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A fresh look at the catenary

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Abstract
A hanging chain takes the familiar form known as the catenary which is one of the most ubiquitous curves students encounter in their daily life. Yet most introductory physics and mathematics texts ignore the subject entirely. In more advanced texts the catenary equation is usually derived as an application of the calculus of variations. Although the variational approach is mathematically elegant, it is suitable for more advanced students. Here we derive the catenary equation in special and rectangular coordinates by considering the equilibrium conditions for an element of the hanging chain and without resorting to the calculus of variations. One advantage of this approach is its simplicity which makes it accessible to undergraduate students; another is the concurrent derivation of a companion equation which gives the tension along the chain. These solutions provide an excellent opportunity for undergraduates to explore the underlying physics. One interesting result is that the shape of a hanging chain does not depend on its linear mass density or on the strength of the gravitational field. Therefore, within a scale factor, all catenaries are copies of the same universal curve. We give the functional dependence of the scale factor on the length and terminal angle of the hanging chain.

Keywords: catenary, hanging chain, catenary equation

(Some figures may appear in colour only in the online journal)

1. Introduction

The catenary (from the Latin *catena* for chain) is the familiar shape of a chain when it hangs loosely from its ends. The mathematical equation of the catenary was unknown until the invention of the calculus of variations in the late 17th century. Indeed the catenary equation was one of the early triumphs of the calculus of variations; it was derived independently by Leibnitz, Huygens, and Johann Bernoulli in response to a challenge posed by Jacob Bernoulli around 1690 [1, 2].

In the first decades of the 18th century, the Bernoulli brothers played an important role in publicizing the calculus of variations and applied it to other problems in mechanics including the brachistochrone [3]. Euler and later Lagrange developed the subject further, and explored its applications in geometry, mechanics and dynamics [4]. For example Euler showed that when the catenary is rotated about the *x*-axis the resulting surface is the minimum surface of revolution (catenoid) bounded by the two end circles [5].

Under stable equilibrium the gravitational potential energy of a hanging chain is a minimum. Therefore, the catenary problem may be restated as follows: given a chain of length L hanging from two given points, what is the shape that minimizes its gravitational energy? This problem is tailor-made for the calculus of variations which provides the method for finding the curve connecting two points which either minimizes or maximizes a given integral.

Although the calculus of variations provides a mathematically elegant method for deriving the catenary equation (see the appendix), the procedure is most suitable for the advanced students and comes with a hidden drawback: it strikes most students as a mathematical recipe for arriving at the desired differential equation without offering much physical insight.

The catenary is one of the most common curves students encounter in their daily life, yet it is often left out of the introductory physics and calculus courses [6]. At the intermediate level, several older texts [7, 8] derive the catenary equation without resorting to the calculus of variations. The treatment in [7] is typical where the catenary equation is the solution of a differential equation which is obtained by considering the equilibrium of forces. The Wikipedia article on '*catenary*' [9] provides ample references to the historical development of the subject and gives the standard non-variational derivation of the catenary equation following Routh [7]. The article also includes another closely related derivation due to Lamb [8]. A modern version of this approach appears in a textbook on classical mechanics by Davis [10].

In more advanced mathematics and physics textbooks the catenary equation appears as an application of the calculus of variations but with little or no discussion of the underlying physics [11]. Sometimes the catenary is mentioned briefly in connection with the derivation of a minimal surface of revolution known as the catenoid [12].

Here we show that the catenary equation can be derived simply in an introductory course by considering the chain under equilibrium and with judicious use of elementary calculus. In section 2 we consider the equilibrium of an element of the chain and derive the catenary equation in its simplest form $s(\theta)$, where s is the arc length measured from the vertex and θ is the angle of the chain with the horizontal. One advantage of this approach is its simplicity which makes it accessible to undergraduate students; another is an auxiliary equation which gives the tension along the chain $T(\theta)$.

In section 3 we derive a pair of parametric equations, $y(\theta)$ and $x(\theta)$, to describe the catenary completely in rectangular coordinates, again without resorting to the calculus of variations. In section 4 we eliminate θ between the parametric equations to arrive at the usual catenary equation in rectangular coordinates $y(x)$. The solutions of the problem in angular, parametric, and rectangular forms provide the undergraduates the opportunity to tackle the problem in its simplest form first and to follow a gentle path to the final solution.

In section 5, we discuss the more interesting features of these solutions. One important feature is the fact that the particular shape assumed by a hanging chain does not depend on the linear mass density of the chain. A consequence of this feature is that all catenaries can be made to fit a universal curve by adjusting a scale factor. We provide the relation that gives the scale factor in terms of the two readily available parameters of a hanging chain: the length and the terminal angle.

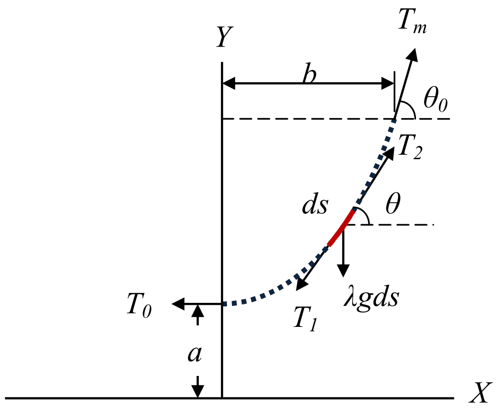

Figure 1. The right half of the chain of length l and linear mass density λ is shown schematically. The element ds is in equilibrium under the action of three forces: the tensions T_2 and T_1 and its weight λgds.

Finally, in both special and rectangular coordinates, we discuss the equations that give the tension in the chain. As expected, for a given hanging chain the tension does depend on the linear mass density as well as the length and terminal angle.

2. Catenary equations in special coordinates

Consider a chain of length L and linear mass density λ which hangs from two points. Let us choose the vertical axis to be the line that bisects the hanging chain through its vertex. Without loss of generality we may consider the right half of the chain with length l as shown in figure 1. Let θ be the angle between the chain and the horizontal. Clearly this angle is zero at the vertex but increases gradually along the chain to reach its maximum value θ_0 at the top end.

The analysis is vastly simplified by choosing θ as the independent variable to specify the position of a point along the chain. In this scheme, $s(\theta)$ gives the arc length from the vertex to any point along the chain and $T(\theta)$ gives the tension at θ.

To derive the functions $s(\theta)$ and $T(\theta)$ we begin with the equations which describe the equilibrium of an element of the chain. Referring to figure 1, the equilibrium of forces along the element ds is given by,

$$\mathrm{d}T = \lambda g \sin\theta \, \mathrm{d}s, \tag{1}$$

where, g is the acceleration of gravity and

$$\mathrm{d}T = T_2 - T_1. \tag{2}$$

The equilibrium of forces normal to the element gives,

$$T\mathrm{d}\theta = \lambda g \cos\theta \, \mathrm{d}s. \tag{3}$$

Equations (1) and (3) are two coupled differential equations whose solutions give the two equations $s(\theta)$ and $T(\theta)$. First, the differential equations may be decoupled by dividing equation (1) by equation (3)

$$\mathrm{d}T/T = \mathrm{d}\theta \tan\theta. \tag{4}$$

Equation (4) can be integrated to give the tension $T(\theta)$,

$$lnT = ln\,(1/\cos\theta) + ln\,(\lambda ga). \tag{5}$$

Where, for later convenience, λga is the constant of integration to be determined from the initial conditions l and θ_0. Equation (5) may be recast into a more convenient form,

$$T\,(\theta) = \lambda ga/(\cos\theta). \tag{6}$$

We can now substitute equation (6) for T in equation (3) to get a decoupled differential equation,

$$\mathrm{d}s = a\ \mathrm{d}\theta/\cos^2\theta, \tag{7}$$

which can be integrated to give,

$$s = a\ \tan\theta + C_1. \tag{8}$$

Note that $s = 0$ at the vertex where $\theta = 0$, implying that $C_1 = 0$. Consequently, we have,

$$s\,(\theta) = a\ \tan\theta. \tag{9}$$

Equations (6) and (9) give the tension and arc length of the chain at a given angle along the chain. For a chain of half length l, and linear density λ, we can determine the constant a from equation (9) by applying the appropriate boundary conditions.

At the chain terminal where $\theta = \theta_0$, and $s(\theta_0) = l$, equation (9) gives,

$$a = l/\tan\theta_0. \tag{10}$$

Equations (6) and (9) completely describe the catenary curve and the tension in the curve. It is interesting to note that the shape of a hanging chain is not affected by the chain's linear mass density or by the strength of the gravitational field. So a chain will have the same shape on Earth and the Moon regardless of its linear mass density as long as we ensure that the geometrical parameters (length and terminal angles) are the same. Needless to say, the tension in the chain does depend on both the linear mass density of the chain and the strength of the gravitational field.

Furthermore, equation (9) gives the minimum tension T_0, and the maximum tension T_m,

$$T_0 = \lambda ga = \lambda gl/\tan\theta_0 \tag{11}$$

and,

$$T_m = \lambda ga/\cos\theta_0 = \lambda gl/\sin\theta_0. \tag{12}$$

In this formulation the shape of the chain, as described by equation (9), is determined only by the length l and the terminal angle θ_0. Note, however, that the linear mass density λ appears in equation (6) which gives the tension along the chain.

3. Parametric equations of the catenary

Referring to figure 1, we may immediately write,

$$\mathrm{d}x = \mathrm{d}s\ \cos\theta = a\ \mathrm{d}\theta/\cos\theta, \tag{14}$$

and,

$$\mathrm{d}y = \mathrm{d}s\ \sin\theta = a\ \mathrm{d}\theta\ \sin\theta/\cos^2\theta, \tag{15}$$

where, equation (7) is used to eliminate ds from both equations.

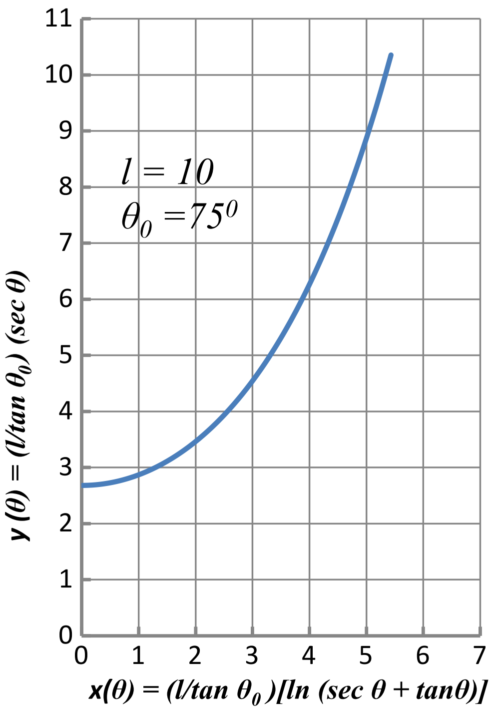

Figure 2. The catenary graph from the parametric equation $y(\theta)$ and $x(\theta)$ for chain length $l = 10$, and terminal angle $\theta_0 = 75°$. The intercept is at $a = l/\tan\theta_0 = 2.68$.

Equations (14) and (15) may be integrated to give,

$$x(\theta) = a\ ln\,(\sec\theta + \tan\theta) + C_1, \tag{16}$$

and,

$$y(\theta) = a\,(1/\cos\theta) + C_2. \tag{17}$$

Equations (16) and (17) are the two parametric equations of the catenary. If we choose the y-axis to pass through the vertex of the catenary, then $x(0) = 0$, and $C_1 = 0$. Similarly if we choose the y-intercept $y(0) = a$, we have $C_2 = 0$. Therefore,

$$x(\theta) = a\ ln\,(\sec\theta + \tan\theta), \tag{18}$$

and,

$$y(\theta) = a\,(1/\cos\theta). \tag{19}$$

Clearly the y-intercept $a \equiv l/\tan\theta_0$ sets the scale. As discussed before, this scale factor is determined by the length of the chain and the terminal angle θ_0. However, as seen in figure 1, the terminal angle θ_0 is related to parameter b, the distance from the chain end to the y-axis, since we have,

$$x(\theta_0) = b = a\ ln\left(\sec\theta_0 + \tan\theta_0\right). \tag{20}$$

Therefore the scale factor a for a chain of half-length l may also be set by the choice for b.

We may graph the catenary by computing the values of $y(\theta)$ and $x(\theta)$ from equations (18) and (19) for angle θ ranging from $\theta = 0$ to $\theta = \theta_0$. Figure 2 is the catenary plot for the case when $l = 10$, and $\theta_0 = 75°$.

4. The catenary equation in rectangular coordinates

By eliminating θ between the two parametric equations for $x(\theta)$ and $y(\theta)$, it is simple to obtain $y(x)$ as follows. From equation (18), we have,

$$e^{x/a} = (1 + \sin\theta)/(\cos\theta), \tag{21}$$

and thus,

$$e^{-x/a} = (\cos\theta)/(1 + \sin\theta). \tag{22}$$

Therefore,

$$\left(e^{x/a} + e^{-x/a}\right) = \left[(1 + \sin\theta)^2 + \cos^2\theta\right]\Big/[\cos\theta(1 + \sin\theta)], \tag{23}$$

which simplifies to,

$$\left(e^{x/a} + e^{-x/a}\right) = 2/\cos\theta. \tag{24}$$

Finally, substitution of equation (19) in equation (24) results in,

$$y = a\left[e^{x/a} + e^{-x/a}\right]\Big/2 = a\cosh(x/a). \tag{25}$$

Equation (25) is the familiar catenary equation.

We may now turn our attention to finding $s(x)$, and $T(x)$. From equation (21) we have,

$$e^{x/a} = (1/\cos\theta) + \tan\theta. \tag{26}$$

Substitution of equations (9) and (19) into equation (26) gives,

$$e^{x/a} = (y + s)/a. \tag{27}$$

Therefore,

$$s = a\,e^{x/a} - y = (a/2)\left[e^{x/a} - e^{-x/a}\right], \tag{28}$$

or,

$$s(x) = a\sinh(x/a). \tag{29}$$

To find $T(x)$, we substitute equation (19) in equation (6) to get,

$$T(\theta) = \lambda g\, y(\theta). \tag{30}$$

And therefore,

$$T(x) = \lambda g y(x) = \lambda g a\cosh(x/a). \tag{31}$$

We now have all the relevant equations to describe the catenary completely in the rectangular coordinate system:

$$y(x) = a\cosh(x/a), \tag{32}$$

$$s(x) = a\sinh(x/a), \tag{33}$$

$$T(x) = \lambda g a\cosh(x/a) = \lambda g\, y(x). \tag{34}$$

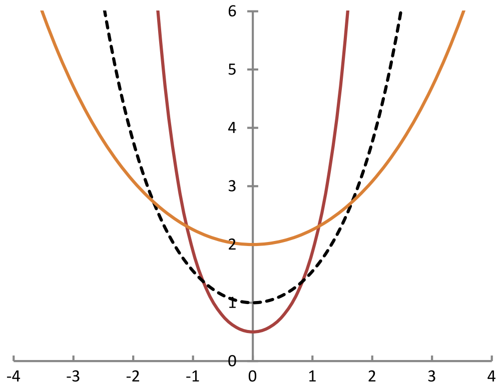

Figure 3. Graphs of the catenary equation $y = a \cosh x/a$, with three values (0.5, 1.0, and 2.0) for the y-intercept a. The dashed graph for which $a = 1$ represents the universal catenary $y = \cosh x$. The upper graph results when the dashed graph expands by a factor of 2 ($a = 2$). The lower graph results when the dashed graph shrinks by a factor of 2 ($a = 0.5$).

Equations (32) and (33) lead to,

$$y^2 = s^2 + a^2, \tag{35}$$

or,

$$y = \left(s^2 + a^2\right)^{1/2}. \tag{36}$$

Substitution of equation (36) in equation (34) results in,

$$T(s) = \lambda g \left(s^2 + a^2\right)^{1/2}. \tag{37}$$

Equation (37) gives the tension along the chain as a function of the arc length s. Therefore, at the vertex where $s = 0$, the minimum tension is

$$T_0 = \lambda g a. \tag{38}$$

The maximum tension occurs at the top end of the chain where $s = l$,

$$T_m = \lambda g \left(l^2 + a^2\right)^{1/2}. \tag{39}$$

5. Discussion

When we hang a chain by holding its two ends at the same level, two parameters are paramount: the length of the chain L, and the separation B between the two ends. Given a chain of length L, and end separation B, what is the function $y(x)$ that describes the shape of the chain? We have already established that when the y-axis is chosen to pass through the vertex of the catenary, the equation is,

$$y = a \cosh(x/a). \tag{40}$$

Figure 3 shows three plots of equation (40) with $a = 0.5$, 1, and 2. The middle graph (dashed line), associated with $a = 1$, may be considered the universal graph for the catenary.

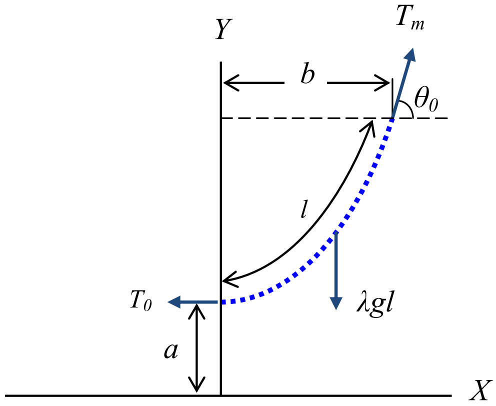

Figure 4. The right half of a hanging chain with length l, linear density λ, and end distance b from the y-axis. The intercept a is related to l and b, through the relation: $l = a \sinh b/a$.

As is evident in figure 3, the catenary is a symmetric function, i.e., $y(x) = y(-x)$, while a is the y-intercept. Furthermore, the linear scale of the catenary is set by a. These points are evident when equation (40) is recast in its exponential form,

$$\frac{y}{a} = \cosh\frac{x}{a} = \frac{1}{2}\left(e^{x/a} + e^{-x/a}\right). \tag{41}$$

Therefore any catenary can be recast into the universal configuration $y = \cosh x$ by choosing a to be the unit of length.

In practice, when a chain of length L is hanging from two points, the resulting catenary configuration is characterized by the separation distance B between its two ends. This point is evident in figure 3, where the catenary graph widens as a is increased and narrows when a is decreased. A more suitable interpretation of this trend is to note that the graphs *appear* to widen simply because the scale factor is increased. So the upper graph in figure 3 associated with $a = 2$ may be made to coincide with the universal graph by shrinking it by a factor of 2. In contrast the lower graph associated with $a = 0.5$ may be enlarged by a factor of 2 to fit the universal graph. Furthermore, since the chain configuration does not depend on its linear density, all chains of the same length L will hang in the same configuration if their end points are separated by the same distance B.

So the question becomes: How is the constant a in equation (40) related to the two physical parameters L and B? More specifically for a hanging chain of length L, we must relate B to the y-intercept a.

Figure 4 shows the right half of a hanging chain with length l, end distance b from y-axis, and linear density λ as before. From equation (33), we have,

$$l/a = \sinh(b/a). \tag{42}$$

Equation (42) gives the intercept a in terms of b for a given chain of half-length l. To plot a universal graph showing the dependence of a on b for a chain of any length we use the chain half length l as a scale factor. Figure 5 is a plot of (a/l versus b/l) which is a normalized graph for displaying the dependence of a on b for all chains. The trend in figure 5 is clear: for a given chain a increases as b is increased, a fact that students can demonstrate to their own satisfaction.

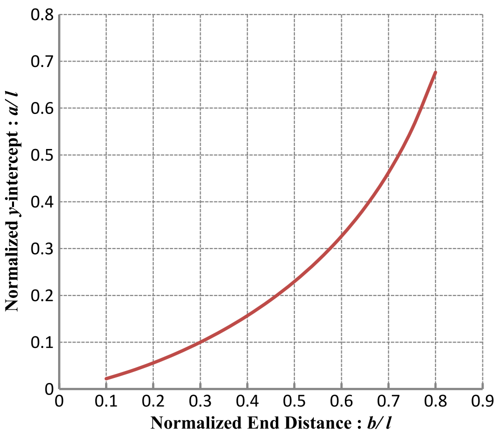

Figure 5. A plot of the normalized intercept *a*/*l* versus the normalized end distance *b*/*l*. This plot is a universal graph showing the dependence of *a* on *b*.

So far we have considered the case of a hanging chain with its end points at the same level. What about the case when the two ends of the chain are held at different levels? More specifically consider a hanging chain of total length L. The vertical axis passes through the vertex and divides the chain into two sections. Let the right hand section of the chain be characterized by length l_1 and its end point located at a distance b_1 from the y-axis. Similarly the left section is characterized by length l_2 and end point distance b_2. In light of equation (24), each half may be treated independently without loss of generality.

$$l_1 = a \sinh \frac{b_1}{a}, \tag{43}$$

$$l_2 = a \sinh \frac{b_2}{a}. \tag{44}$$

Clearly each half can be treated separately and the intercept a may be determined from either of the two sections.

Appendix A

Mathematically, calculus of variations provides the most direct tool for deriving the equation of the catenary. The shape assumed by a catenary minimizes the gravitational potential U of the hanging chain. For a hanging chain of linear mass density λ, the gravitational potential energy is given by,

$$U = \lambda g \int y \, \mathrm{d}s. \tag{A1}$$

Therefore, we must find the function $y(x)$ in the above expression that minimizes U. Since,

$$\mathrm{d}s = \left(\mathrm{d}x^2 + \mathrm{d}y^2\right)^{1/2}. \tag{A2}$$

We have,

$$U = \lambda g \int y \, \mathrm{d}x \left(1 + y'^2\right)^{1/2}, \tag{A3}$$

where,

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x}. \tag{A4}$$

To find the function $y(x)$ that minimizes U, the integrand

$$f = y\left(1 + y'^2\right)^{1/2}, \tag{A5}$$

must satisfy the Euler–Lagrange equation. However, since the integrant is not an explicit function of x, we may use an alternative form of the Euler–Lagrange equation to simplify the analysis,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f - \mathrm{y}'\frac{\partial \mathrm{f}}{\partial \mathrm{y}'}\right) - \frac{\partial \mathrm{f}}{\partial \mathrm{x}} = 0. \tag{A6}$$

Which yields,

$$\left(f - \mathrm{y}'\frac{\partial \mathrm{f}}{\partial \mathrm{y}'}\right) = a. \tag{A7}$$

Where a is a constant. Referring to equation (A5),

$$\frac{\partial f}{\partial y'} = yy'\left(1 + y'^2\right)^{-1/2}. \tag{A8}$$

Substitution of equations (A5) and (A8) into equation (A7) followed by some algebraic simplification results in a simple differential equation,

$$y^2 = a^2\left(1 + y'^2\right). \tag{A9}$$

This differential equation may be recast into,

$$\frac{\mathrm{d}y}{\left(y^2 - a^2\right)^{1/2}} = \frac{\mathrm{d}x}{a}. \tag{A10}$$

Which is integrated to give,

$$\cosh^{-1}\left(\frac{y}{a}\right) = \frac{x + k}{a}, \tag{A11}$$

or,

$$y = a \cosh\left(\frac{x + k}{a}\right). \tag{A12}$$

Two constants appear in equation (A12). When the y-axis passes through the vertex of the catenary as in figure 3, $k = 0$. The constant a is clearly the y-intercept.

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