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ANALYTICAL AND CLOSED FORM SOLUTIONS FOR DEEP WATER RISER-LIKE EIGENVALUE PROBLEM

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ABSTRACT

The analytical solution of the suspended heavy cable eigenvalue problem, given in terms of Bessel functions, has been known for a long time. Also well known and superbly discussed by Irvine & Caughey (1974) is the physical inadequacy concerning the inextensibility hypothesis. Eventhough, firstly motivated by design procedures and by VIV and Mathieu-like instabilities analysis, which demand insight and expeditious evaluations, this work recovers the Bessel solution approach, presenting some elementary but useful solutions for non-vertical risers. Also, an analytical and closed form WKB approximate solution is derived for the general riser-like problem, given the tension function along the length. A simple formula for the natural frequencies and eigenmodes of a catenary riser under no current is then derived from WKB general solution. Compared to Bessel's approximations and to numerical results obtained by a standard Finite Element Method formulation, where extensibility is taken into account, the present solution shows a rather good agreement for typical free-hanging catenary risers. The effect of extensibility is illustrated and discussed.

Keywords: riser, eigenvalue, eigenmodes, WKB, analytical solution, extensibility

INTRODUCTION

This is a simple work on a classical and well-known matter in offshore engineering: *the riser-like eigenvalue problem*. Those who have been involved with the design of risers (umbilical cables, flexible pipes, steel catenary risers), jumpers, TLP tethers or any other kind of cable structures, certainly have faced this eigenvalue problem. Vortex-Induced Vibrations and lock-in analysis (e.g., Lyons & Patel, 1986, Martins, 1989), Mathieu instabilities problems (e.g., Patel & Park, 1991, Simos & Pesce, 1997) or inverse design procedures (Bernitsas et al., 1985) are some examples where the eigenvalue problem plays a fundamental role. Such a problem, however, is standard enough to induce little attention from Patel & Seyed, 1995, in their review paper on *Flexible Riser Modelling and Analysis Techniques*, where other important topics are discussed as fundamental research points. Nevertheless, though most numerical codes on risers analysis have an eigenproblem solver, comparisons among their respective results show, sometimes, poor results (Larsen, 1992). However, to the authors knowledge (or, eventually, ignorance), despite some simplicity of this linearised problem, compared, for instance, to the complexity of full nonlinear approaches (see, e.g., Leissa & Saad, 1994), or to specific topics as tangential resonant mechanisms driven by transversal excitation (Newberry & Perkins, (1997)), no systematic attempts have been done through analytical approaches, aiming to present practical results on typical eigenvalue problems in offshore engineering. On the other hand, analytical results are often presented for static problems (e.g., Langer, 1985). Recently, riser's dynamic problems have received some attention from the analytical point of view. Examples are the frequency-domain analytical solution for the dynamic tension (Aranha et al, 1993) or the analytical formula for the dynamic curvature at the touchdown point (TDP) of catenary risers (Aranha et al., 1997), both experimentally verified (Andrade, 1995 and Pesce et al., 1998). Within the field of analytical approaches, perturbation techniques and asymptotics are methods which fit best, of course.

It is worthwhile to mention that all these recent analysis have been pioneered by the works of Burridge et al., 1982 and Triantafyllou et al., 1985, after the fundamental paper by Irvine & Caughey, 1974. In this thorough analysis a number of interesting physical and mathematical questions are discussed. One of most importance is the effect of axial deformation on transversally dominated modes and so to the mathematical ill-posedeness of the problem when this effect is not properly taken into account. Inextensibility hypothesis is physically inadequate, as pointed out by Irvine & Caughey, (1974), axial displacement boundary conditions being then lost. As clearly shown by those authors, the most intriguing consequence of properly considering extensibility is that the first 'symmetric' eigenmode may present two internal nodes, depending on the value taken by a non-dimensional parameter that measures the ratio of axial to geometric rigidities. To the axial deformation effect, within the context of marine cables, Chucheepsakul & Huang, 1997, have dedicate an entire study, by formulating the two-dimensional problem on a virtual-work principle basis, but solving it numerically.

In the present work, a standard perturbation analysis is conducted. In some sense, it can be seen as an exercise on applied mathematics. Though aware of the physical inadequacy concerning the inextensibility hypothesis, we decided to work upon this basis. The purposes are twofold. Firstly, this assumption drastically simplifies the mathematical analysis. Secondly, we intend to evaluate the error this hypothesis causes, in typical riser cases. Despite the simplifications, the results showed themselves interesting and practical enough to give rise to a piece of work on this specific subject.

The analysis shows that the *inextensible tensioned-and-curved-heavy-string* equations apply fairly well to represent the riser-like eigenvalue problem. Equations are written in the Frenét-Serret intrinsic coordinates, i.e., along the normal and tangential directions, around the static equilibrium configuration. This choice of coordinates will be shown to be crucial in the analytical derivation. The dynamic equations are coupled through curvature. Tension and angle functions fully represent the static configuration, giving 'support' to the dynamics. Observing that curvature is usually a small quantity, coupling is weak and the normal displacement dynamic equation dominates the overall behaviour. Not considering extensibility, the tangential displacement is written as a linear operation on the normal displacement. As previously pointed out, axial displacement boundary conditions are lost, however.

In the classical vertical riser case, the tension is a linear function of the length (see, e.g., Bowman, 1958), and a modified Bessel's equation represents such a particular case, the solution being given in terms of Bessel's functions of first kind and zero-order. Further observing that, for risers and cables problems, tension is usually a quasi-linear function of the arch-length, a 'naive' linear approximation of the actual tension function leads, as pointed out by Irvine & Caughey, to a similar Bessel form solution. This is straightforward, in fact, and is shown afterwards, at the end of the paper.

Instead, first we search for an approximate solution for the general riser-like eigenvalue problem, given a tension function. By looking at the pure catenary problem, written in the Frenét coordinates, a non-dimensional arch-length variable is introduced, $z = \tan q$, being $q(s)$ the angle with respect to the horizontal. This allows one to write the inextensible tensioned-and-curved-heavy-string equation in a rather familiar form, $\varphi'' + \left(F'/F\right)\varphi' + \left(\Lambda^2/F\right)\varphi = 0$, where $F(z)$ is the tension function. Such a form is very well suited to the WKB method, a classical perturbation technique; see, e.g., Bender & Orszag, 1978. The eigenvalue problem solution is then straightforward, the eigenfunctions being given, as it should be expected, as a sinusoidal function, modulated in amplitude and phase. These modulation functions are respectively $F^{-1/4}(z)$ and $F^{-1/2}(z)$, and the solution takes a very simple form for the catenary case. Particularly, a formula for the natural frequencies and a closed form solution for the modes of a catenary-riser are presented. These formulae can be easily applied in practical situations, when the designer (or the field engineer) searches for a quick evaluation.

Few examples are shown, comparing WKB to the modified Bessel's equation solution. The agreement is very good, as it should be expected. Not only eigenvalues but also eigenfunctions predicted by these two approximate solutions agree to a very high degree. Both, high-order and low-order eigenmodes, are well predicted. Finally a typical real flexible-pipe riser case is taken, in order to exemplify the comparison of the WKB solution, derived for the (inextensible) tensioned-and-curved-cable equation, to numerical results obtained by a standard finite element method formulation, where extensibility is taken into account. The comparison results help to elucidate some important aspects concerning simplifying assumptions. For instance, as axial rigidity is rather large, in this typical case, the first numerically calculated mode is the first 'assymmetric' one, in accordance to Chucheepsakul & Huang, 1997 analysis. The effect of the extensibility is addressed. Nevertheless, WKB solution proves to be a fairly good approximation to the real problem, with the advantage of, since purely analytical, being easily incorporated in design oriented codes.

THE GENERAL TWO-DIMENSIONAL RISER-LIKE DYNAMIC EQUATIONS

Consider the general two-dimensional riser-like problem. The static configuration is supposed to be known, given by the functions, $q(s)$ and $T(s)$, respectively the angle with the horizontal and the static tension along the arc-length coordinate s, that can be measured from a convenient (but otherwise arbitrary position, e.g. the touchdown point (TDP)). Under small perturbation assumption, let $u(s,t)$ and $v(s,t)$ be the *tangential* and *normal* displacements at s. We assume standard structural mechanics theory and the usual constitutive equations, neglecting geometrically and dynamically nonlinear terms, disconsidering hydrostatic terms and considering only the case of a inextensible line, i.e., the dynamic tension variation[1] is not considered at this moment. As previously pointed out,

[1] (eventually responsible for Mathieu-like instabilities)

we are aware of the fact that disconsidering axial deformation, being physically inadequate, leads to a mathematical ill-posedeness, since axial boundary conditions cannot be enforced anymore. A proper account of this subject is given, for small sagged cables, by Irvine & Caughey, 1974 and, for marine cables, by Triantafyllou et al., 1985. Under this restrictive assumptions the following dynamic equilibrium equations can be derived,

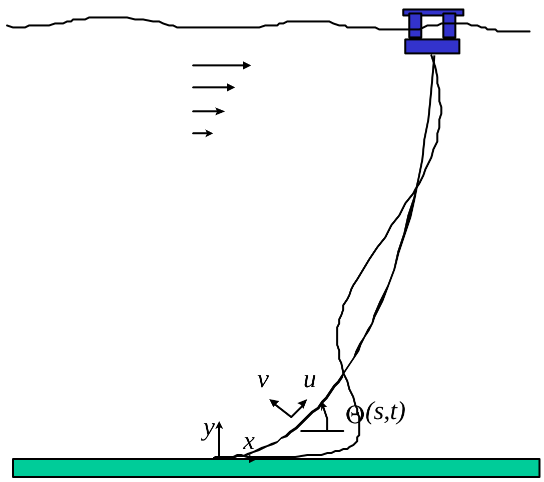

Figure 1 Two-dimensional problem

$$-\left(T - EJ\frac{\partial^2}{\partial s^2}\right)\left(\frac{\partial v}{\partial s} + u\frac{d\theta}{ds}\right)\frac{d\theta}{ds} - $$
$$- EJ\frac{d^3\theta}{ds^3}\left(\frac{\partial v}{\partial s} + u\frac{d\theta}{ds}\right) - EJ\frac{d^2\theta}{ds^2}\frac{\partial}{\partial s}\left(\frac{\partial v}{\partial s} + u\frac{d\theta}{ds}\right) + v_u = m\frac{\partial^2 u}{\partial t^2}$$

(1).

$$- EJ\frac{\partial^3}{\partial s^3}\left(\frac{\partial v}{\partial s} + u\frac{d\theta}{ds}\right) + EJ\frac{d^2\theta}{ds^2}\left(\frac{\partial v}{\partial s} + u\frac{d\theta}{ds}\right)\frac{d\theta}{ds} +$$
$$+ \frac{\partial}{\partial s}\left(T\left(\frac{\partial v}{\partial s} + u\frac{d\theta}{ds}\right)\right) + v_v = m\frac{\partial^2 v}{\partial t^2}$$

In Eq. (1), *EJ* is the bending stiffness, *m* is the mass and $v_{u,v}(s,t)$ represent the dynamic parcels of the hydrodynamic forces acting on the line, due the relative motion with respect to the current and wave flow, all per unit length.

Notice that equations (1) are coupled through static curvature. In the particular case of a inextensible and perfectly vertical line, under no current action, equation (1a) has no meaning and we get the familiar *beam-under-tension* equation,

$$- EJ\frac{\partial^4 v}{\partial s^4} + \frac{\partial}{\partial s}\left(T\frac{\partial v}{\partial s}\right) + v_v = m\frac{\partial^2 v}{\partial t^2} \qquad (2).$$

Instead, if a ideal heavy cable (inextensible and infinitely flexible (EJ=0)) is considered we obtain the following simplified coupled linear equations

$$-T\frac{dq}{ds}\left(\frac{\P v}{\P s}+u\frac{dq}{ds}\right)+v_{u}=m\frac{\P^{2}u}{\P t^{2}}$$

(3).

$$\frac{\P}{\P s}\left(T\left(\frac{\P v}{\P s}+u\frac{dq}{ds}\right)\right)+v_{v}=m\frac{\P^{2}v}{\P t^{2}}$$

Globally neglecting flexible rigidity can be presented in a more proper way. By defining $x=s/L$, $u=u/L$, $h=v/L$ and $t=t\,c_0/L$ as non-dimensional quantities, where

$$c_0=\sqrt{\frac{T_0}{(m+m_a)}} \tag{4}$$

is the transversal cable wave celerity and T_0 the static tension at the TDP (or any other convenient section) and m_a is the added mass per unit length (corresponding to the inertial part of the hydrodynamic force v_v), and if terms of order $e_F{}^2=\left(EJ/T_0\right)L^{-2}$ and viscous terms are neglected (see, Appendix or Pesce et al, 1998), we get the associate non-dimensional and non-damped form of Eq. (3) (the inextensible and non-damped outer equation that dominates the overall dynamic behaviour),

$$F(x)\left(\frac{\partial h}{\partial x}+u\frac{dq}{dx}\right)\frac{dq}{dx}+\frac{1}{1+a}\frac{\P^{2}u}{\P t^{2}}=0$$

(5)

$$-\frac{\partial}{\partial x}\left(F(x)\left(\frac{\partial h}{\partial x}+u\frac{dq}{dx}\right)\right)+\frac{\partial^{2}h}{\partial t^{2}}=0$$

where $a=m_a/m$ is here defined as the added mass coefficient[2],

$$F(x)=\frac{T(x)}{T_0} \tag{6}$$

is the tension function. Notice that $F(x)$ and the non-dimensional curvature $c(x)=dq/dx$ carry information about the static configuration. For a catenary riser, for example, curvature is maximum at TDP and takes the non-dimensional value $c_0=qL/T_0$, where L is the suspendend length and q the immersed weigth for unit length. We call Eq. (5) the linear (inextensible) tensioned-and-curved-string equations, which can be considered as a good first-order approximation (outer problem) for the riser-like problem analysis. If we take now, $u(x,t)=y(x)e^{iwt}$ and $h(x,t)=\varphi(x)e^{iwt}$, the eigenvalue problem, associated with equation (5), is

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[2] Usually $C_M=m_a/m_d$, where m_d is the displaced mass, is taken.

$$F(x)\left(\varphi' + y\,c(x)\right)c(x) - \frac{w^2}{1+a}y = 0$$

(7).

$$\frac{d}{dx}\left[F(x)\left(\varphi' + y\,c(x)\right)\right] + w^2\varphi = 0$$

Eq. (7) can be written in the following form,

$$y(x) = \frac{F(x)c(x)}{\frac{w^2}{1+a} - F(x)c^2(x)}\varphi'(x)$$

(8)

$$\frac{d}{dx}\left[F(x)\varphi'\right] + \frac{d}{dx}\left[\frac{F^2(x)c^2(x)}{\frac{w^2}{1+a} - F(x)c^2(x)}\varphi'\right] + w^2\varphi = 0$$

Notice that Eq. (8b) is, strictly speaking, the eigenvalue equation. Equation (8a) gives the tangential displacement $y(x)$, as a direct linear operation on $\varphi(x)$. This is a direct result for not considering extensibility, what leads to the impossibility of satisfying any general axial boundary condition that would be physically imposed (actually this a first real drawback of the present analysis). Therefore, *close to extremities* we should expect poor results for the longitudinal displacement function $y(x)$. This will be numerically confirmed later on. More important than this point: depending on the actual value taken by the non-dimensional axial-to-geometric-ratio rigidity parameter, as defined by Irvine & Caughey, the inextensibility hypothesis can lead to a complete misinterpretation concerning the first eigenmode (as probably has occurred in Larsen, 1992, when comparing natural periods determined by a number of program codes).

Notice also that approximation (8) gets worse as higher is the eigenmode, such that the typical corresponding wavelength is of same order of magnitude of the flexural length parameter. In such cases, bending stiffness effect would not be confined to small boundary-layers anymore.

If quadratic terms in curvature could be neglected, Eq. (8) would then be written,

$$y(x) = \frac{1}{w^2}\left[(1+a)F(x)c(x)\right]\varphi'(x)$$

(9).

$$\left(F(x)\varphi'\right)' + w^2\varphi = 0$$

This could be done, for a free-hanging catenary, e.g., where $c(x) = c_0\cos^2 q(x)$, but only far from TDP.

In the next section we shall return to equations (7) and (8) in order to derive a general WKB approximate solution, valid for general static configurations.

THE 'CATENARY' RISER-LIKE CASE: A WKB CLOSED FORM SOLUTION AND AN ANALYTICAL APPROXIMATION

We turn our attention to the 'catenary' riser-like eigenvalue case[3]. We focus on the *inextensible tensioned-and-curved-heavy-string* equation, Eq. (7) (or (8)). First we observe that tension is usually dominated by geometric (or catenary) rigidity, unless a strong current is present. In this latter case tension fuction is supposed to be previously known. In the particular but important case of a free-hanging and pure catenary (no current) the (non-dimensional) tension function can be easily shown to be

$$F_c(x) = \sqrt{1 + \tan^2 q_c(x)} \qquad (10),$$

where c stands for catenary. This suggests us to introduce a new variable,

$$z = \tan q(x) \qquad (11).$$

Notice that $z = \tan q_L$, at the upper end, where $x = 1$.

In the pure catenary case,

$$F_c(z) = \sqrt{1 + z^2} \qquad (12)$$

Notice that, in this particular case, $F_c(z) \approx z$, for $z >> 1$, i.e., in the region close to the upper end, for top angles $q_L > p/4$, and $F_c(z) \approx 1$ in the touchdown point (TDP) region, where $z << 1$. On the other hand, for very low values of q_L (very tight cables), the tension function can be written $F_c(z) \cong 1 + \mathrm{O}(z^2)$. Figure 3 show the tension function $F_c(z)$. Notice that the curve is almost linear. It should also be noticed that

$$F(z) = \left(\frac{c(z)}{c_0}\right)^2 \qquad (13),$$

where $c(z) = \sqrt{T(z)/(m + m_a)}$ is the local transversal wave celerity. Naturally, if a linear approximation (of the least square type, for example) is taken, a modified Bessel equation is got and an approximate solution can be obtained. This is straightforward, however, and will be done latter on.

Instead, motivated by the above stated considerations, we shall proceed with the 'general' two-dimensional tensioned-and-curved-string eigenvalue equation, aiming to construct a closed form solution, given $F(z)$. We shall apply the well-known WKB technique.

[3] We should distinguish the pure catenary, when no current exists, from the 'catenary'-like case.

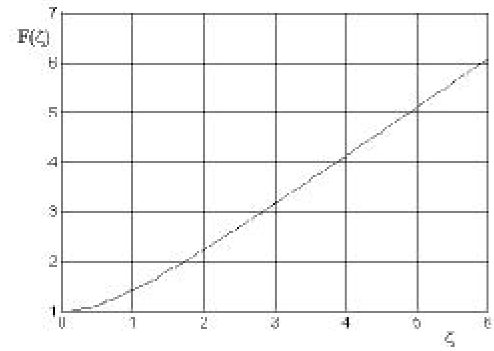

Figure 2 Non-dimensional tension function, $F(z)$***, along a free-hanging catenary.***

A WKB closed form solution for the inextensible tensioned-and-curved-heavy-string equation

WKB technique can be applied easily; see, e.g. Bender & Orszag, chapter 10. Let $m = \tan q_L$. Equation (8) transforms, under the variable substitution given by (11), into

$$y(z) = \left(\frac{1}{g(z)-1}\right)\frac{1}{c(z)}\varphi'(z)$$

$$\left\{1+\frac{1}{g(z)-1}\right\}\varphi'' + \left\{1+\left(\frac{1}{F}+\frac{2}{F'}\frac{c'}{c}\right)\frac{g(z)}{\left(g(z)-1\right)^2}\right\}\frac{F'}{F}\varphi' + \frac{\Lambda^2}{F}\varphi = 0 \tag{14}$$

where

$$g(z) = \frac{\Lambda^2}{1+a}\frac{1}{F(z)c^2(z)}$$

$$\Lambda = \frac{w}{m} \tag{15}$$

Obviously, a singular problem would arise wherever $F(z) = 0$, for any particular z. This means a section where wave celerity is zero; see Eq. (13). Though tractable, we can take $F(z) \neq 0$, for all z, in the present problem. On the other hand, a 'turning-point' problem would arise whenever Λ^2/F has a zero; Abramowitz & Stegun, Eq. 10.4.108. Such cases, where turning points would exist, will not be treated here, either.

The following reasoning is strictly valid for finite $m = \tan q_L$ and large w (higher-order eingenmodes), such that we can assume $\Lambda >> 1$. Surprisingly[4], the approximate WKB solution will show to be rather good, even for low-order eigenmodes. Equation (14) is almost in the form shown in a number of classical text books on applied mathematics, as in

[4] Not really surprising, as shown in Bender & Orszag, by means of a standard eigenvalue problem.

Bender & Orszag's, page, 490. In this circumstances, when $\Lambda >> 1$, we get from Eqs. (14-15), with an error of order $O(\Lambda^{-2})$, the following simple equation,

$$y(z) = \frac{(1+a)}{\Lambda^2} F(z) c(z) \varphi'(z)$$

(16).

$$\varphi'' + \frac{F'}{F}\varphi' + \frac{\Lambda^2}{F}\varphi = 0$$

Notice that this is exactly the form that would be obtained from Eq. (9), where quadratic terms in curvature were neglected.

A classical WKB exponential series solution is taken,

$$\varphi(z) \approx \exp\left[\frac{1}{d}\sum_0^\infty d^k S_k(z)\right] \quad (17),$$

and placed directly into Eq. (16), with $e = 1/\Lambda$ a small quantity. After the standard 'dominant balance' argument is used (allowing to take $d = e$), we come up, to first-order in $e = 1/\Lambda$, with

$$\varphi(z) \cong F^{-1/4}(z)\left[C_1 \sin\left(\Lambda\int^z F^{-1/2}(u)du\right) + C_2 \cos\left(\Lambda\int^z F^{-1/2}(u)du\right)\right] \quad (18),$$

a rather fairly form. Equation (18) gives a general closed form solution for the inextensible tensioned-and-curved-heavy-string problem (not only for the riser-like problem but also for jumpers, for instance). Notice that eigenmodes are sinusoidal functions, modulated in phase and amplitude and resembling Bessel's functions. Also, from Eq. (18), being $f(z) = \Lambda\int^z F^{-1/2} dz$, the phase angle, the local non-dimensional wave-number is given by $k = \frac{df}{dz} = \frac{\Lambda}{\sqrt{F(z)}}$. Hence, from Eq. (13), we get $\frac{c(z)}{c_0} = \frac{\Lambda}{k}$, as should be expected, since this is a classical result that tells us local wave length is linearly proportional to wave-celerity (phase velocity); see, e.g. Whitham, p. 365. (The corresponding dispersion relation is given by $k^2 + \frac{F'}{F}k - \frac{\Lambda^2}{F} = 0$). Notice that if $F(z) = F_0$, a constant, there would be no dispersion at all, Eq. (16) being transformed into the classical string equation under constant tension.

Applying, e.g., a pinned-pinned boundary condition, such that, $\varphi(0) = \varphi(m) = 0$, we get $C_2 = 0$ and

$$\varphi_n(z) \cong A_n F^{-1/4}(z)\sin\left(\Lambda_n \int_0^z F^{-1/2}(u)du\right)$$

$$\Lambda_n \cong np\left(\int_0^m \frac{dz}{\sqrt{F(z)}}\right)^{-1} \qquad (19).$$

Therefore, the (dimensional) natural frequencies are given by

$$\Omega_n = \Lambda_n m \frac{c_0}{L} \cong \Lambda_n \tan q_L \sqrt{\frac{T_0}{(m+m_a)L}} \qquad (20).$$

If a *free-hanging catenary* with a touch-down point[5] is taken, Eq. (12) applies. Then, by using Eq. (11),

$$\varphi_n(q;q_L) \cong A_n(\cos q)^{1/4}\sin\left\{\Lambda_n \int_0^{q_L} \frac{dq}{(\cos q)^{3/2}}\right\}$$

$$\Lambda_n = \Lambda_n(q_L) \cong \frac{np}{\int_0^{q_L} \frac{dq}{(\cos q)^{3/2}}} \qquad (21)$$

and, from the well-known catenary relationship, $T_0 = qL/\tan q_L = qL/m$, where T_0 is the tension at TDP, we get the following simple form for the natural frequencies,

$$\Omega_n \cong \Lambda_n \sqrt{\frac{q\tan q_L}{(m+m_a)L}} \qquad (22).$$

Further observing that, for a *circular* section, $m_a \cong r\,pD^2/4$, where r is the water mass density and D the external diameter, we can write $q \cong (m-m_a)g$ and, defining the 'added mass coefficient' $a = m_a/m$, Eq. (22) can be written in the following form,

$$\Omega_n \cong \Lambda_n \sqrt{\tan q_L \frac{(1-a)}{(1+a)}}\sqrt{\frac{g}{L}} \qquad (23),$$

being g the acceleration of gravity. Notice that for a neutrally buoyant line $(q = 0; a = 1)$, geometric rigidity (the only source here considered) is null, breaking down the proposed eigenvalue problem. Recovering, from catenary equations, that $L = H\sin q_L/(1-\cos q_L)$, where H is the waterdepth, we finally get

5 It has been formally shown, Aranha et al., 1997, that in the dynamic problem of a free-hanging cable, with a simple touchdown point, and correct to first-order in $a_0 c_0$, being a_0 the TDP non-dimensional excursion amplitude and $c_0 = qL/T_0$ the non-dimensional static curvature at TDP, boundary condition at TDP can be properly modeled as a hinge. The local bending stiffness effect (and proper null curvature and tangency boundary conditions at TDP) can be incorporated by means of boundary-layer technique.

$$\Omega_n \cong \Lambda_n \sqrt{\frac{(1-\cos q_L)}{\cos q_L}} \sqrt{\frac{(1-a)}{(1+a)}} \sqrt{\frac{g}{H}} \qquad (24),$$

that gives a formula for evaluating the natural frequencies of a catenary line, written solely in terms of waterdepth H and of the upper end angle with respect to horizontal, q_L. For practical and immediate usage of formula (24), Fig. 3 gives $(\Lambda_n/n)\sqrt{(1-\cos q_L)/\cos q_L}$ as a function of q_L. Numerical examples will be shown in the next two sections.

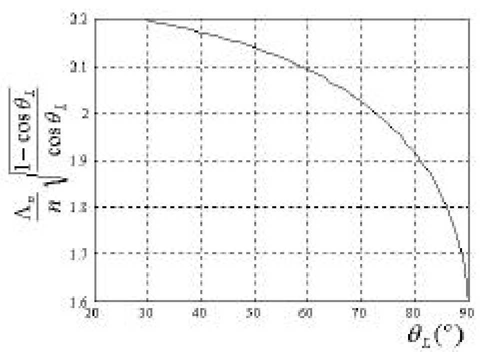

Figure 3 Eigenvalues $(\Lambda_n/n)\sqrt{(1-\cos q_L)/\cos q_L}$ for a catenary-riser under no current as a function of q_L, the angle at the upper end, with respect to the horizontal.

An analytical approximation for the 'catenary' riser-like problem

A straightforward analytical solution can be derived from Eq. (16) with a linear approximation for the tension function, in the form $F(z) \approx a^2 + bz$. Then, Eq. (16) transforms into a modified Bessel equation, as in the classical case of a vertical line (see, e.g., Bowman, 1958). For, let $b = b/a^2$, $z^2 = 1 + bz$ and Eq. (9b) reads,

$$z^2\varphi'' + z\varphi' + 4\mathrm{K}^2 z^2 \varphi = 0$$

$$\mathrm{K} = \frac{\Lambda}{ab} = \frac{w}{abm} \qquad (25),$$

solution being given in terms of Bessel's functions of first kind and zero order,

$$\varphi(z) = C_1 J_0(2\mathrm{K}z) + C_2 Y_0(2Kz) \qquad (26)$$

or, in terms of z ,

$$\varphi(z) = C_1 J_0\left(2\frac{\Lambda}{ab}(1+bz)^{1/2}\right) + C_2 Y_0\left(2\frac{\Lambda}{ab}(1+bz)^{1/2}\right) \qquad (27),$$

or else, in terms of $x = \dfrac{z}{m}$; $m = \tan q_L$,

$$\varphi(x) = C_1 J_0\left(2\frac{w}{abm}(1+bmx)^{1/2} \right) + C_2 Y_0\left(2\frac{w}{abm}(1+bmx)^{1/2} \right) \qquad (28).$$

For a hinged-hinged boundary condition, eigenvalues satisfy the characteristic equation,

$$J_0\left(2\frac{w}{abm}(1+bm)^{1/2} \right) Y_0\left(2\frac{w}{abm} \right) - J_0\left(2\frac{w}{abm} \right) Y_0\left(2\frac{w}{abm}(1+bm)^{1/2} \right) = 0 \qquad (29),$$

the corresponding eigenfunctions being then written,

$$\varphi_n(x) = J_0\left(2\frac{w_n}{abm}(1+bmx)^{1/2} \right) + \frac{J_0\left(2\frac{w_n}{abm} \right)}{Y_0\left(2\frac{w_n}{abm} \right)} Y_0\left(2\frac{w_n}{abm}(1+bmx)^{1/2} \right) \qquad (30).$$

Again we take the case of a catenary riser (no current), such that $F(z) = F_c(z) = \sqrt{1+z^2}$. If a standard least-square error approximation is used, with the additional condition $F(0) = 1$, such that tension at TDP is preserved, we get for this particular case $a^2 = 1$, b=0.4789 (m=2) *or* b=0.7395 (m=5.7), such that b=0.4789 or 0.7395.

Figures 4 and 5 show a comparison between WKB (Eq. (21)) and Bessel's (Least Square Approximation - Eq. 30) solutions. We took $m = \tan q_L = 0.2$ and 5.7, corresponding to upper end angle values $q_L = p/3 = 60^o$ and $q_L = 4p/9 = 80^o$. The agreement is very good, indeed. Notice that (non-dimensional) natural frequencies values agree up to three digits and results get even better for higher eigenmodes, as predicted when applying WKB technique. Notice that the 'tangential' displacement function, $y(x)$, is determined from Eq. (16a), being $dq/dz = c(z) = c_c(z) = c_0/(1+z^2) = c_0/F_c^2(z)$ the corresponding static curvature. As anticipated, by not considering axial deformation we miss the boundary condition for the tangential displacement, leading to poor results for the axial displacement, particularly close to a extremity where tension is low and curvature is somewhat large, as TDP, in this case. More than that, according to Irvine & Caughey's (1974) analysis, the fundamental (inextensible) eigenmode (no internal nodes) determined from both approximate solutions of the inextensible tensioned-and-curved-heavy-string equations, can be physically incongruous. For this reason we call such a fundamental mode the zeroth-mode in figures 5 and 6, redefining a mode counter $k = n-1$.

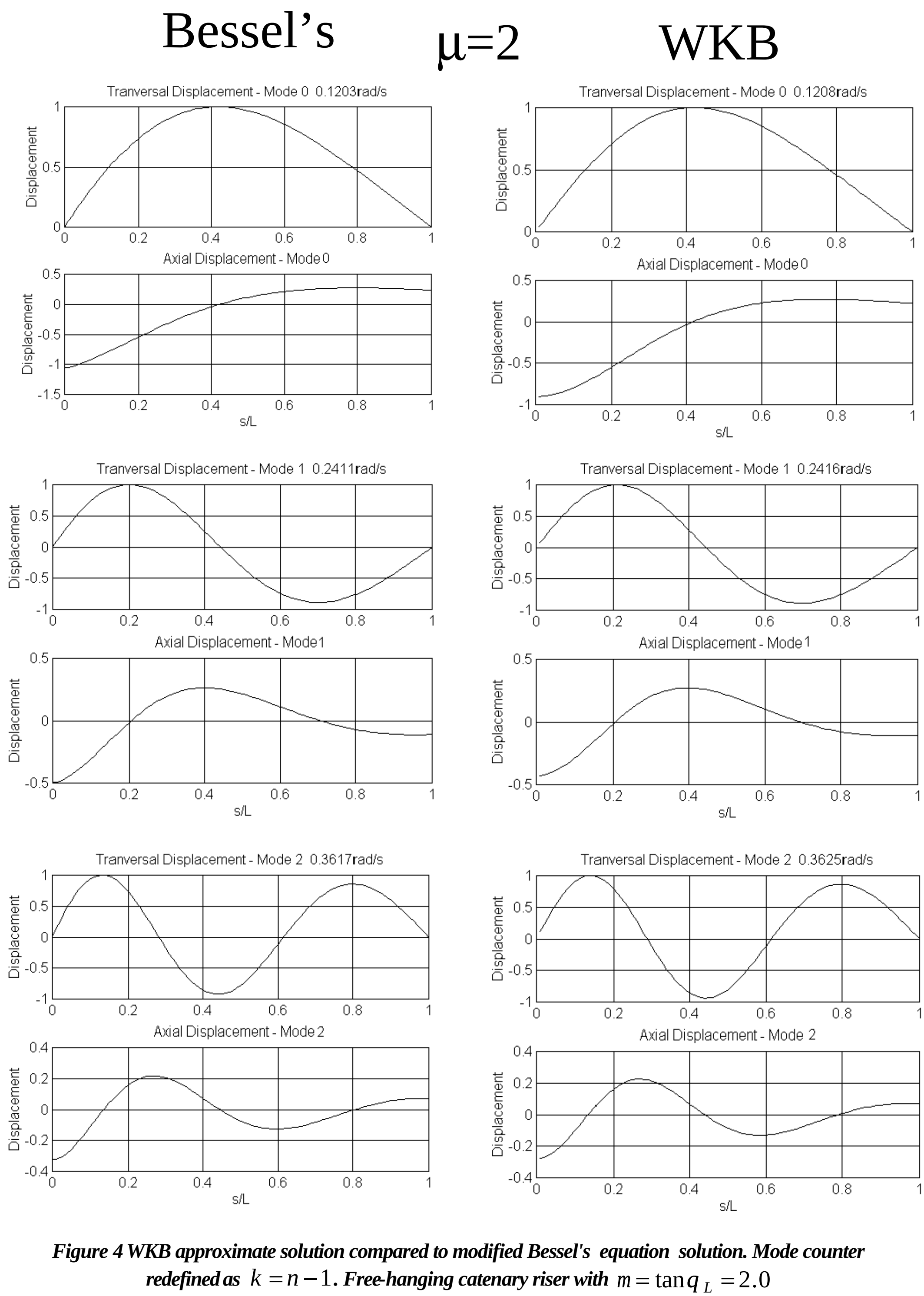

Figure 4 WKB approximate solution compared to modified Bessel's equation solution. Mode counter redefined as $k = n - 1$. ***Free-hanging catenary riser with*** $m = \tan q_L = 2.0$

Bessel’s μ=5.7 WKB

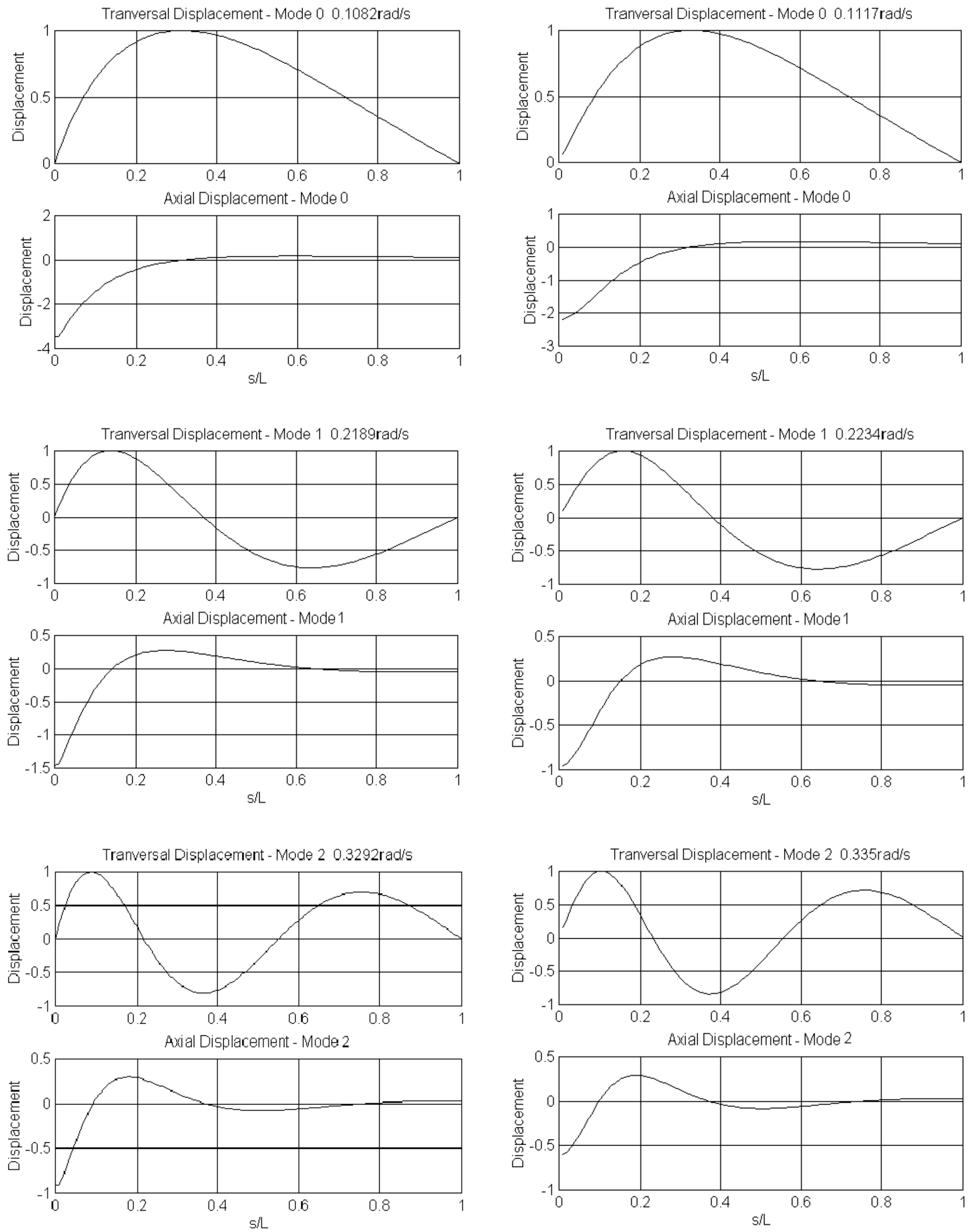

Figure 5 WKB approximate solution compared to modified Bessel's equation solution. Mode counter redefined as $k = n - 1$. Free-hanging riser with $\mu = \tan \theta_L = 5.7$.

Numerical Comparison

Figure 6 and 7 show WKB approximations for eigenmodes and corresponding eigenvalues for a free-hanging catenary riser. Table 1 presents general data. A standard flexible pipe in 785 meters depth has been taken. Again we took $m = \tan q_L = 0.2$ and 5.7, corresponding to upper end angle values $q_L = p/3 = 60^o$ and $q_L = 4p/9 = 80^o$. Notice that bending stiffness is small whereas axial rigidity is very large. Solutions are compared to the numerical results, calculated with POLIFLEX, Martins, 1998, an in-house made computer code, that takes extensibility into account. WKB solution was applied considering riser hinged at TDP. Actually it can be verified (see Appendix) that the axial-to-geometric rigidity ratio parameter of Irvine & Caughey is rather large in the present analysis, taking the values $l = 14.35$ and 9.37, corresponding to $q_L = p/3 = 60^o$ and $q_L = 4p/9 = 80^o$. According to those authors' analysis for a similar case but symmetric case, first 'symmetric' mode is expected to occur with two internal nodes whenever $l \geq 2p$. So, the first POLIFLEX eigenmode showed, as it should, as an 'anti-symmetric' one. The fundamental WKB mode (no nodes) should be disregarded. On another hand, as pointed out earlier, the approximate WKB solution gives poor results concerning the axial displacement, since axial boundary conditions are lost in virtue of the inextensibility hypothesis. Shape, however, is preserved.

Notice that POLIFLEX was run under three conditions:

(i) Riser hinged at TDP, with hinge mounted on a linearly elastic horizontal spring, whose rigidity is the same as for the effective length of cable laid on the sea floor, assuming a Coulomb friction law, with friction coefficient 0.4. Actual value for the axial rigidity is taken.

(ii) Riser hinged at TDP, but with the actual value for the axial rigidity.

(iii) Riser hinged at TDP, taking the axial rigidity 100 times larger than the actual value (here referred to as 'infinite' axial rigidity).

Total length (the sum of suspended and supported-on-the soil parts) is 3000 m, and that is the reason why eigenmodes are plotted along the arch-length coordinate, from the TDP to the top; Figure 6 and 7 refer to condition (iii).

Figures 8 and 9 present eigenvalues calculated for each upper end angle condition, $m = \tan q_L = 0.2$ and 5.7, comparing WKB (inextensible) solution to POLIFLEX results obtained under conditions (i), (ii) and (iii). The agreement is good, for both upper angle conditions, particularly for 'low-order' eigenmodes, although, strictly speaking, WKB technique assumes large eigenvalues. We can see that natural frequencies are lower as we consider axial extensibility in POLIFLEX solution (conditions (i) and (ii)). Particularly, if we consider an effective length laid on the sea floor (condition (i)), extensibility effect is even greater, as would be expected. For both conditions (i) and (ii) WKB results agree worse the larger the mode order. When we consider condition (iii), in which we take, in POLIFLEX, a value for the axial rigidity that is 100 times larger than the actual rigidity, results match nicely with WKB (inextensible) approximate results, as they should, therefore verifying POLIFLEX code. Also, comparing figures 8 and 9, extensibility effect is larger for taught risers, as it should be expected.

Figure 10, at last, presents an example of a high-order mode, the 29th eigenmode, for both cases where $m = \tan q_L = 0.2$ and 5.7.

Table 1 Flexible-pipe riser data

EA (kN)	312500
EJ (kNm^2)	49.61
q (kN/m)	0.914
m (t/m)	0.218
D (m)	0.3934
H (m)	785
Total length (m)	3000
L (m) for $q_L = 80^o$	935.5
L (m) for $q_L = 60^o$	1359.6

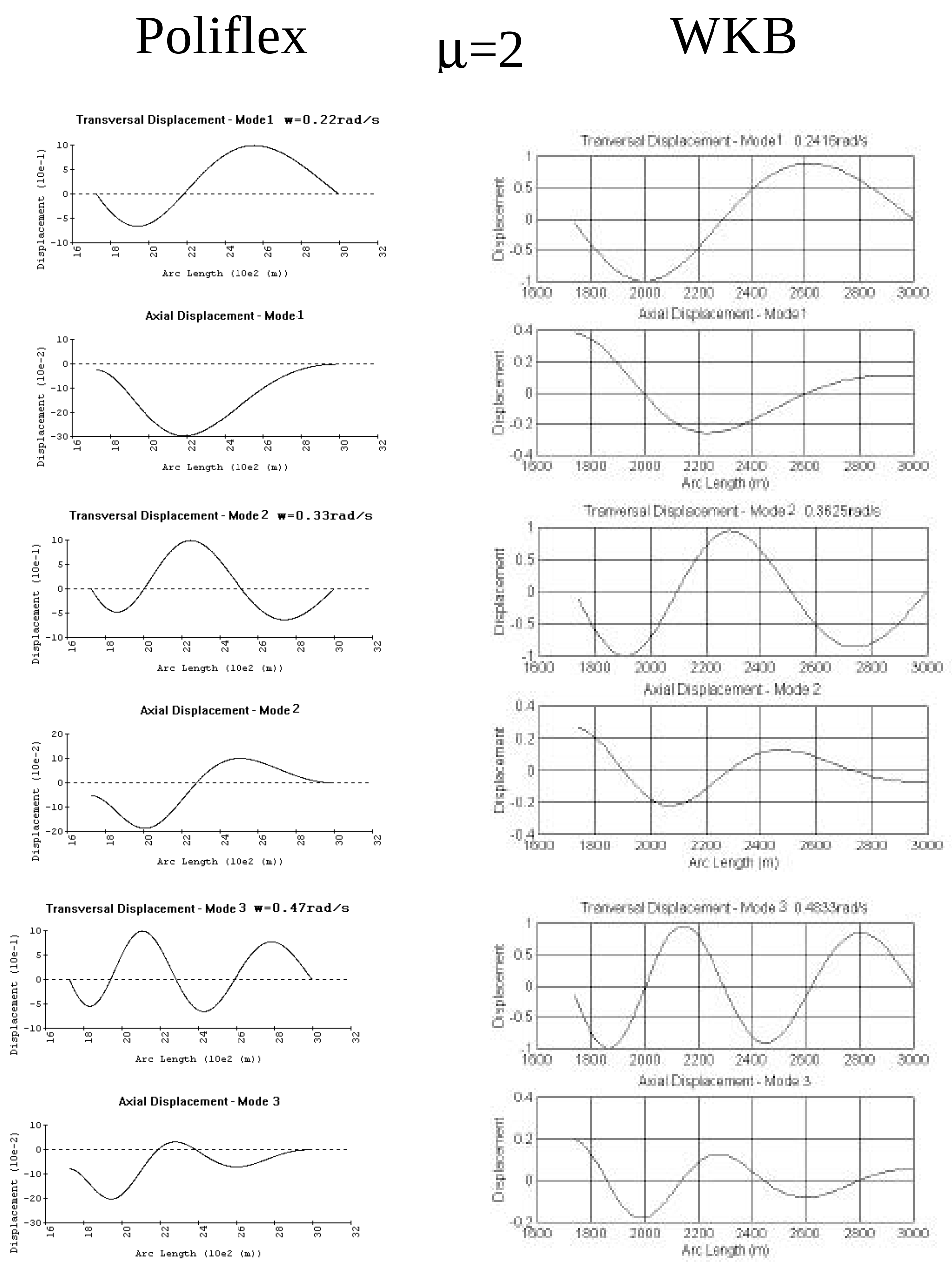

Figure 6 WKB approximate solution compared to numerical results obtained by a standard Finite Element Three first eigenmodes. Finite Element Method Formulation: POLIFLEX code. Free-hanging catenary-riser. $m = \tan q_L = 2.0$.

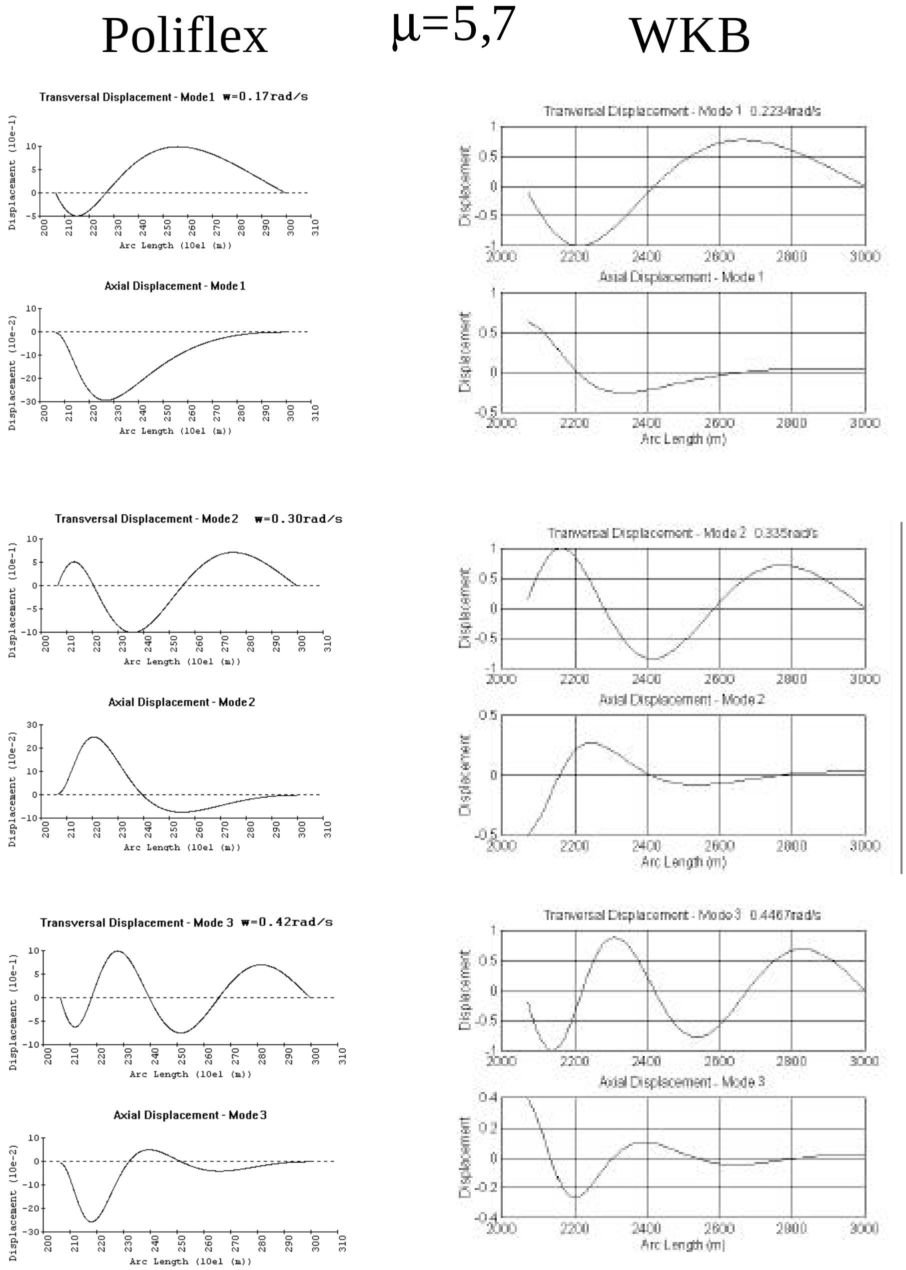

Figure 7 WKB approximate solution compared to numerical results obtained by a standard Finite Element Three first eigenmodes. Finite Element Method Formulation: POLIFLEX code. Free-hanging catenary-riser. $\mu = \tan\theta_L = 5.7$

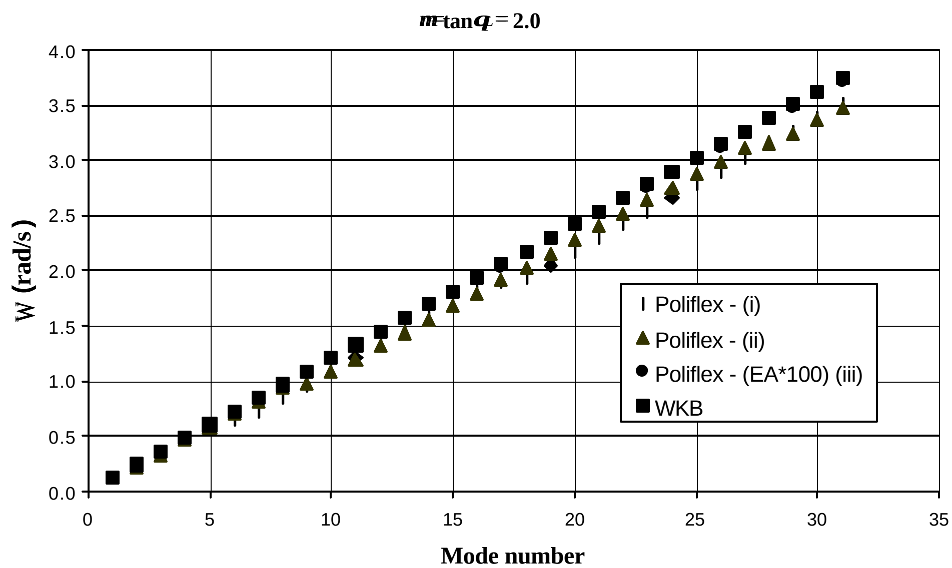

Figure 8 Eigenvalues for a catenary (flexible-pipe) riser. Inextensible WKB approximate solution compared to POLIFLEX results. $m = \tan q_L = 2.0$..

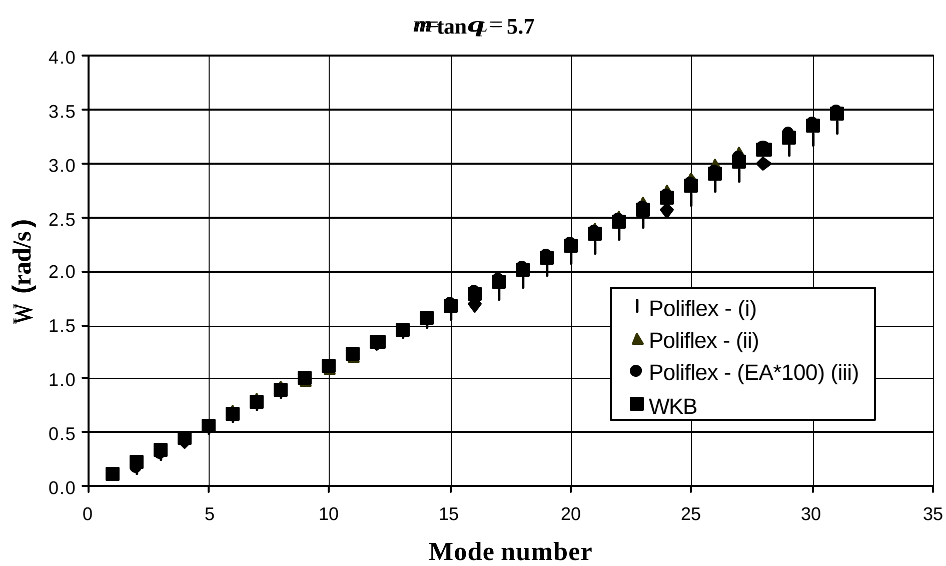

Figure 9 Eigenvalues for a catenary (flexible-pipe) riser. Inextensible WKB approximate solution compared to POLIFLEX results. $m = \tan q_L = 5.7$

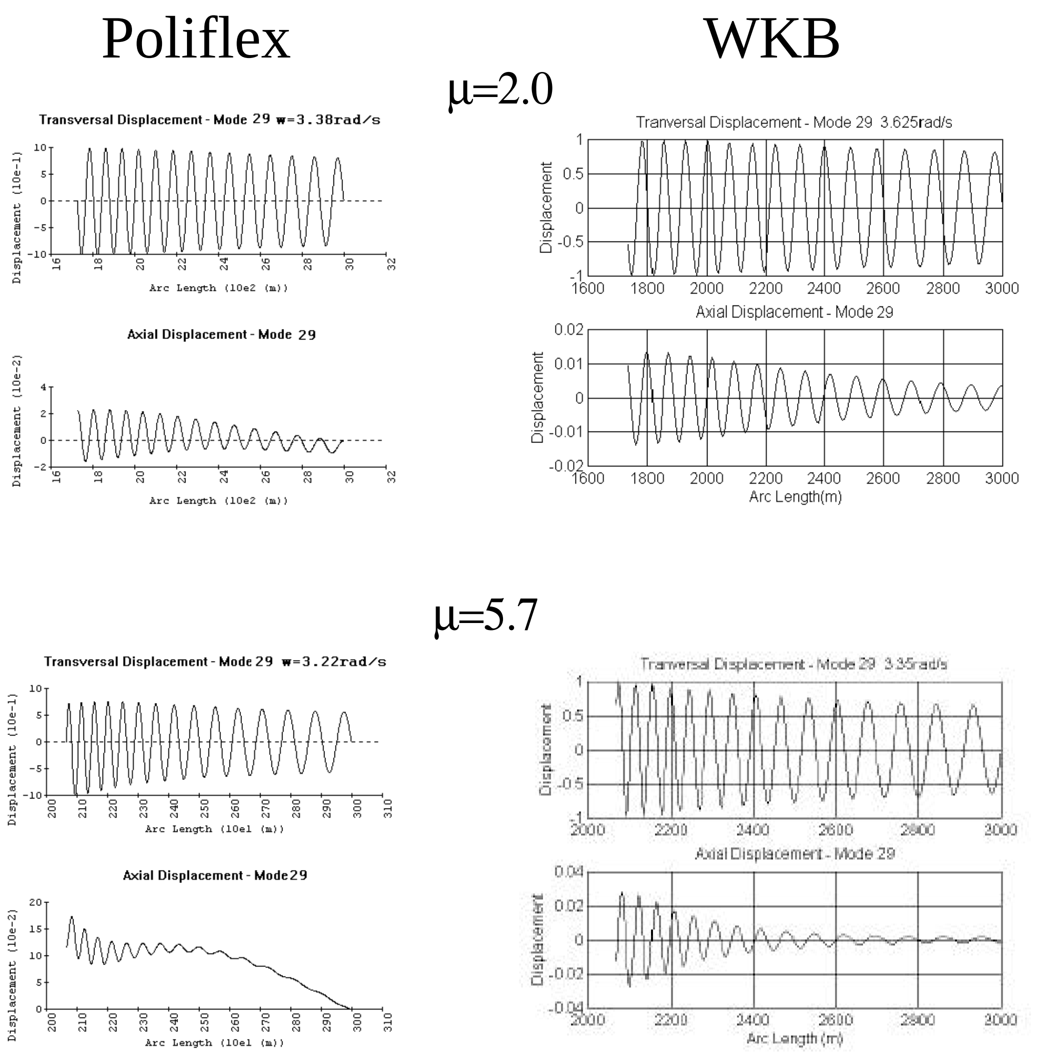

Figure 10 WKB approximate solution compared to numerical results obtained by a standard Finite Element A high-order eigenmode: k=29. Finite Element Method Formulation: POLIFLEX code. Free-hanging catenary-riser. $m = \tan q_L = 2.0;5.7$.

CONCLUSIONS

Though being aware of the physical inadequacy concerning the inextensibility assumption, as shown by Irvine & Caughey, the two-dimensional riser-like problem have been addressed on this basis. The corresponding eigenvalue problem has been posed and discussed, although tangential boundary condition is missed in this approximation. An analytical and closed form WKB approximate solution has been derived for the general (inextensible) eigenvalue riser-like problem. Such solution is given, as it should, in the form of sinusoidal functions, modulated in amplitude and phase,

$$\varphi(z) \cong F^{-1/4}(z)\left[C_1 \sin\left(\Lambda\int^z F^{-1/2}(u)du\right) + C_2\cos\left(\Lambda\int^z F^{-1/2}(u)du\right)\right].$$

These modulation functions are, respectively, $F^{-1/4}(z)$ and $F^{-1/2}(z)$, where $F(z)$ is the non-dimensional tension along the archlength z. The solution takes a very simple form for the catenary case. Particularly, a formula for the natural frequencies and a closed form solution for the modes of a catenary-riser have been presented. These formulae can be easily applied in practical situations, when the designer (or the field engineer) searches for a quick evaluation. Two simple formulae, given only in terms of local depth and angle at upper end, have also been derived, from WKB general solution, for the natural frequencies and eigenmodes of a catenary riser under no current. A free-catenary riser has been taken as example. Care must be taken, however, as the (inextensible cable) WKB solution always provides the fundamental 'symmetric' mode (no internal nodes), that might not appear for typical risers.

The classical analytical solution of the suspended heavy cable eigenvalue problem, based on a Bessel's modified equation approximation, and given in terms of Bessel functions of zero order, has been also derived for the catenary riser case and some elementary but useful solutions have been exemplified. Comparison of WKB solution and Bessel's approximation to numerical results obtained by a standard Finite Element Method formulation showed very good agreement. Extensibility effect has been addressed through a typical flexible pipe riser case in 785 meters depth. The examples have shown that, from a practical point of view, WKB approximation gives a good and expeditious estimate for the eigenvalues and eigenfunctions of a catenary riser. The present mathematical analysis applies not only to construct paradigms for numerical solutions but, particularly, enables one to properly address the validity of other physical intuitive arguments that are usually assumed, such as the small global effect of flexural rigidity. Corrections, close to extremities, would be necessary, however. The WKB solution is somewhat general and can be applied straightforwardly for 'jumpers' and can be promptly adapted to take into account other risers shapes, as 'lazy-wave', 'steep-wave' or multi-leg configurations.

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APPENDIX A: DISCONSIDERING GLOBAL EFFECT OF FLEXURAL RIGIDITY

In nondimensional form, Eq. (1) is written,

$$-\left(F(x)-\varepsilon_F^2\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial\eta}{\partial x}+\upsilon\frac{d\theta}{dx}\right)\frac{d\theta}{dx}-$$
$$-\varepsilon_F^2\frac{d^3\theta}{dx^3}\left(\frac{\partial\eta}{\partial x}+\upsilon\frac{d\theta}{dx}\right)-\varepsilon_F^2\frac{d^2\theta}{ds^2}\frac{\partial}{\partial s}\left(\frac{\partial\eta}{\partial x}+\upsilon\frac{d\theta}{dx}\right)=\frac{1}{1+\alpha}\frac{\partial^2\upsilon}{\partial t^2}$$

(31).

$$-\varepsilon_F^2\frac{\partial^3}{\partial x^3}\left(\frac{\partial\eta}{\partial x}+\upsilon\frac{d\theta}{dx}\right)+\varepsilon_F^2\frac{d^2\theta}{dx^2}\left(\frac{\partial\eta}{\partial x}+\upsilon\frac{d\theta}{dx}\right)\frac{d\theta}{ds}+$$
$$+\frac{\partial}{\partial x}\left(T\left(\frac{\partial\eta}{\partial x}+\upsilon\frac{d\theta}{dx}\right)\right)=\frac{\partial^2\eta}{\partial t^2}$$

where

$$\varepsilon_F=\frac{l_F}{L} \tag{32}$$

is the small flexural non-dimensional parameter, being

$$l_F = \sqrt{\frac{EJ}{T_0}} \tag{33}$$

the local (here at TDP) flexural length (see Aranha et al., 1997) that gauges the bending stiffness importance with respect to geometric rigidity. Equation (31) is a singular perturbation problem, of the beam-string type (see, e.g., Kevorkian & Cole, 1981). It tell us that bending stiffness effect is important only inside small regions of length $O(l)$, i.e. boundary-layers, either external (at the ends) or internal ones, around sections where curvature attains local maxima (TDP, for instance). Boundary-layer technique has been applied in order to study the dynamic curvature at the extremities (TDP and top end) of a catenary riser; Aranha et al., 1997, Pesce, 1997. Notice that in Eq. (31) we took the viscous damping terms off. If terms of order $e_F^{\;2}$ are neglected, we get the associate non-dimensional and non-damped form of Eq. (3) (the inextensible and non-damped outer equation that dominates the overall dynamic behaviour), Eq. (5).

APPENDIX B: IRVINE & CAUGHEY 'S RIGIDITY PARAMETER

In Irvine & Caughey's paper a similar (actually a geometrically symmetric case) is analysed. An axial-to-geometric rigidity parameter is defined as

$$l^2 = \left(\frac{8d}{l}\right)^2 \frac{l}{(\bar{T}L_e / EA)} \tag{34}$$

where, d is the sag, l the span, $\bar{T}$ is the horizontal tension (exactly the tension at midsection in their analysis), and L_e is a form length parameter given by,

$$L_e = \int_0^l \left(\frac{ds}{dx}\right)^3 dx \tag{35}.$$

They showed that the first symmetric eigenmode has no internal nodes, whenever $l \le 2p$. Otherwise, the first symmetric mode has two internal nodes and the first antisymmetric mode appears as the lowest one, with one internal node. If the definition above were applied to the present catenary riser problem, taking $\bar{T} = T_D = T_o \sec q_D$ (refer to the figure below), we easily would get from classical catenary equations

$$L_e = L + \left(\frac{q}{T_o}\right)^2 \frac{L^3}{3} \tag{36}$$

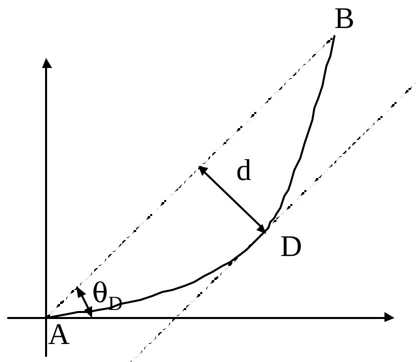

Table 2 values calculated for both exemplified cases.

m	x_B (m)	y_B (m)	L (m)	l (m)	d (m)	T_0 (kN)	T_D (kN)	L_e (m)	l^2
2	916.8	785	1270	1207	167.8	580	764KN	2963	206.11
5.7	400.3	785	934.7	881	118.8	150	329KN	11058	87.85

APPENDIX C: CONSIDERING VIRTUAL WORK DONE BY HYDROSTATIC TERMS

Notice that if the (virtual) work done by hydrostatic pressure were considered, (as in Chucheepsakul & Huang (1997)), in the manner explained by means of the usual concept of 'effective' tension in marine cables and risers, the inextensible dynamic equation would be written,

$$\left[F(x)\frac{dq}{dx} - g_h \cos q\right]\left(\frac{\partial h}{\partial x} + u\frac{dq}{dx}\right) + \frac{1}{1+a}\frac{\P^2 u}{\P t^2} = 0$$

(37),

$$-\frac{\partial}{\partial x}\left(F(x)\left(\frac{\partial h}{\partial x} + u\frac{dq}{dx}\right)\right) - g_h\left(u \sin q + h\cos q\right) + \frac{\partial^2 h}{\partial t^2} = 0$$

where

$$g_h = \frac{(r_a S_o - r_i S_i) g L}{T_0}$$

is the nondimensional 'effective' hydrostatic effect per unit length, being r_a the density of water, r_i the density of a possible internal fluid, S_o and S_i the outer and inner cross sectional areas, respectively. The approximate eigenvalue problem would then be written,

$$\left(F(x)c(x) - g_h \cos q(x)\right)\left(\varphi' + y\,c(x)\right) - \frac{w^2}{1+a} y = 0$$

(38)

$$\frac{d}{dx}\left[F(x)\left(\varphi' + y\,c(x)\right)\right] + g_h\left(y \sin q(x) + \varphi \cos q(x)\right) + w^2 \varphi = 0$$