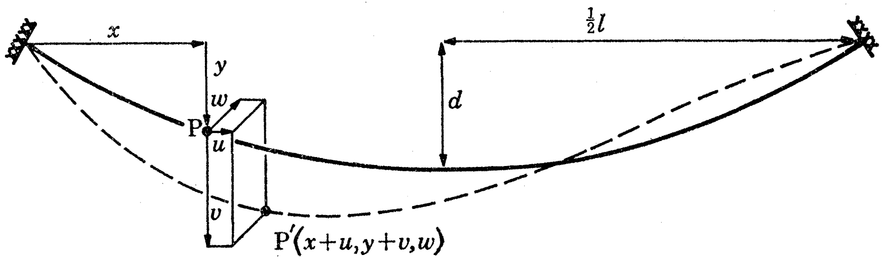

FIGURE 1. Definition diagram for cable vibrations.

These equations may be simplified for the problem at hand. Each equation is expanded, substitutions are made for the equations of static equilibrium, and terms of the second order are neglected. In addition, since the analyses are to be valid for cables with ratios of sag to span of about 1:8, or less, the longitudinal component of the equations of motion is unimportant and may be ignored (i.e. the first equation in equation (3) is dropped). Consequently, the equations of motion are reduced to

$$H\frac{\partial^2 v}{\partial x^2}+h\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = m\frac{\partial^2 v}{\partial t^2}, \tag{4}$$

and

$$H\frac{\partial^2 w}{\partial x^2} = m\frac{\partial^2 w}{\partial t^2}, \tag{5}$$

where h is defined as the additional horizontal component of cable tension and is a function of time alone.

The linearized cable equation, which provides for the elastic and geometric compatibility of the cable element, reads (Irvine 1974)

$$\frac{h\,(\mathrm{d}s/\mathrm{d}x)^3}{E_\mathrm{c}A_\mathrm{c}} = \frac{\partial u}{\partial x}+\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\partial v}{\partial x}, \tag{6}$$

where E_c is the modulus of elasticity of the cable and A_c is the area of the cable. The cable equation gives the closure condition for the symmetric vertical motion and it also allows for the calculation of the longitudinal motion. Thus (4), (5) and (6) are the linearized equations which govern the problem.

It will be noticed that the transverse horizontal motion has uncoupled from the in-plane motion because, to first order, the transverse horizontal motion involves no additional cable tension. This is consistent with experience since, for a chain hanging across a driveway, the only mode of vibration easily excited is its first, swinging mode. Therefore, to first order, a disturbance which has no in-plane components will induce only transverse horizontal motion, and vice-versa..

Under the restrictions placed here on cable geometry it is the vertical component of motion which is most apparent when the cable vibrates in an in-plane mode. The amplitude of the corresponding longitudinal modal component is always substantially less than the amplitude of the vertical motion. (It is for this reason that equation (6) is, by itself, adequate to describe the longitudinal motion.) Consequently, a symmetric in-plane mode is defined as one in which the vertical component of the mode is symmetric, and vice-versa.

The transverse horizontal motion will be considered first because it is the easiest to analyse.

(a) *The transverse horizontal motion*

By writing $w(x,t) = \tilde{w}(x)\,\mathrm{e}^{\mathrm{i}\omega t}$, where ω is the natural circular frequency of vibration, equation (5) is reduced to

$$H\,\mathrm{d}^2\tilde{w}/\mathrm{d}x^2 + m\omega^2\tilde{w} = 0. \tag{7}$$

The boundary conditions are

$$\tilde{w}(0) = \tilde{w}(l) = 0$$

from which it is found that the natural frequencies of vibrations are

$$\omega_n = \frac{n\pi}{l}\sqrt{\left(\frac{H}{m}\right)} \quad n = 1, 2, 3, \ldots, \tag{8}$$

and the transverse horizontal modes are given by

$$\tilde{w}_n(x) = A_n \sin(n\pi x/l) \quad n = 1, 2, 3, \ldots, \tag{9}$$

where $n = 1, 2, 3, \ldots,$ signify the first, second, third, etc., modes, respectively.

The frequency of the first transverse horizontal mode (i.e. $n = 1$) is the lowest natural frequency of any given parabolic cable.

(b) *The in-plane motion*

As defined previously, antisymmetric in-plane modes consist of antisymmetric vertical components and (as will be shown) symmetric longitudinal components, while symmetric in-plane modes consist of symmetric vertical components and antisymmetric longitudinal components. In the former case, to first order, no additional cable tension is induced by the motion; however, additional cable tension is induced by the motion in the latter case.

(i) *Antisymmetric in-plane modes*

Since, to first order, the additional horizontal component of cable tension is zero, equation (4) becomes

$$H\frac{\mathrm{d}^2\tilde{v}}{\mathrm{d}x^2}+m\omega^2\tilde{v}=0, \tag{10}$$

where the substitution, $v(x,t)=\tilde{v}(x)\,\mathrm{e}^{\mathrm{i}\omega t}$, has been made. The cable equation is reduced to a statement of geometric compatibility

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}x}+\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}\tilde{v}}{\mathrm{d}x}=0, \tag{11}$$

where the substitution, $u(x,t)=\tilde{u}(x)\,\mathrm{e}^{\mathrm{i}\omega t}$, has also been made. Together with the boundary conditions, $\tilde{v}(0)=\tilde{v}(\frac{1}{2}l)=0$, equations (10) and (11) are sufficient to obtain the natural frequencies and modal components of the antisymmetric in-plane modes.

It is easily shown that the natural frequencies are given by

$$\omega_n=\frac{2n\pi}{l}\sqrt{\left(\frac{H}{m}\right)},\quad n=1,2,3,\ldots, \tag{12}$$

where $n=1,2,3,\ldots$, signify the natural frequencies of the first, second, third, etc., antisymmetric in-plane modes.

The antisymmetric vertical modal components are given by

$$\tilde{v}_n(x)=A_n\sin{(2n\pi x)}/l),\quad n=1,2,3,\ldots. \tag{13}$$

The longitudinal components of motion in these modes are found from equation (11), and it is seen that these components are symmetric since $\mathrm{d}y/\mathrm{d}x$ is zero at mid-span. After substitution of equation (13), integration and rearrangement, it is found that

$$\tilde{u}_n(x)=-4\left(\frac{d}{l}\right)A_n\left\{\left(1-2\left(\frac{x}{l}\right)\right)\sin\left(\frac{2n\pi x}{l}\right)+\frac{1}{n\pi}\left(1-\cos\left(\frac{2n\pi x}{l}\right)\right)\right\}, \tag{14}$$

where, as before, A_n is the amplitude of the nth antisymmetric vertical component of the mode.

It is clear that the amplitudes of the longitudinal components become very small as the cable becomes flatter (i.e. as $d/l\rightarrow 0$). However, these longitudinal components have some peculiar properties. The maximum displacement of the first component occurs at the quarter span points and not at midspan. The displacement is a local minimum at midspan (see figure 2*a*). Also, both the slope and displacement are zero at midspan for the second component (see figure 2*b*). This pattern repeats itself for the higher longitudinal components.

Equations (12), (13) and (14) are similar to those given by Routh (1868), which he deduced from an exact analysis of the inextensible, cycloidal cable by allowing the ratio of sag to span to become small.

A more exact analysis of the antisymmetric in-plane modes of the parabolic cable, done concurrently by the second author, has shown that the assumption, $h = 0$, is a very good one. For a cable with a ratio of sag to span of 1:8, this assumption results in an error of less than 4 % in the determination of the natural frequencies and modal components. The error may be attributed to the longitudinal component of the equations of motion which has been neglected. When this is accounted for small, second-order changes in cable tension do occur.

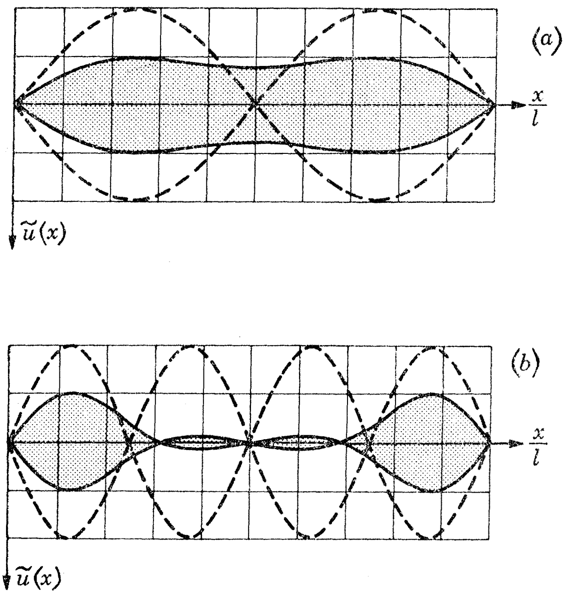

FIGURE 2. Longitudinal components and associated vertical components of the first two antisymmetric in-plane modes (vertical scale arbitrary). (a) First symmetric longitudinal component. (b) Second symmetric longitudinal component.

(ii) *Symmetric in-plane modes*

Here, additional cable tension is induced by the motion and equation (4) becomes

$$H\frac{\mathrm{d}^2\tilde{v}}{\mathrm{d}x^2}+m\omega^2\tilde{v}=\frac{8d}{l^2}\tilde{h}, \tag{15}$$

where the substitutions, $v(x,t)=\tilde{v}(x)\,\mathrm{e}^{\mathrm{i}\omega t}$, $h(t)=\tilde{h}\,\mathrm{e}^{\mathrm{i}\omega t}$ and $\mathrm{d}^2y/\mathrm{d}x^2=-8d/l^2$, have been made. The cable equation is

$$\frac{\tilde{h}(\mathrm{d}s/\mathrm{d}x)^3}{E_\mathrm{c}A_\mathrm{c}}=\frac{\mathrm{d}\tilde{u}}{\mathrm{d}x}+\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}\tilde{v}}{\mathrm{d}x} \tag{16}$$

where, again, the substitution, $u(x,t)=\tilde{u}(x)\,\mathrm{e}^{\mathrm{i}\omega t}$, has been made. Together with the boundary conditions $\tilde{u}(0)=\tilde{u}(l)=0$, $\tilde{v}(0)=\tilde{v}(l)=0$, equations (15) and (16) are sufficient to obtain the natural frequencies and modal components of the symmetric in-plane modes.

It proves convenient to proceed with a detailed discussion of the natural frequencies and the vertical components. A discussion of the longitudinal components will be given later.

The solution of equation (15), with the given boundary conditions, is

$$\frac{\tilde{v}(x)}{8d} = \frac{\tilde{h}}{H}\frac{1}{(\beta l)^2}\{1-\tan(\tfrac{1}{2}\beta l)\sin\beta x-\cos\beta x\}, \tag{17}$$

where $\beta = (m\omega^2/H)^{\frac{1}{2}}$ and the value of (βl) specifies the particular (symmetric) vertical modal component. With the boundary conditions, equation (16) may be integrated to give

$$\frac{\tilde{h}L_e}{E_c A_c} = \frac{8d}{l^2}\int_0^l \tilde{v}(x)\,dx, \tag{18}$$

where $L_e = \int_0^l (ds/dx)^3\,dx \simeq l\{1+8(d/l)^2\}$. Use is now made of equation (18) to eliminate $\tilde{h}$ and obtain the following transcendental equation from which the natural frequencies of the symmetric in-plane modes may be found

$$\tan(\tfrac{1}{2}\beta l) = (\tfrac{1}{2}\beta l) - (4/\lambda^2)(\tfrac{1}{2}\beta l)^3, \tag{19}$$

where $$\lambda^2 = \left(\frac{8d}{l}\right)^2 \frac{l}{(HL_e/E_c A_c)}.$$

This equation is of fundamental importance in the theory of cable vibrations. It is seen that λ^2, the parameter involving cable geometry and elasticity, governs the nature of the roots of the equation. The eigenvalue problem specified by equation (19) is strongly non-linear with respect to this parameter.

In order to illustrate the following discussion, reference should be made to figure 3 where a graphical solution for the first non-zero root of equation (19) is presented.

When λ^2 is very large, the cable may be assumed inextensible and equation (19) is reduced to

$$\tan(\tfrac{1}{2}\beta l) = \tfrac{1}{2}\beta l. \tag{20}$$

This is the transcendental equation first given by Rohrs (1851) and later by Routh (1868). The equation occurs in other problems in mechanics, including the flexural and torsional buckling of struts under certain boundary conditions.

By using a more exact analysis, the second author has shown that equation (20) is in error by less than 0.2 % for an inextensible cable with a ratio of sag to span of 1:8.

The first two roots of equation (20) are

$$(\beta l)_{1,2} \simeq 2.86\pi,\ 4.92\pi \tag{21}$$

and higher roots are quite accurately given by

$$(\beta l)_n \simeq (2n+1)\pi, \quad n = 3, 4, 5, \ldots, \tag{22}$$

where $(\beta l)_n$ contains the frequency of the nth symmetric in-plane mode of an inextensible cable.

The other limiting value of λ^2 occurs when the ratio of sag to span becomes very small. The cable approaches a taut string and λ^2 is very small. The general transcendental equation is then reduced to

$$\tan\left(\tfrac{1}{2}\beta l\right) = -\infty \tag{23}$$

and the roots of this equation correspond to those of the symmetric modes of the taut string, namely,

$$(\beta l)_n = (2n-1)\pi, \quad n = 1, 2, 3, \ldots. \tag{24}$$

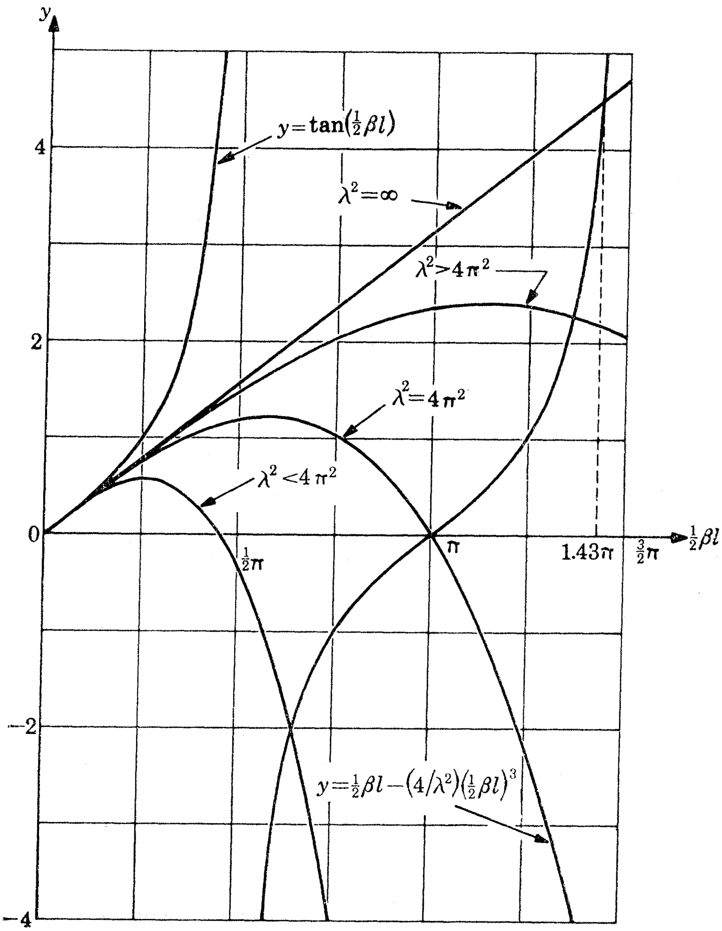

Figure 3. Graphical solution for first non-zero root of equation (19).

A comparison of equations (22) and (24) shows that the condition of inextensibility causes a shift of almost 2π in the roots obtained from the transcendental equation governing the symmetric modes of a taut string.

The preceding analysis indicates that the first non-zero root of equation (19) lies between

$$\tfrac{1}{2}\pi \quad \text{and} \quad 1.43\pi;$$

the second root lies between

$$\tfrac{3}{2}\pi \quad \text{and} \quad 2.46\pi;$$

the third root lies between

$$\tfrac{5}{2}\pi \quad \text{and} \quad \approx \tfrac{7}{2}\pi;$$

and so on. The actual values of the roots depend on λ^2. Three important cases are now considered:

(*a*) If $\lambda^2 < 4\pi^2$, then the frequency of the first symmetric in-plane mode is less than the frequency of the first antisymmetric in-plane mode. The first symmetric, vertical modal component has no internal nodes along the span (see figure 4*a*).

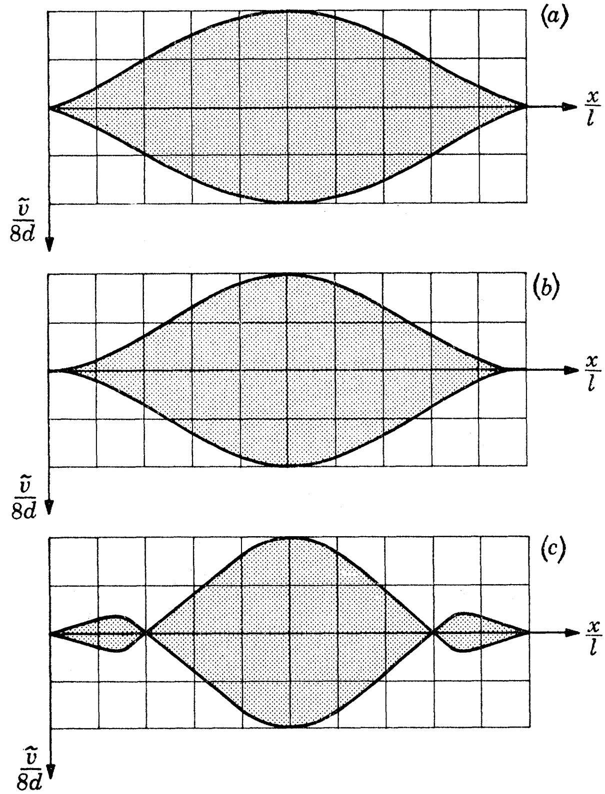

FIGURE 4. Possible forms of the first symmetric vertical modal component. (*a*) $\lambda^2 < 4\pi^2$; (*b*) $\lambda^2 = 4\pi^2$; (*c*) $\lambda^2 > 4\pi^2$.

(*b*) If $\lambda^2 = 4\pi^2$, then the frequency of the first symmetric in-plane mode is equal to the frequency of the first antisymmetric in-plane mode. This value of λ^2 gives the first 'cross-over' point. The first symmetric, vertical modal component is tangential to the static cable profile at each support (see figure 4*b*).

(*c*) If $\lambda^2 > 4\pi^2$, then the frequency of the first symmetric in-plane mode is greater than the frequency of the first antisymmetric in-plane mode. The first symmetric, vertical modal component has two internal nodes along the span (see figure 4*c*).

It may also be noted that, if $4\pi^2 < \lambda^2 < 16\pi^2$, both the first and second symmetric, vertical modal components have two internal nodes along the span. If $\lambda^2 = 16\pi^2$, then the frequency of the second symmetric in-plane mode is equal to the frequency of the second antisymmetric in-plane mode. This value of λ^2 gives the second cross-over point, and so on.

It is obvious from the above discussion that the solutions of equation (19) are strongly influenced by the value of the characteristic parameter, λ^2. This parameter can appear in the analysis only if the effects of cable elasticity are accounted for. If the cable is assumed inextensible at the outset, the correct solution cannot be found. In most practical problems it is the cable geometry term, $(8d/l)^2$, rather than the cable elasticity term, $l/(HL_e/E_c A_c)$, which dictates the size of λ^2. For example, suspension bridges and transmission lines typically have values of $(8d/l)^2 \approx 1$, while $l/(HL_e/E_c A_c \approx 10^3$, which give $\lambda^2 \approx 10^3$. On the other hand, to ensure the structural efficiency of the cables supporting a flat sag cable roof, $l/(HL_e/E_c A_c) \approx 10^3$, but $(8d/l)^2 \approx 10^{-4} \rightarrow 10^{-2}$, which give rise to values of λ^2 in the range $10^{-1} \rightarrow 10^1$. By assuming the cable to be inextensible, Saxon & Cahn (1953), and others, were led to the wrong conclusion regarding the symmetric in-plane modes as the ratio of sag to span becomes small.

The longitudinal modal components are found from equation (16). After the substitution for equation (17) is made and the integration is performed, the following equation is obtained

$$\frac{\tilde{u}(x)}{8d} = \left(\frac{8d}{l}\right)\frac{\tilde{h}}{H}\frac{1}{(\beta l)^2}\left[\frac{(\beta l)^2}{\lambda^2}\frac{L_x}{L_e} - \frac{1}{2}\left(1 - 2\left(\frac{x}{l}\right)\right)\{1 - \tan\left(\tfrac{1}{2}\beta l\right)\sin\beta x - \cos\beta x\} - \frac{1}{\beta l}\{(\beta x) - \tan\left(\tfrac{1}{2}\beta l\right)(1 - \cos\beta x) - \sin\beta x\}\right], \quad (25)$$

where
$$L_x \simeq l\left[\frac{x}{l} + 24\left(\frac{d}{l}\right)^2\left\{\frac{x}{l} - 2\left(\frac{x}{l}\right)^2 + \frac{4}{3}\left(\frac{x}{l}\right)^3\right\}\right].$$

These longitudinal components are antisymmetric since equations (25) and (16) show that the longitudinal displacement and slope are always zero and non-zero at midspan, respectively. Like the symmetric vertical modal components, the nature of the antisymmetric longitudinal modal components depends on the value of the characteristic parameter, λ^2.

(c) *General remarks*

The theories presented here are abridged from a section of a report (Irvine, 1974), and represent one way to approach the solution. Several other aspects of cable vibrations are treated in this report and mention may be made of them here.

The use of Fourier series permits the analysis of the symmetric in-plane modes to be approached from a different viewpoint. A proof of the orthogonality of the modes of vibration is given, and several numerical examples are worked out to illustrate the applications of the theories to practical problems.

(*a*)

(*b*)

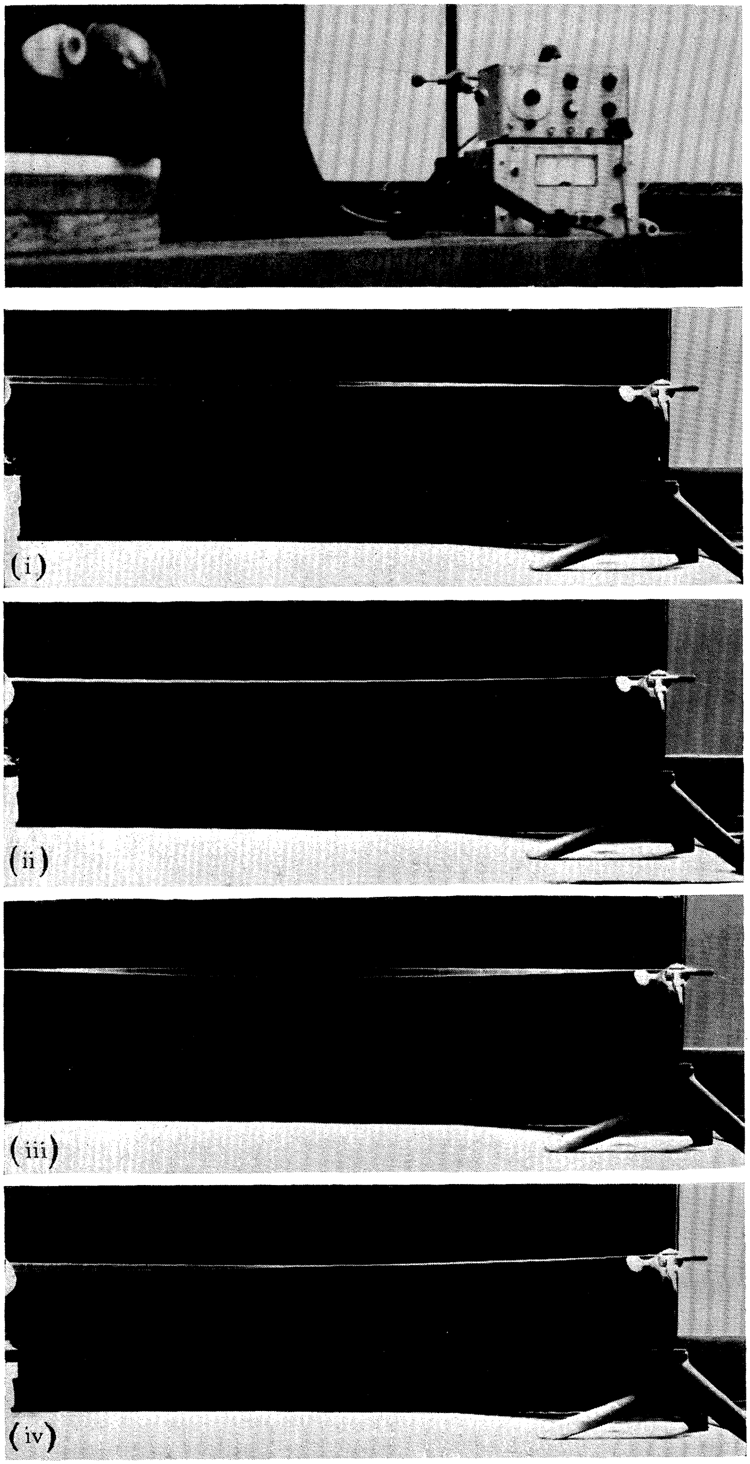

FIGURE 5. (*a*) Experimental set up. (*b*) Possible forms of the first-in-plane mode of a suspended cable (mid-span is at left-hand edge).

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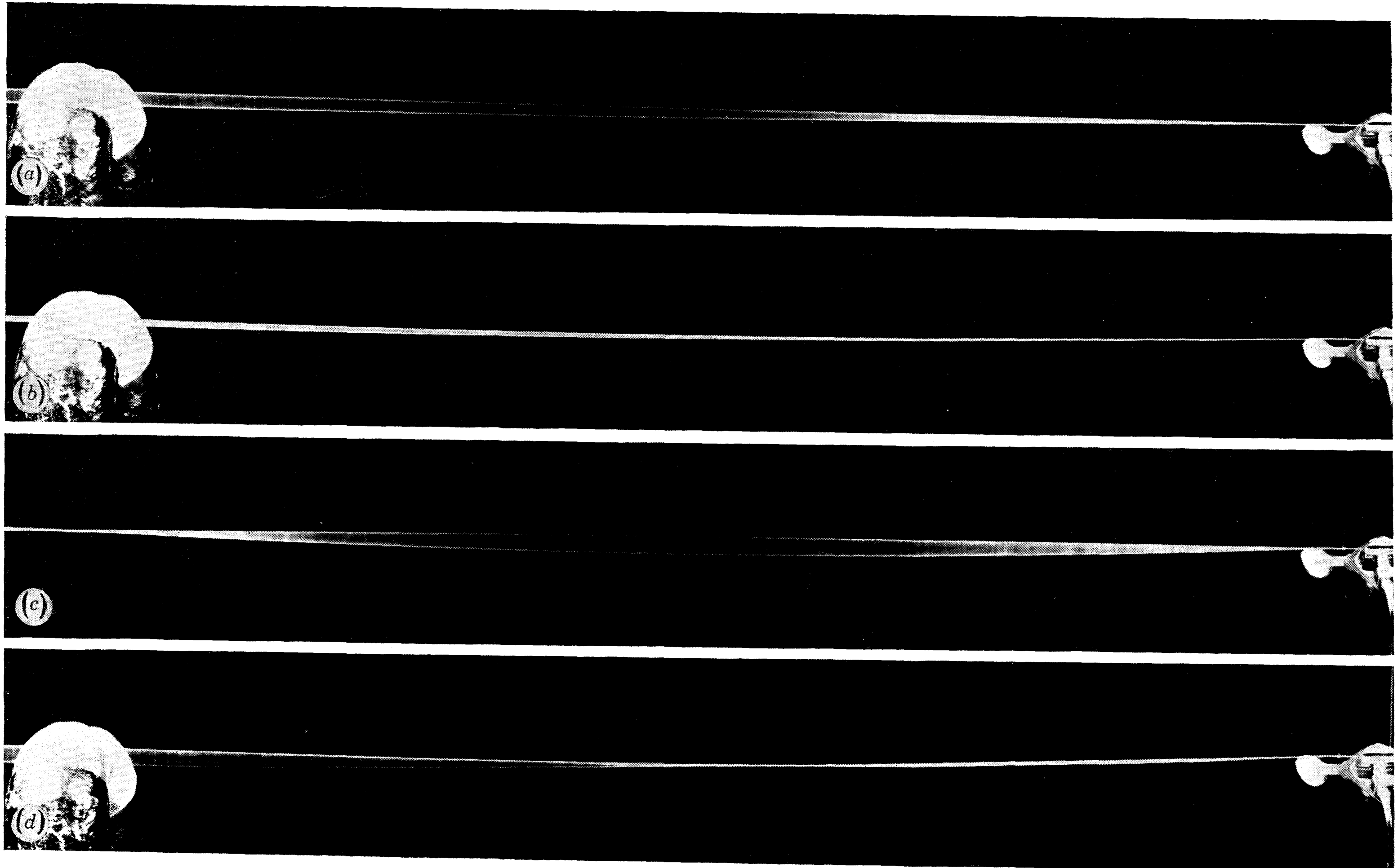

FIGURE 6. Enlarged versions of figure 5*b*. (*a*) $\lambda^2 \ll 4\pi^2$, first symmetric mode of a taut string; (*b*) $\lambda^2 \simeq 4\pi^2$, first symmetric in-plane mode at first 'cross-over' point; (*c*) $\lambda^2 \simeq 4\pi^2$, first antisymmetric in-plane mode at first 'cross-over' point; (*d*) $\lambda^2 > 4\pi^2$, first symmetric in-plane mode of an (essentially) inextensible suspended cable.

Cable vibration experiments

Vibration experiments were conducted on a model cable in order to test the linear theory presented in the preceding section. The experiments were made very simple because only a qualitative verification of the theory seemed necessary. It was felt that, if the first cross-over point of the in-plane modes of vibration could be shown to exist, the validity of the theory would be established.

With this in mind, the following procedure was adopted.

(*a*) *Experimental procedure*

A fine copper wire was fastened between two supports which were at the same level and about 2 m apart. The cable was placed between the poles of a magnet and an amplifier, connected to an oscillator, was used to pass a small alternating current along the cable. Modes of vibrations were found by tuning the oscillator so that the frequency of the alternating current coincided with the natural frequency of the required mode. By ensuring that the gain of the amplifier was kept low, nonlinear effects owing to large amplitudes of vibration were avoided. When a mode had been isolated, a long-exposure photograph was taken to capture the envelope of vibration. The cable was painted white, strong overhead lighting was used, and the photographs were taken against a black back-ground to ensure that high contrast was achieved (see figure 5*a*, plate 10).

(*b*) *Experimental results*

The results are shown in figure 5*b* and enlarged versions (in the same order) are shown in figure 6, plate 11. Only the first in-plane modes appear. Since the form of the first antisymmetric in-plane mode was found to be constant (as predicted by the theory), only one such mode has been shown. The other modes are various forms of the first symmetric in-plane mode.

It is clear that changes in the value of the characteristic parameter, λ^2, caused substantial changes in the nature of the first symmetric in-plane mode. Changes in λ^2 were brought about by varying the sag of the cable. With the cable pulled taut, λ^2 was very small, and the first mode excited was that of the classical taut string (see figure 6*a*). In order to search for the first cross-over point (i.e. $\lambda^2 = 4\pi^2$), the cable was gradually slackened off until a point was reached where the frequencies of the first symmetric and first antisymmetric in-plane modes were substantially equal (see figures 6*b*, *c*). With a further increase in sag, the frequencies of the first symmetric and first antisymmetric in-plane modes diverged again. However, here the frequency of the first symmetric mode was greater than the frequency of the first antisymmetric mode. Only a very small increase in sag was necessary before the ratio of the natural frequencies settled down to a value of about 1.4:1, as required by the inextensible theory (see figure 6*d*).

It will be noticed from the photographs in figures 5*b* and 6 that the first cross-over point occurred at a very small value of the sag to span ratio; the ratio was about 1:50. It should not be inferred from this that such a value of the ratio of sag to span is typical of parabolic cables in general. It is not. For example, if the same copper wire had spanned 200 m instead of 2 m then, provided that the wire did not break under its own weight, it can readily be shown that the first cross-over point would occur at a ratio of sag to span of about 1:11. The photographs give striking visual proof of the validity of the theory.

Conclusions

A linear theory has been presented for the free vibrations of uniform, suspended cables where the ratios of sag to span are about 1:8, or less. This restriction covers most cases of practical importance.

It was shown that, to first order, the transverse horizontal motion uncouples from the in-plane motion. The transverse horizontal motion is particularly easy to analyse.

A full description of the in-plane motion can be given by ignoring the longitudinal component of the equations of motion and concentrating on just two equations. These are: the equation of motion for the vertical component, and the cable equation, which provides for the elasticity of the cable and for geometric compatibility between the longitudinal and vertical components of the in-plane modes.

The antisymmetric in-plane modes consist of antisymmetric vertical components and symmetric longitudinal components. To first order, no additional cable tension is induced by this motion.

The symmetric in-plane modes consist of symmetric vertical components and antisymmetric longitudinal components. Additional cable tension is induced by this motion. It was found that a parameter involving cable geometry and elasticity, λ^2, is of fundamental importance. This parameter can appear only if the extensibility of the cable is allowed for in the cable equation. When λ^2 is very large, the cable may be assumed inextensible and the results of previous theories are obtained. However, when λ^2 is very small the classical theory of the taut string is obtained. Previous theories, in which the cable was assumed inextensible, cannot be reconciled with the theory of the taut string as the ratio of sag to span becomes very small (i.e. λ^2 becomes very small).

The theory of the symmetric in-plane modes predicts the existence of cross-over points. For example, at the first cross-over point (i.e. $\lambda^2 = 4\pi^2$), the frequencies of the first symmetric and first antisymmetric in-plane modes are equal. Cable vibration experiments were reported in which the first cross-over point was isolated. It may be concluded that the validity of the theory has been established.

Discussions held with P. C. Jennings of the California Institute of Technology are gratefully acknowledged.

Appendix. The elastic parabola

If the cable is assumed to be inextensible then, provided that the ratio of sag to span is about 1:8, or less, the basic relations are

$$\left.\begin{aligned} y &= 4d\left\{\frac{x}{l}-\left(\frac{x}{l}\right)^2\right\}, \\ H &= \frac{mgl^2}{8d}, \\ L &\simeq l\left\{1+\frac{8}{3}\left(\frac{d}{l}\right)^2-\frac{32}{5}\left(\frac{d}{l}\right)^4+\dots\right\}, \end{aligned}\right\} \tag{26}$$

where L is the length of the inextensible cable and the other quantities are as defined previously. If m and l are known only one of the three remaining variables (H, d, L) needs to be specified in order to obtain a complete solution. This is a convenient point at which to consider the effect that cable elasticity has on the properties of inextensible parabola.

For a given inextensible sag d, an unstressed length of cable L is laid out. When this cable is hung between two supports, a distance $l(< L)$, apart, stretching occurs; the sag increases to $(d+\Delta d)$ and the horizontal component of tension reduces from its inextensible value to $(H-\Delta H)$. Provided that the additional movements are small, vertical equilibrium of an element in the stretched configuration is given by

$$(H-\Delta H)\frac{\mathrm{d}^2}{\mathrm{d}x^2}(y+\delta) = -mg, \tag{27}$$

where δ is the additional vertical deflexion of the element. Equation (27) may be integrated directly to give

$$\delta = \frac{H_*}{(1-H_*)}y = \frac{H_*}{(1-H_*)}4d\left\{\frac{x}{l}-\left(\frac{x}{l}\right)^2\right\}, \tag{28}$$

where $H_* = \Delta H/H$. Hence, the fractional increase in sag owing to cable stretch is

$$d_* = H_*/(1-H_*), \tag{29}$$

where $d_* = \Delta d/d$.

In order to evaluate H_*, recourse must be made to the cable equation which relates the stretching of the cable to the additional displacements. In the present context the equation reads (Irvine 1974)

$$(1-H_*)\frac{HL_e}{E_c A_c} = \frac{4d}{l^2}\left[\frac{2-H_*}{1-H_*}\right]\int_0^l \delta\,\mathrm{d}x, \tag{30}$$

where E_c is the modulus of elasticity, A_c is the area of the cable and $L_e \simeq l\{1+8(d/l)^2\}$. After substitution, integration and rearrangement the following cubic is obtained

$$(1-H_*)^3 = \tfrac{1}{24}\lambda^2(2H_*-H_*^2), \tag{31}$$

where
$$\lambda^2 = \left(\frac{8d}{l}\right)^2\frac{l}{(HL_e/E_cA_c)}.$$

The dimensionless variable λ^2 is the fundamental parameter of the extensible cable. From equation (31) it is seen that

$$0 < H_* < 1$$

an intuitively obvious result. Consider now the two limits for λ^2.

(i) λ^2 *large* (*i.e.* $\lambda^2 > 100$)

This covers most freely hanging cables which have small, although appreciable, sag to span ratios. It is easily shown that here

$$\left.\begin{aligned} H_* &\simeq 1/(3+\tfrac{1}{12}\lambda^2), \\ d_* &\simeq 1/(2+\tfrac{1}{12}\lambda^2). \end{aligned}\right\} \tag{32}$$

The corrections represented by equation (32) are usually so small that they may be ignored. However, the absolute increase in sag may be important for the calculations involving the free-hanging cables of long-span suspension bridges.

(ii) λ^2 *small* (*i.e.* $\lambda^2 \ll 1$)

Cables for which λ^2 is small may be flat (in which case d/l is very small) and/or they may be very extensible (in which case E_c is small). Here

$$\left.\begin{aligned} H_* &\simeq 1-(\tfrac{1}{24}/\lambda^2)^{\frac{1}{3}}, \\ \frac{\Delta d}{l} &\simeq \tfrac{1}{2}\times 3^{\frac{1}{3}}\left\{\left(\frac{d}{l}\right)\left(\frac{L_e}{l}\right)\left(\frac{H}{E_c A_c}\right)\right\}^{\frac{1}{3}}. \end{aligned}\right\} \tag{33}$$

The only case of any practical importance is when d/l is very small and E_c is large as in a taut, flat steel cable. Then $\Delta d/l$ is small, as expected. It is noticed that a large change in tension occurs since $\Delta d/l$, although small, will be several times larger than d/l.

The validity of these remarks depends on the adoption of a construction procedure whereby a length of cable is cut and then hung in place. Because d/l is so small, the unstressed length of the cable will be only a minute fraction longer than the span. Consequently, taut, flat cables are not usually constructed in this way. Normally, the cable is placed in supports, prestressed to a given tension and then anchored off. Since the mass and span of the cable are also known, the final sag is equal to the classical expression, $mgl^2/8H$.

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(a)

(b)

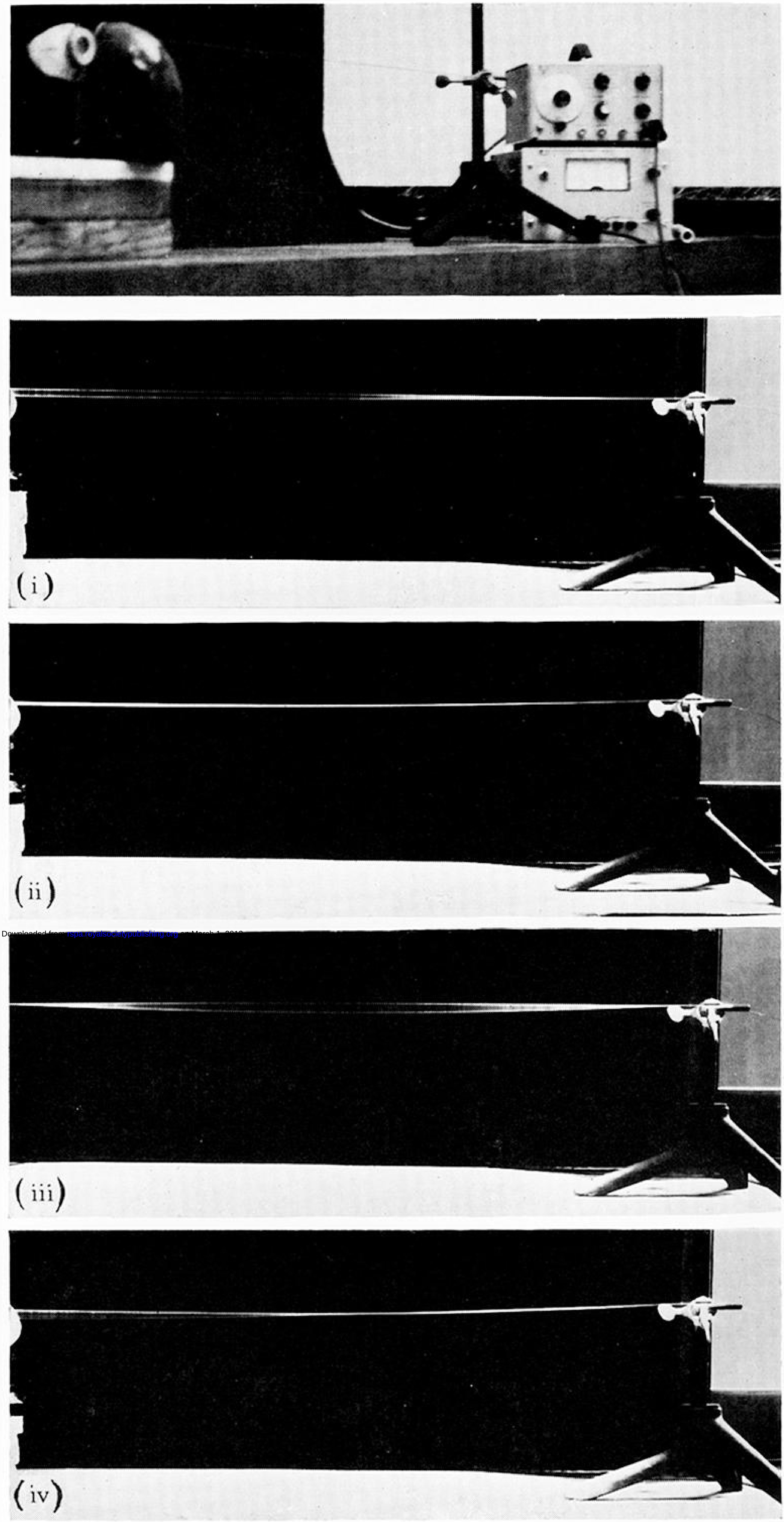

FIGURE 5. (a) Experimental set up. (b) Possible forms of the first-in-plane mode of a suspended cable (mid-span is at left-hand edge).

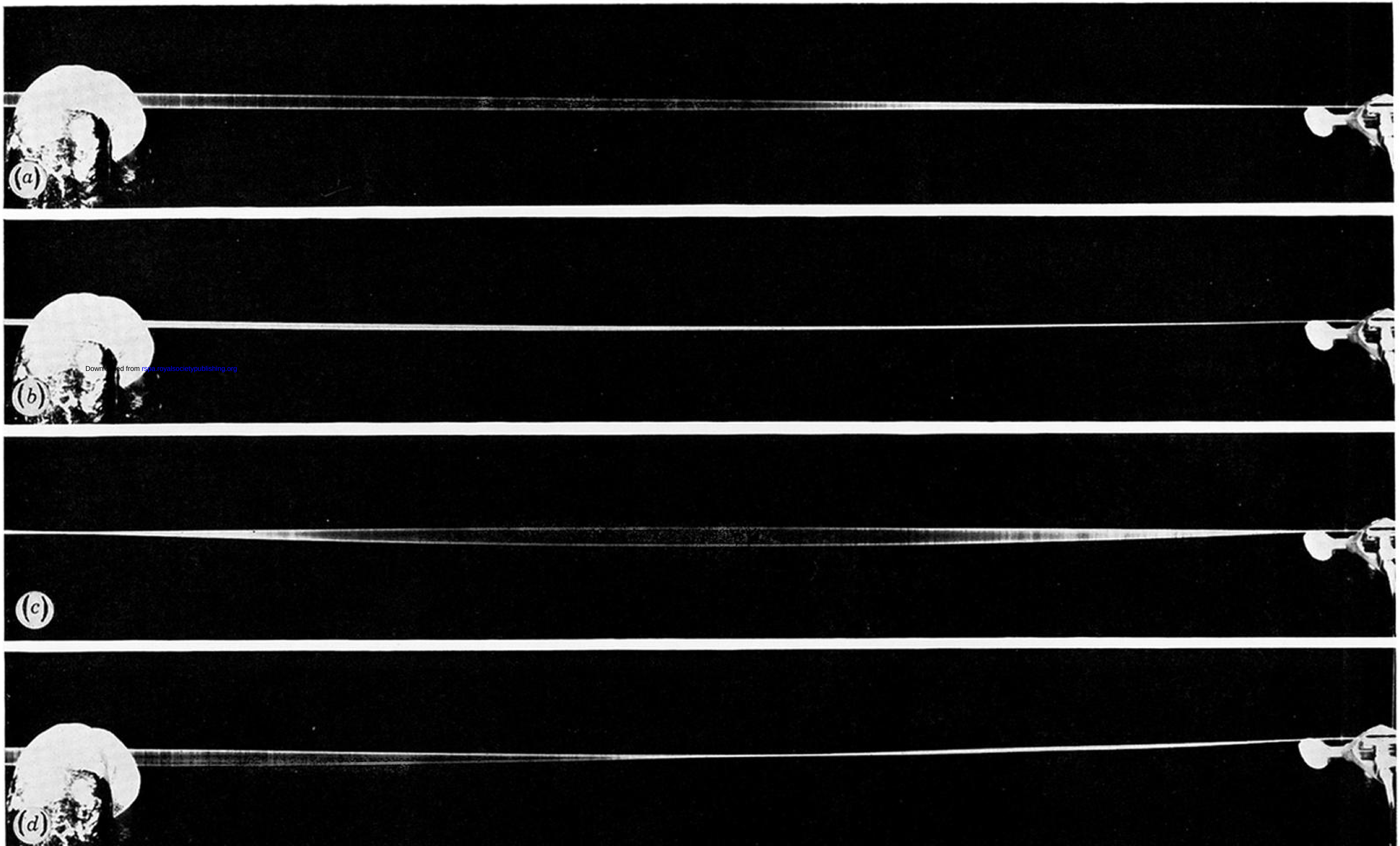

FIGURE 6. Enlarged versions of figure 5*b*. (*a*) $\lambda^2 \ll 4\pi^2$, first symmetric mode of a taut string; (*b*) $\lambda^2 \simeq 4\pi^2$, first symmetric in-plane mode at first 'cross-over' point; (*c*) $\lambda^2 \simeq 4\pi^2$, first antisymmetric in-plane mode at first 'cross-over' point; (*d*) $\lambda^2 > 4\pi^2$, first symmetric in-plane mode of an (essentially) inextensible suspended cable.