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F. Clementi · L. Demeio · C. E. N. Mazzilli · S. Lenci

Nonlinear vibrations of non-uniform beams by the MTS asymptotic expansion method

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Abstract The frequency response curves of a non-uniform beam undergoing nonlinear oscillations are determined analytically by the multiple time scale method, which provides approximate, but accurate results. The axial inertia in neglected, and so the equations of motion are statically condensed on the transversal displacement only. The nonlinearity due to the stretching of the axis of the beam is considered. The effects of variable cross-section, of variable material properties and of the distributed axial loading are taken into account in the formulation. They have been illustrated by means of two examples and are also compared with existing results. The main result of this work is that the effects of any type of non-uniformity can be detected by simple formulas.

Keywords Non-uniform beam · Nonlinear curvature · Nonlinear oscillations · Multiple time scale method · Frequency response curves

1 Introduction

This paper continues in more detail the discussion on the vibrations in non-uniform beams initiated in [1], where simple formulas for the determination of the natural frequencies were proposed.

With respect to the formulation introduced in [1], in the present paper the differential equation of motion for the non-uniform beam—which takes into account the variation of the flexural stiffness and/or the geometric stiffness (due to normal force variation) and/or the unit mass along the beam length—differs by the addition (i) of an ad hoc viscous damping, (ii) of longitudinal and transversal loads, which can also resonate with a certain natural mode, and (iii) of the geometric nonlinear terms. Thus, we investigated also nonlinear forced vibrations, whereas in [1] only linear free vibrations are considered. Further, instead of the Lindstedt–Poincaré method, the multiple time scales (MTS) method is used here.

In the free vibration analysis, we here recover the same backbone curve (which describes the nonlinear relationship between the natural frequencies and the modal amplitudes) which was found in [1], although following an altogether different approach.

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F. Clementi · S. Lenci (✉)
Department of Civil and Buildings Engineering, and Architecture, Polytechnic University of Marche, Ancona, Italy
E-mail: lenci@univpm.it

L. Demeio
Department Industrial Engineering and Mathematical Sciences, Polytechnic University of Marche, Ancona, Italy

C. E. N. Mazzilli
Department of Structural and Geotechnical Engineering, Escola Politécnia, University of São Paulo, São Paulo, Brazil

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In the forced vibration analysis under resonant conditions, the influence of each non-uniform beam property function (flexural stiffness, geometric stiffness and unit mass) is studied, so that its effect can be apprehended directly from the obtained results.

In order to illustrate the general theory, two examples are addressed in this paper: in the first example, we assess the influence of the transversal load per unit length (submerged weight) on the linear frequency of a riser, whereas in the second example the nonlinear effects of a varying cross-section of a tapered beam are studied.

As already mentioned in [1], non-uniform beams are used in helicopter rotor blades, airplane wings and wind turbine blades, to mention just a few mechanical engineering applications; non-prismatic pylons of cable-stayed bridges, and beams and columns of varying cross-sections, to mention just a few civil engineering applications that, besides serving to structural or optimization purposes, nowadays are widely used by renowned architects due to esthetic reasons. Offshore vertical risers are also within the range of application of this study, provided the non-uniformity of the beam is mild, especially with respect to the tensile force variation that affects the riser's geometric stiffness.

Modal analysis of non-uniform beams has attracted the attention of various researchers from the fifties to the eighties [2–7]. Subsequently, linearly tapered beams subjected to axial forces were studied by Sato [8]; elastic support of the Winkler type was considered by Filipich et al. [9]; elastic boundary conditions were introduced by Goel [10] and Lee and Ke [11].

More recently, Dugush and Eisenberg [12] investigated the vibrations of non-uniform beams under moving loads; Raj and Sujith [13] discussed free longitudinal vibrations; Shahba et al. [14] looked at curved Timoshenko beams; Bambill et al. [15] studied rotating Timoshenko beams, a problem of particular interest in helicopter blade dynamics. Tapered beams of functionally graded materials were also addressed [16,17].

Exact solutions were obtained for some special cases of tapering: Bessel's functions were used in [10,18] and hypergeometric functions in [13,19]. Abrate [20] discussed a special class of tapering in rods and beams, for which the governing equation of motion could fall into the same pattern of a uniform media.

Yet, exact solutions are restricted to a few simple cases that can hardly be applied to more realistic geometries, material properties, boundary conditions or loading.

Searching for approximate solutions is the other approach, followed, for instance, by Grossi and Bhat [21]. Auciello and Nolè [18] used the Rayleigh quotient, while Sato [8] chose the Ritz method. Sakiyama [22] solved numerically an integral equation that replaced the differential equation of transverse motion of tapered beams of "any class." Chen and Xie [23] and Laura et al. [24] also proposed numerical solutions for non-uniform beams.

Approximate solutions for nonlinear oscillations of tapered beams have been investigated, for example, by Karimpour et al. [25] and by Abdel-Jaber et al. [26], which considered a Galerkin-like reduction on one mode, and by Katsikadelis and Tsiatas [27], which used the analog equation method that permits to study also the transient dynamics.

In this paper, an approximate analytical solution is sought for by means of the MTS method [28], in parallel to the one obtained in [1] by the Poincaré–Lindstedt method [28] for the linear case. It is worth underlining that we consider together the effects of the non-uniformity and of the nonlinearity, by applying the MTS method directly to the partial differential equation, and not to the ordinary differential equation ensuing from a preliminar Galerkin reduction. This permits to have more reliable results, although requiring more mathematical effort.

Nayfeh [29] used the MTS method, but he was concerned with the wave propagation in cables, and not with the free vibrations of finite-length beams and cables. An asymptotic analysis was developed in [30], but the chosen smallness parameter was the thickness of the beam and not, as in the present paper, the difference with respect to a uniform continuum. The WKB and other approximate methods were used in [31] for the rod (i.e., the cable) problem, while the WKB method was used for beams in [32].

The paper is organized as follows. In Sect. 2, the model is obtained by starting from the Euler–Bernoulli kinematics hypotheses; a static condensation is used to eliminate the axial displacement, under the hypothesis that the axial inertia is negligible. In Sect. 3, we apply the MTS method, by considering the first two terms of the asymptotic expansion. The main findings are illustrated with two examples in Sect. 4, while in Sect. 5 we state our conclusions and suggestions for further developments.

2 The model

Let us consider a beam-type solid with the following nonlinear Euler–Bernoulli kinematics

$$\varepsilon_x = \varepsilon_y = \gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0, \quad \varepsilon_z = U' + \frac{1}{2}(W')^2 - YW'', \tag{1}$$

where $U(Z,T)$ and $W(Z,T)$ are the longitudinal and the transversal displacements, respectively, of the axis of the beam, and where the primes mean derivative with respect to Z. The axis is the locus of the centroids of the Z-varying cross-sections, and it is assumed to be straight and of length L. Y is the coordinate in a principal direction perpendicular to the axis of the beam; the displacements are assumed to be planar.

The elastic energy is given by

$$E_e = \frac{1}{2}\int_V \sigma_z \varepsilon_z \mathrm{d}V = \frac{1}{2}\int_V E\varepsilon_z^2 \mathrm{d}V, \tag{2}$$

where $E(X,Y,Z)$ is a variable (non-constant or constant alike) Young modulus, so that (2) becomes

$$E_e = \frac{1}{2}\int_0^L \left[EA\left(U' + \frac{1}{2}W'^2\right)^2 + EJW''^2\right]\mathrm{d}Z, \tag{3}$$

where

$$EA = (EA)(Z) = \int_A E\mathrm{d}A, \quad EJ = (EJ)(Z) = \int_A Y^2 E\mathrm{d}A. \tag{4}$$

The following boundary conditions are considered in this work:

$$\begin{aligned} U(0,T) &= 0, \quad U(L,T) = U_L(T), \\ W(0,T) &= 0, \quad W(L,T) = 0, \\ W''(0,T) &= 0, \quad W''(L,T) = 0, \end{aligned} \tag{5}$$

where $U_L(T)$ is an imposed displacement, possibly time dependent, in the axial direction.

By taking into account (5), the first variation of (3) becomes

$$\begin{aligned} \delta E_e = &-\int_0^L \left[EA\left(U' + \frac{1}{2}W'^2\right)\right]' \delta U \mathrm{d}Z \\ &-\int_0^L \left[EA\left(U' + \frac{1}{2}W'^2\right)W'\right]' \delta W \mathrm{d}Z + \int_0^L (EJW'')''\delta W \mathrm{d}Z. \end{aligned} \tag{6}$$

The kinetic energy is given by

$$E_k = \frac{1}{2}\int_V \rho[\dot{U}^2 + \dot{W}^2]dV = \frac{1}{2}\int_0^L \rho A[\dot{U}^2 + \dot{W}^2]\mathrm{d}Z, \tag{7}$$

where the dots mean derivative with respect to T, $\rho(X,Y,Z)$ is a variable (non-constant or constant alike) density, and

$$\rho A = (\rho A)(Z) = \int_A \rho \mathrm{d}A. \tag{8}$$

The expressions (4) and (8) allow to consider beams made of composite as well as of any anisotropic materials.

The virtual work done by the external loads is

$$\delta E_p = \int_0^L (Q\delta U + P\delta W)\mathrm{d}Z, \tag{9}$$

where $Q(Z,T)$ and $P(Z,T)$ are the applied loads per unit length of the beam axis in the longitudinal and the transversal direction, respectively.

The equations of motion can be obtained by the extended Hamilton's principle assuming holonomic constraints

$$\int_{T_1}^{T_2} (\delta E_k - \delta E_e + \delta E_p)\mathrm{d}T = 0, \tag{10}$$

which, by assuming

$$\begin{aligned} \delta U(Z, T_1) = 0, \quad \delta U(Z, T_2) = 0, \\ \delta W(Z, T_1) = 0, \quad \delta W(Z, T_2) = 0, \end{aligned} \tag{11}$$

provides

$$\begin{cases} \rho A\ddot{U} - \left[EA\left(U' + \frac{1}{2}W'^2\right)\right]' - Q = 0, \\ \rho A\ddot{W} - \left[EA\left(U' + \frac{1}{2}W'^2\right)W'\right]' + (EJW'')'' - P = 0. \end{cases} \tag{12}$$

As is usually done, the axial inertia $\rho A\ddot{U}$ is neglected, so that (12) gives

$$\begin{cases} EA\left(U' + \frac{1}{2}W'^2\right) = C_1 + F, \\ (EJW'')'' - [(C_1 + F)W']' + \rho A\ddot{W} - P = 0, \end{cases} \tag{13}$$

where

$$F = F(Z, T) = -\int_0^Z Q(\zeta, T)\mathrm{d}\zeta. \tag{14}$$

It is worth to remark that F is *known*.

Upon integrating $(13)_1$ and considering the boundary conditions $(5)_1$, we have

$$C_1 = C_2 + C_3 \int_0^L W'^2 \mathrm{d}Z, \tag{15}$$

where the C_2 and C_3 are given by

$$C_2(T) = \frac{U_L(T) - \int_0^L \frac{F(Z,T)}{EA(Z)}\mathrm{d}Z}{\int_0^L \frac{\mathrm{d}Z}{EA(Z)}}, \quad C_3 = \frac{1}{2\int_0^L \frac{\mathrm{d}Z}{EA(Z)}}. \tag{16}$$

Remark The axial force in the beam is

$$\begin{aligned} N(Z, T) &= \int_A \sigma_z \mathrm{d}A = \int_A E\varepsilon_z \mathrm{d}A = EA\left(U' + \frac{1}{2}W'^2\right) \\ &= C_1 + F = C_2 + C_3 \int_0^L W'^2 \mathrm{d}Z + F. \end{aligned} \tag{17}$$

By computing this expression for $Z = L$ and by rearranging, we obtain

$$\begin{aligned} U_L(T) = N_L(T) \int_0^L \frac{\mathrm{d}Z}{EA(Z)} + \int_0^L \frac{F(Z, T)}{EA(Z)}\mathrm{d}Z \\ -\frac{1}{2}\int_0^L W'^2 \mathrm{d}Z + \left(\int_0^L \frac{\mathrm{d}Z}{EA(Z)}\right)\left(\int_0^L Q(Z, T)\mathrm{d}Z\right). \end{aligned} \tag{18}$$

This expression can be used to determine $U_L(T)$ if a known force $N_L(T)$ is given at the boundary instead of $U_L(T)$. Note that in this case, we have $C_1 = N_L(T) + \int_0^L Q(Z, T)\mathrm{d}Z$, and the nonlinear effect due to the stretching of the beam disappears in this model.

The difference between movable and immovable ends was studied in [33], where it was highlighted that, when the stretching disappears, the nonlinear curvature remains the unique source of nonlinearity and thus cannot be omitted. But in our case, due to the considered boundary conditions, the stretching is present. □

Using (15), the condensed governing equation is finally obtained

$$(EJW'')'' - (FW')' - \left(C_2 + C_3 \int_0^L W'^2 \mathrm{d}Z\right) W'' + \rho A \ddot{W} - P = 0. \tag{19}$$

It is convenient to cast (19), in dimensionless form. Let EJ_0, F_0 and ρA_0 be the reference bending stiffness, axial force and unit mass, respectively. They can be, for example, the values of the corresponding functions at $Z = 0$, but other choices can be made as well. Let $EJ(Z) = EJ_0 + \widetilde{EJ}(Z)$, $F(Z,T) = F_0 + \widetilde{F}(Z,T)$ and $\rho A(Z) = \rho A_0 + \widetilde{\rho A}(Z)$. By introducing the dimensionless space variable, defined by $Z = zL$, the dimensionless time, defined by $T = tL^2\sqrt{\frac{\rho A_0}{EJ_0}}$ and the dimensionless displacement, defined by $W = wL$, Eq. (19) can be written in the form

$$\begin{aligned}&\{[1+f_1]w''\}'' - \{[\alpha + f_2]w'\}' - c_3 \left(\int_0^1 w'^2 \mathrm{d}z\right) w'' \\ &+[1+f_3]\ddot{w} + 2c_4\dot{w} - p = 0,\end{aligned} \tag{20}$$

where

$$\alpha(t) = \frac{F_0 L^2}{EJ_0} + \frac{C_2(t)L^2}{EJ_0}, \quad c_3 = \frac{C_3 L^3}{EJ_0} = \frac{L^2}{2\,EJ_0 \int_0^1 \frac{\mathrm{d}z}{EA(z)}}, \quad p(z,t) = \frac{P(z,t)L^3}{EJ_0} \tag{21}$$

and where

$$\begin{aligned} f_1(z) &= \frac{\widetilde{EJ}(z)}{EJ_0} = \frac{EJ(z) - EJ_0}{EJ_0}, \\ f_2(z,t) &= \frac{\widetilde{F}(z,t)L^2}{EJ_0} = \frac{[F(z,t) - F_0]L^2}{EJ_0}, \\ f_3(z) &= \frac{\widetilde{\rho A}(z)}{\rho A_0} = \frac{\rho A(z) - \rho A_0}{\rho A_0}, \end{aligned} \tag{22}$$

are the dimensionless varying parts of the bending stiffness, of the axial force and of the mass per unit length. A damping term $2c_4\dot{w}(z,t)$ has been introduced to take into account the dissipation which is always present in real structures.

The unforced ($p = 0$) undamped ($c_4 = 0$) linear version of (20) has been studied by the Poincaré–Lindstedt asymptotic development method in [1].

3 The MTS solution

In this section, we study the nonlinear oscillations of (20) by the multiple time scale asymptotic expansion method. With this goal, a small book-keeping parameter ε is introduced and Eq. (20) becomes

$$\begin{aligned}&\{[1+\varepsilon f_1]w''\}'' - \{[\alpha + \varepsilon f_2]w'\}' - \varepsilon c_3 \left(\int_0^1 w'^2 \mathrm{d}z\right) w'' \\ &+[1+\varepsilon f_3]\ddot{w} + \varepsilon 2c_4\dot{w} - \varepsilon p = 0,\end{aligned} \tag{23}$$

which reduces to (20) for $\varepsilon = 1$. Using a perturbation method for solving differential equations with weakly variable coefficients, like (23), has been previously done both in mathematical and in engineering literature, see for example [34].

We also assume that the dimensionless transversal load has the form of a harmonic excitation, namely

$$p(z,t) = p_0(z)\sin(\Omega t). \tag{24}$$

Furthermore, we assume that f_2 and C_2 (and therefore α) do not depend on time. Since we are interested in analyzing the response of the system near resonance, we introduce a detuning parameter σ so that $\Omega = \Omega_0 + \varepsilon\sigma$, where Ω_0 is a natural frequency (to be determined later), and σ measures the frequency shift of the excitation with respect to the natural frequency Ω_0. Equation (24) then becomes

$$p(z,t) = p_0(z)[\sin(\Omega_0 t_0)\cos(\sigma t_1) + \cos(\Omega_0 t_0)\sin(\sigma t_1)], \tag{25}$$

where $t_i = \varepsilon^i t$, $i \in \mathbb{N}$, are the various time scales.

As prescribed by the MTS method, the solution is sought in the form

$$w(z,t) = w_0(z,t_0,t_1) + \varepsilon w_1(z,t_0,t_1) + \cdots \tag{26}$$

Before we proceed, a comment on the choice of the smallness parameters is in order. Actually, in our problem, the unique smallness parameter that appears "naturally" is the one taking into account the non-uniformity. The smallness of damping and forcing is straightforward and commonly used.

The main hypothesis, indeed, is that of assuming the nonlinear term to be small. We do this because we wish to start with a ε^0 term in the expansion (26), i.e., the zero-order solution is *not* necessarily small. We could have assumed that the nonlinear term is not small, but in this case we had to start with ε^1 term in (26), i.e., we had to *assume* that the oscillation amplitude is small, otherwise the first-order equation would not be linear, which is unpleasant.

The second strong hypothesis is that we use the *same* smallness parameter for the non-uniformity and for nonlinearity, while we could have used two different parameters. This would need a two-term expansion that could possibly lead to different results, and which would be far too much involved in our opinion. From a mechanical point of view, we are assuming that non-uniformity and nonlinearity have the same order of (small) magnitude, and we are aware that different conclusions could possibly be reached should different orders of smallness parameters be adopted. This is left for future works.

3.1 Zero-order solution

The zero-order equation of motion is

$$\frac{\partial^4 w_0}{\partial z^4} - \alpha\frac{\partial^2 w_0}{\partial z^2} + \frac{\partial^2 w_0}{\partial t_0^2} = 0, \tag{27}$$

and the periodic zero-order solution satisfying the boundary conditions is given by

$$w_0(z,t_0,t_1) = \sin(kz)[w_{0s}(t_1)\sin(\Omega_0 t_0) + w_{0c}(t_1)\cos(\Omega_0 t_0)], \tag{28}$$

where

$$k = n\pi, \quad \Omega_0 = k\sqrt{\alpha + k^2}. \tag{29}$$

3.2 First-order solution

The first-order equation of motion is

$$\frac{\partial^4 w_1}{\partial z^4} - \alpha\frac{\partial^2 w_1}{\partial z^2} + \frac{\partial^2 w_1}{\partial t_0^2} + g = 0, \tag{30}$$

where

$$g = \sin(kz)\left\{\sin(\Omega_0 t_0)\left[f_2 k^2 w_{0s} + \frac{3c_3k^4}{8} w_{0s}(w_{0s}^2 + w_{0c}^2) - f_1'' k^2 w_{0s} - 2\frac{\partial w_{0c}}{\partial t_1}\Omega_0 - f_3 w_{0s}\Omega_0^2 - 2c_4 w_{0c}\Omega_0 + f_1 k^4 w_{0s}\right] + \cos(\Omega_0 t_0)\left[f_2 k^2 w_{0c} + \frac{3c_3k^4}{8} w_{0c}(w_{0s}^2 + w_{0c}^2) - f_1'' k^2 w_{0c} + 2\frac{\partial w_{0s}}{\partial t_1}\Omega_0 - f_3 w_{0c}\Omega_0^2 + 2c_4 w_{0s}\Omega_0 + f_1 k^4 w_{0c}\right] + \sin(3\Omega_0 t_0)\left[-\frac{c_3k^4}{8} w_{0s}(w_{0s}^2 - 3w_{0c}^2)\right] + \cos(3\Omega_0 t_0)\left[-\frac{c_3k^4}{8} w_{0c}(3w_{0s}^2 - w_{0c}^2)\right]\right\} + \cos(kz)\{-\sin(\Omega_0 t_0)[f_2' k + 2 f_1' k^3] w_{0s} - \cos(\Omega_0 t_0)[f_2' k + 2 f_1' k^3] w_{0c}\}. \tag{31}$$

Accordingly, the solution w_1 is sought in the form

$$w_1(z, t_0, t_1) = \sin(\Omega_0 t_0) w_{11s}(z, t_1) + \cos(\Omega_0 t_0) w_{11c}(z, t_1) + \sin(3\Omega_0 t_0) w_{13s}(z, t_1) + \cos(3\Omega_0 t_0) w_{13c}(z, t_1). \tag{32}$$

After some algebra, the solvability conditions for (30) are:

$$\frac{dw_{0c}}{dt_1} + c_4\, w_{0c} - \Omega_1 w_{0s} - d\, w_{0s}(w_{0c}^2 + w_{0s}^2) + l\cos(\sigma t_1) = 0, \tag{33}$$

$$\frac{dw_{0s}}{dt_1} + c_4\, w_{0s} + \Omega_1 w_{0c} + d\, w_{0c}(w_{0c}^2 + w_{0s}^2) - l\sin(\sigma t_1) = 0, \tag{34}$$

where

$$d = \frac{3c_3k^4}{16\Omega_0}, \tag{35}$$

$$l = \frac{1}{\Omega_0}\int_0^1 [\sin(kz) p_0(z)]\mathrm{d}z,$$

$$\Omega_1 = \frac{1}{\Omega_0}\left\{-\Omega_0^2 \int_0^1 [\sin(kz)^2 f_3(z)]\mathrm{d}z + k^4 \int_0^1 [\sin(kz)^2 f_1(z)]\mathrm{d}z + k^2 \int_0^1 [\cos(kz)^2 f_2(z)]\mathrm{d}z\right\}. \tag{36}$$

Note that, as expected, only the projection of the load $p_0(z)$ along the (first order) modal shape $\sin(kz)$ appears in the equations.

To proceed further, we look for solutions of Eqs. (33) and (34) that correspond to steady-state oscillations. To this aim, we first express w_{0c} and w_{0s} in polar coordinates,

$$w_{0c}(t_1) = \Gamma(t_1)\,\cos\beta(t_1) \qquad w_{0s}(t_1) = \Gamma(t_1)\,\sin\beta(t_1), \tag{37}$$

and then substitute these expressions into (33) and (34), thus obtaining

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t_1}\cos\beta - \Gamma\frac{\mathrm{d}\beta}{\mathrm{d}t_1}\sin\beta + c_4\,\Gamma\cos\beta - \Omega_1\Gamma\sin\beta - d\,\Gamma^3\sin\beta = -l\cos(\sigma t_1),$$

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t_1}\sin\beta + \Gamma\frac{\mathrm{d}\beta}{\mathrm{d}t_1}\cos\beta + c_4\,\Gamma\sin\beta + \Omega_1\Gamma\cos\beta + \mathrm{d}\Gamma^3\cos\beta = l\sin(\sigma t_1), \tag{38}$$

where we have dropped the arguments in the functions Γ and β for brevity. The steady-state oscillations are then obtained by setting $\mathrm{d}\Gamma/\mathrm{d}t_1 = 0$, thus obtaining

$$\Gamma\left[\left(\frac{\mathrm{d}\beta}{\mathrm{d}t_1} + \Omega_1 + \mathrm{d}\ \Gamma^2\right)\sin\beta - c_4\cos\beta\right] = l\cos(\sigma t_1), \tag{39}$$

$$\Gamma\left[c_4\sin\beta + \left(\frac{\mathrm{d}\beta}{\mathrm{d}t_1} + \Omega_1 + \mathrm{d}\Gamma^2\right)\cos\beta\right] = l\sin(\sigma t_1). \tag{40}$$

Two new equations are then obtained by (i) multiplying (39) by $\cos\beta$ and (40) by $\sin\beta$ and subtracting them from each other and by (ii) squaring (39) and (40) and adding them up. This gives:

$$\begin{aligned} &\Gamma\, c_4 = -l\cos(\sigma t_1 + \beta),\\ &\Gamma^2\left[\left(\frac{\mathrm{d}\beta}{\mathrm{d}t_1} + \Omega_1 + \mathrm{d}\,\Gamma^2\right)^2 + c_4^2\right] = l^2, \end{aligned} \tag{41}$$

Note that the condition

$$\left|\frac{\Gamma\, c_4}{l}\right| \le 1 \tag{42}$$

must be fulfilled for the existence of steady-state oscillations. This introduces an upper bound on Γ, $\Gamma_{\max} = l/c_4$, and results in closed resonance curves. The left-hand side of $(41)_1$ is constant with respect to time, so the right-hand side must also be constant, namely $\sigma t_1 + \beta = \theta_0 =$ constant and $\mathrm{d}\beta/\mathrm{d}t_1 = -\sigma$. Substituting into $(41)_2$, we then have

$$\sigma = \sigma(\Gamma) = \Omega_1 + \frac{3c_3k^4}{16\Omega_0}\Gamma^2 \pm \sqrt{\frac{l^2}{\Gamma^2} - c_4^2}, \tag{43}$$

whose solutions exist if and only if (42) is satisfied. The inverse, $\Gamma(\sigma)$, of the previous function $\sigma(\Gamma)$ gives the nonlinear resonance response curves.

The stability of the previous solution can be studied as done in Sect. 4.1 of [35].

For better visualization, we rescale the curves of $\Gamma(\sigma)$. By defining $\sigma = d_1 s$ and $\Gamma = (l/d_1)\gamma$, where

$$d_1 = \frac{1}{4}\sqrt[3]{\frac{12l^2k^4c_3}{\Omega_0}}, \tag{44}$$

equation (43) simplifies to

$$s = \frac{\Omega_1}{d_1} + \gamma^2 \pm \sqrt{\frac{1}{\gamma^2} - \left(\frac{c_4}{d_1}\right)^2}. \tag{45}$$

The effects of the parameters Ω_1/d_1 and c_4/d_1 on the function $\gamma(s)$ are described in Fig. 1.

From Eq. (28) we have, up to the first order and after some algebra,

$$w(z,t) = \sin(kz)\Gamma(\sigma)\cos(\Omega t - \theta_0), \tag{46}$$

which shows that the first-order nonlinear oscillation has the same frequency Ω of the excitation, but a frequency-dependent amplitude $\Gamma(\sigma) = \Gamma(\Omega - \Omega_0)$ given by the inverse of (43).

In the free vibration case ($c_4 = 0$ and $l = 0$), instead, Ω is unknown, and we get that the amplitude-dependent nonlinear frequency is given by

$$\Omega_{nl} = \Omega_0 + \varepsilon\Omega_1 + \varepsilon\frac{3c_3k^4}{16\Omega_0}\Gamma^2. \tag{47}$$

After appropriate rescaling, these are the dashed curves reported in Fig. 1, and they are called "backbone" curves. The fact that the coefficient of Γ^2 is positive means that we have a hardening system, namely that the natural frequency increases by increasing the amplitude of the oscillation.

Equation (47) clearly shows how the term Ω_1 measures the *linear* shift of the natural frequency due to the non-uniformity of the beam (see Fig. 1). It is the *same* value obtained, with a different technique and with a different notation, in [1].

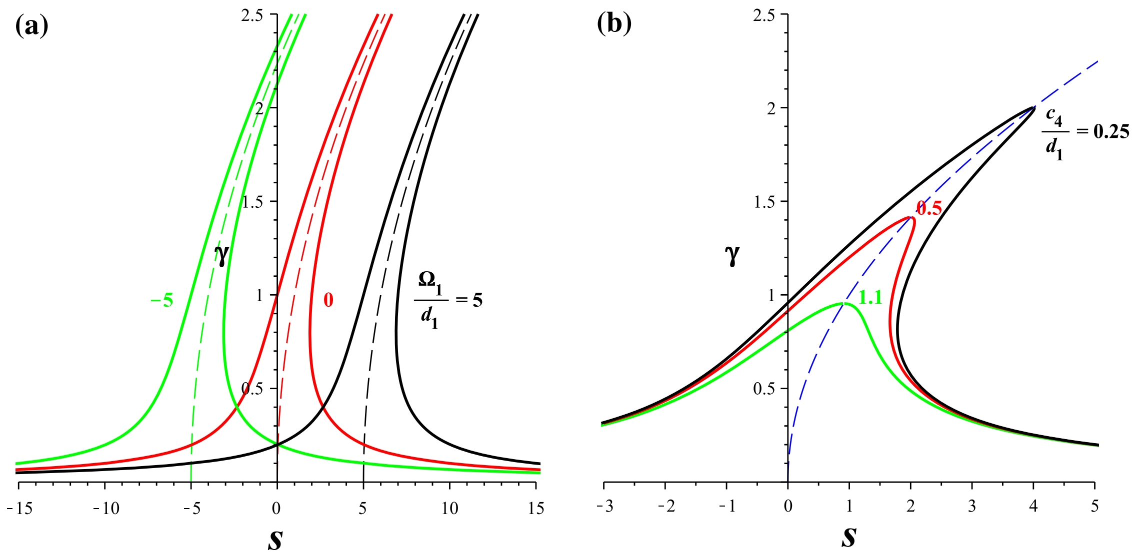

Fig. 1 Frequency response curves $\gamma(s)$ for **a** $c_4 = 0$ and **b** $\Omega_1 = 0$

It is important to underline that the non-uniformity of the beam appears also in the nonlinear parameter

$$c_3 = \frac{L^2}{2EJ_0 \int_0^1 \frac{\mathrm{d}z}{EA(z)}}. \tag{48}$$

Since it is always positive, we conclude that we can never eliminate the nonlinear effects (at least up to the first order) by properly tapering the beam. On the other hand, this can be done for the linear correction Ω_1. In fact, for $f_1(z)$, $f_2(z)$ and $f_3(z)$ satisfying

$$(\alpha + k^2) \int_0^1 [\sin(kz)^2 f_3(z)]\mathrm{d}z = k^2 \int_0^1 [\sin(kz)^2 f_1(z)]\mathrm{d}z + \int_0^1 [\cos(kz)^2 f_2(z)]\mathrm{d}z, \tag{49}$$

we have $\Omega_1 = 0$. In this case, the frequency Ω_0 is accurate up to the second order.

4 Examples

4.1 Risers

We initially consider risers, i.e., beams with a *constant* cross-section, lying in a vertical direction, so that their *constant* weight per unit length Q induces a non-constant axial load. We have $f_1 = f_3 = 0$, $F = -QZ$ and $f_2 = -L^2 (F_0 + QLz)/EJ$, which in turn implies that

$$\Omega_1 = \frac{k}{\sqrt{\alpha + k^2}} \int_0^1 [\cos(kz)^2 f_2(z)]\mathrm{d}z = -\frac{L^2}{2EJ} \frac{k}{\sqrt{\alpha + k^2}} \left(F_0 + \frac{QL}{2} \right). \tag{50}$$

From (50), we see that, if we choose $F_0 = -QL/2$, we have $\Omega_1 = 0$, namely the first-order correction vanishes and $\Omega_0 = n\pi\sqrt{\alpha + n^2\pi^2}$ is valid up to the second order, i.e., even for moderately large values of ε. With this choice of F_0, we obtain

$$\alpha = \frac{EA}{EJ} U_L L. \tag{51}$$

We now distinguish two different situations. In the first one, the riser is statically stretched in the vertical position up to reaching a given axial force N_L^{static} at the top. This provides

Table 1 Characteristics of the risers considered in [36,37]

Riser length	$L = 2{,}000\,\mathrm{m}$
Flexural stiffness	$EJ = 318.6 \times 10^6\,\mathrm{Nm}^2$
Linear weight	$Q = -3433.5\,\mathrm{N/m}$
Bottom tension	$N_0 = 0.6867 \times 10^6\,\mathrm{N}$
Virtual mass (riser mass + added mass)	$\rho A = 1{,}200\ \mathrm{kg/m}$

The cross-section is a circular tube with external diameter equal to 0.5588 m and with thickness equal to 0.0254 m

Table 2 Natural periods in (s)

Mode (n)	Ω_0	$\frac{2\pi L^2}{\Omega_0}\sqrt{\frac{\rho A}{EJ}}$	From [37]	Difference (%)
1	714.59	68.26	77.5	13.5
2	1429.59	34.12	38.7	13.4
3	2145.41	22.73	25.8	13.5
4	2862,45	17.04	19.3	13.3
5	3581.12	13.62	15.4	13.0
6	4301.83	11.34	12.8	12.9
8	5759.98	8.48	9.5	12.0
10	7213.07	6.76	7.6	12.4
15	10945.48	4.46	4.9	9.9
20	14825.74	3.29	3.54	7.6
30	23203.23	2.10	2.21	5.1
35	27777.50	1.76	1.83	4.2
40	32653.27	1.49	1.54	3.1
45	37858.78	1.28	1.32	2.4
50	43418.48	1.12	1.15	2.4

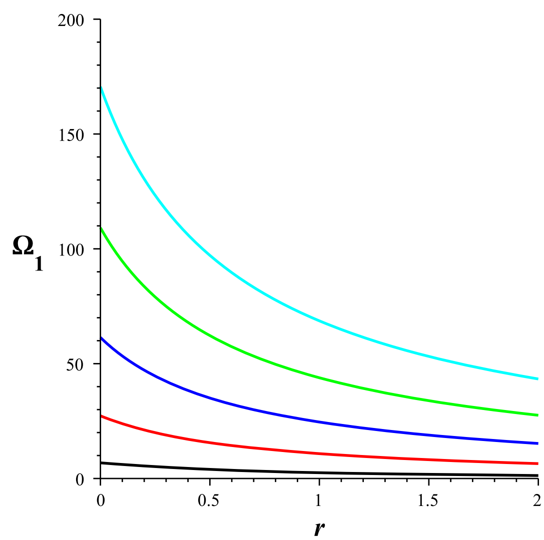

Fig. 2 The function $\Omega_1(r)$ for $n = 1$ (*lower*), 2, 3, 4, 5 (*upper*)

$$\alpha = \frac{L^2}{EJ}\left(N_L^{\text{static}} + \frac{QL}{2}\right), \tag{52}$$

to be used in $\Omega_0 = n\pi\sqrt{\alpha + n^2\pi^2}$. Then, the beam is constrained in the axial direction, and so U_L remains fixed in time while N_L in general varies during the *nonlinear* oscillations.

In the second situation, we suppose that a given, fixed, value N_L of the axial force is applied at the top, while the associated top displacement U_L is allowed to vary during the oscillations. In this case, as noted in the previous remark, the nonlinearity due to stretching disappears, and we have *linear* oscillations. Neglecting the nonlinear term in (18), we still get (52).

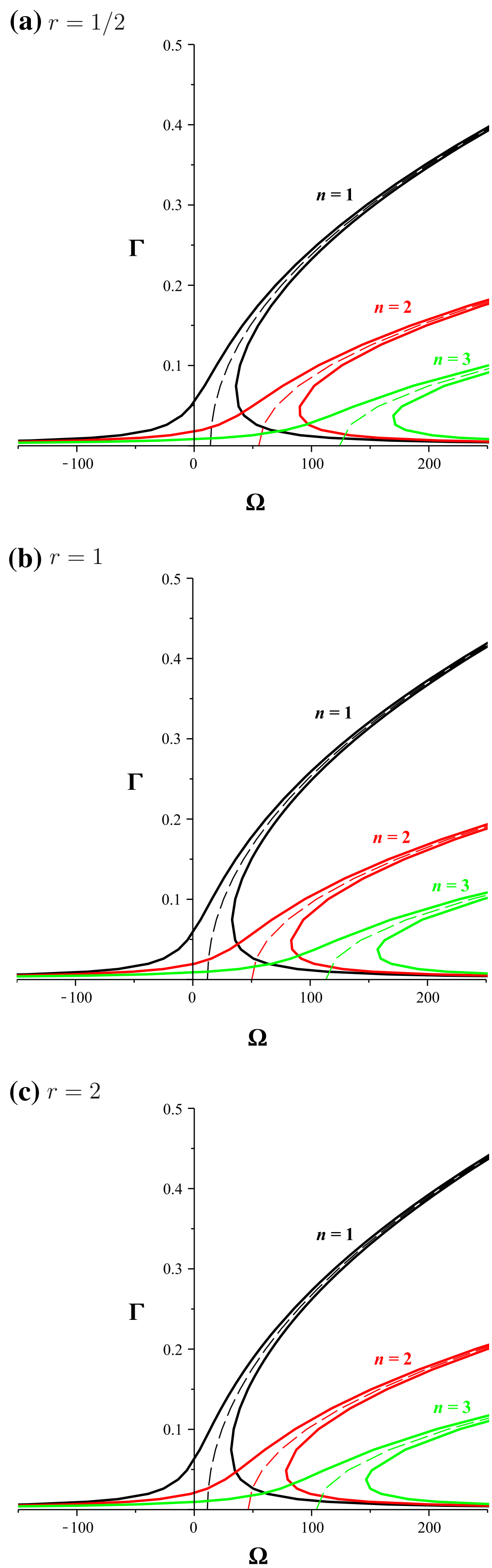

Fig. 3 Frequency response curves $\Gamma(\Omega)$ for different mode number n. $R_0 = 3\,\text{cm}$, $R_L = 4\,\text{cm}$, $L = 50\,\text{cm}$, $c_4 = 0.1$, $l = 1$, $\varepsilon = 1$. **a** $r = 1/2$, **b** $r = 1$, **c** $r = 2$

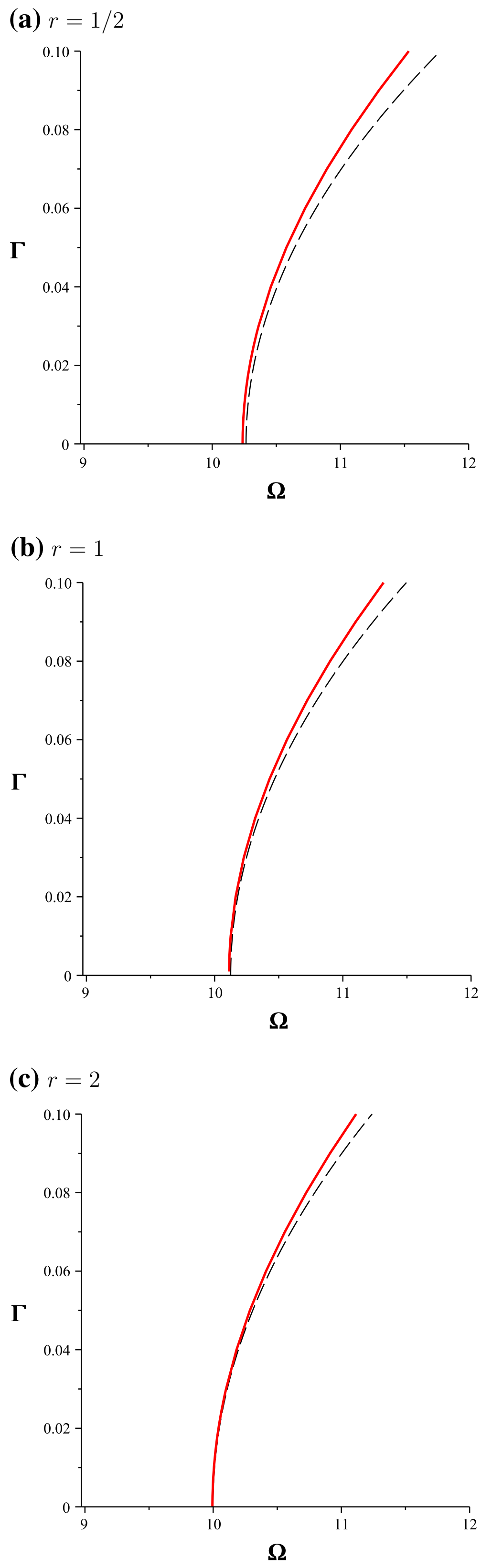

Fig. 4 Comparison of MTS (*dashed*, *thin*) and numerical (*continuous*, *thick*) backbone curves $\Gamma(\Omega)$. $n = 1$, $R_0 = 3\,\text{cm}$, $R_L = 4\,\text{cm}$, $L = 50\,\text{cm}$, $\varepsilon = 0.1$. **a** $r = 1/2$, **b** $r = 1$, **c** $r = 2$

The conclusion is that the linear frequency is the same, but in the first case the system is expected to undergo nonlinear oscillations, while in the second case only linear oscillations are expected (unless different types of nonlinearities are considered, which is *not* the case in this paper).

As a practical example, we consider the riser analyzed in [36,37], whose characteristics are reported in Table 1. Using these values in (52), we get $\alpha = 51729$, from which we compute the natural periods reported in Table 2. When compared with "exact" values taken from [37] (where also finite elements are used), we see that the difference is about 13 % for lower modes, and rapidly decreases for increasing mode number.

The error in the lower modes is due to the assumption that the longitudinal load (submerged weight) p is of order ε. Yet, in practice, p is far from being small. Even so, for preliminary analysis purposes, the possibility of computing the riser's natural periods "by hand" is very attractive. Furthermore, the accuracy greatly increases as the upper modes are considered.

The error can be reduced by considering higher-order terms in the asymptotic expansion (26).

4.2 Tapered beams

We now consider beams without axial force ($\alpha = f_2 = 0$) but with a varying cross-section. More precisely, we consider the beam with a circular cross-section whose radius has a power law tapering:

$$R(Z) = R_0 + aZ^r; \qquad a = \frac{R_L - R_0}{L^r}, \tag{53}$$

where $r \in \mathbb{R}$ is a real number governing the tapering ratio ($r = 1$ corresponds to linear tapering), and R_0 and R_L are the radius for $Z = 0$ and $Z = L$, respectively. The expression of Ω_1 can be computed in closed form. However, it involves hypergeometric functions and it is very complicated, so we do not report it here explicitly.

Assuming $EJ_0 = EJ(0) = E\pi R_0^4/4$ and $\rho A_0 = \rho A(0) = \rho\pi R_0^2$, and for $R_0 = 3\,\text{cm}$, $R_L = 4\,\text{cm}$, $L = 50\,\text{cm}$, we have that the function $\Omega_1(r)$ is reported in Fig. 2 for different values of the mode number n. This figure clearly shows that the increment of the tapering ratio reduces the linear corrections of Ω_0, according to the fact that for large values of r the beam tends to have the radius R_0 almost everywhere except for a neighborhood of $Z = L$, and so the non-uniformity of the beam tends to become negligible.

Some frequency response curves are reported in Fig. 3 for different values of the tapering ratio r and for different values of the mode number n. From these figures, we see that increasing the mode number increases the bending of the backbone curve, namely the effect of the nonlinearity. On the contrary, increasing r tends to reduce the bending of the curves, so reducing the nonlinear effects, although to a minor extent.

To check the reliability of the proposed MTS approximate solution, a comparison with a numerical solution is reported in Fig. 4. The dashed (analytical) backbone curve is obtained by (47), while the continuous (numerical) backbone curve is obtained by a Galerkin approximation $w(z,t) = \sum_{i=0}^{3} q_{2i+1}(t)\sin[(2i+1)\pi z]$ of the solution of (23). This provides a system of 4 ordinary differential equations, which has been integrated numerically by means of the Runge–Kutta method.

Figure 4 shows a very good agreement between our solution and the numerical results, even for "large" values of Γ. We note that the accuracy (very slightly) decreases from $r = 2$ to $r = 1/2$, according to the fact that decreasing r the non-uniformity becomes more and more important, as noted above.

5 Conclusions and further developments

The nonlinear oscillations of a non-uniform beam have been investigated by means of the multiple time scale method, which yields an analytical (and simple, indeed) expression for the quantities of interest. The frequency-response curves are obtained, and the frequency-dependent amplitude of the nonlinear oscillation is determined.

The results of the present work show that the non-uniformity of the beam largely influences the linear natural frequency of the beam, while it is less important for the nonlinear behavior.

The general theory has been illustrated by means of two alternative examples. In the first one, the riser, the beam has constant cross-section but varying axial load. In the second one, the tapered beam, the axial force vanishes, but the cross-section varies according to a power law.

In the present analysis, we have reported only the first-order term of the asymptotic expansion, which entails having the linear spatial mode shape only. Thus, the first development that can be conceived is that of considering also the second-order term, with the aim of both having a better approximation to the nonlinear frequency, and of having a nonlinear correction to the spatial mode shape, i.e., obtaining the nonlinear normal modes.

Another possible development consists in considering a parametric instead of an external excitation, which entails assuming a different detuning parameter. Finally, it would be interesting to investigate the case in which non-uniformity and nonlinearity have different order of smallness, a hypothesis that would require a two-term asymptotic expansion. These developments are left for future works.

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References

1. Lenci, S., Clementi, F., Mazzilli, C.E.N.: Simple formulas for the natural frequencies of non-uniform cables and beams. Int. J. Mech. Sci. **77**, 155–163 (2013)
2. Granch, E.T., Adler, A.A.: Bending vibrations of variable section beams. ASME J. Appl. Mech. **23**, 103–108 (1956)
3. Conway, H.D., Dubil, J.F.: Vibration frequencies of truncated cones and wedge beams. ASME J. Appl. Mech. **31**, 329–331 (1965)
4. Mabie, H., Rogers, C.: Transverse vibrations of double-tapered cantilever beams. J. Acoust. Soc. Am. **51**, 1771–1774 (1972)
5. Klein, L.: Transverse vibrations of nonuniform beams. J. Sound Vib. **37**, 491–505 (1974)
6. Williams, F.W., Banerjee, J.R.: Flexural vibration of axially loaded beams with linear or parabolic taper. J. Sound Vib. **99**, 121–138 (1985)
7. Taber, L.A., Viano, D.C.: Comparison of analytical and experimental results for free vibration of non-uniform composite beams. J. Sound Vib. **83**, 219–228 (1982)
8. Sato, K.: Transverse vibrations of linearly tapered beams with ends restrained elastically against rotation subjected to axial force. Int. J. Mech. Sci. **22**, 109–115 (1980)
9. Filipich, C.P., Sonenblum, M.J., Gil, E.A.: Transverse vibrations of a tapered beam embedded in a non-homogeneous Winkler foundation. Appl. Acoust. **26**, 67–72 (1989)
10. Goel, R.P.: Transverse vibrations of tapered beam. J. Sound Vib. **47**, 1–7 (1976)
11. Lee, S.Y., Ke, H.Y.: Free vibrations of a non-uniform beam with general elastically restrained boundary conditions. J. Sound Vib. **136**, 425–437 (1990)
12. Dugush, Y.A., Eisenberg, M.: Vibrations of non-uniform continuous beams under moving loads. J. Sound Vib. **254**, 911–926 (2002)
13. Raj, A., Sujith, R.I.: Closed-form solutions for the free longitudinal vibration of inhomogeneous rods. J. Sound Vib. **283**, 1015–1030 (2005)
14. Shahba, A., Attarnejad, R., Jandaghi Semnani, S., Honarvar Gheitanbaf, H.: New shape functions for non-uniform curved Timoshenko beams with arbitrarily varying curvature using basic displacement functions. Meccanica **48**, 159–174 (2013)
15. Bambill, D.V., Rossit, C.A., Rossi, R.E., Felix, D.H., Ratazzi, A.R.: Transverse free vibration of non uniform rotating Timoshenko beams with elastically clamped boundary conditions. Meccanica **48**, 1289–1311 (2013)
16. Shahba, A., Rajasekaran, S.: Free vibration and stability of tapered Euler–Bernoulli beams made of axially functionally graded materials. Appl. Math. Model **36**, 3094–3111 (2012)
17. Rajasekaran, S.: Buckling and vibration of axially functionally graded nonuniform beams using differential transformation based dynamic stiffness approach. Meccanica **48**, 1053–1070 (2013)
18. Auciello, N.M., Nolè, G.: Vibrations of a cantilever tapered beam with varying section properties and carrying a mass at the free end. J. Sound Vib. **214**, 105–119 (1998)
19. Wang, H.C.: Generalized hypergeometric function solutions on transverse vibrations of a class of nonuniform beams. ASME J. Appl. Mech. **34**, 702–708 (1968)
20. Abrate, S.: Vibration of non-uniform rods and beams. J. Sound Vib. **185**, 703–716 (1995)
21. Grossi, R.O., Bhat, R.B.: A note on vibrating tapered beams. J. Sound Vib. **147**, 174–178 (1991)
22. Sakiyama, T.: A method of analyzing the bending vibration of any type of tapered beams. J. Sound Vib. **101**, 267–270 (1985)
23. Chen, Y.Z., Xie, J.R.: Evaluation of natural frequencies of non-uniform beams by numerical integration. Comput. Struct. **29**, 693–697 (1988)
24. Laura, P.A.A., Valerga De Greco, B., Utjes, J.C., Carnicer, R.: Numerical experiments on free and forced vibrations of beams of nonuniform cross-section. J. Sound Vib. **120**, 587–596 (1988)
25. Karimpour, S., Ganji, S.S., Barari, A., Ibsen, L.B., Domairry, G.: Nonlinear vibration of an elastically restrained tapered beam. Sci. China Phys. Mech. Astron. **55**, 1925–1930 (2012)
26. Abdel-Jaber, M.S., Al-Qaisia, A.A., Abdel-Jaber, M., Beale, R.G.: Nonlinear natural frequencies of an elastically restrained tapered beam. J. Sound Vib. **313**, 772–783 (2008)
27. Katsikadelis, J.T., Tsiatas, G.C.: Non-linear dynamic analysis of beams with variable stiffness. J. Sound Vib. **270**, 847–863 (2004)
28. Nayfeh, A.H., Mook, D.T.: Nonlinear Oscillations. Wiley, New York (1979)
29. Nayfeh, A.H.: Finite-amplitude longitudinal waves in non-uniform bars. J. Sound Vib. **42**, 357–361 (1975)
30. Tarnopolskaya, T., Hoog, F.de , Fletcher, N.H., Thwaites, S.: Asymptotic analysis of the free in-plane vibrations of beams with arbitrarily varying curvature and cross-section. J. Sound Vib. **196**, 659–680 (1996)
31. ShuQi, G., ShaoPu, Y.: Free longitudinal vibrations of non-uniform rods. Sci. China Technol. Sci. **10**, 2735–2745 (2011)
32. Firouz-Abadi, R.D., Haddadpour, H., Novinzadeh, A.B.: An asymptotic solution to transverse free vibrations of variable-section beams. J. Sound Vib. **304**, 530–540 (2007)

33. Luongo, A., Rega, G., Vestroni, F.: On nonlinear dynamics of planar shear-indeformable beams. ASME J. Appl. Mech. **108**, 619–624 (1986)
34. Luongo, A.: Mode localization in dynamics and buckling of linear imperfect continuous structures. Nonlinear Dyn. **25**, 133–156 (2001)
35. Lenci, S., Warminski, J.: Free and forced nonlinear oscillations of a two-layer composite beam with interface slip. Nonlinear Dyn. **70**, 2071–2087 (2012)
36. Senjanovic, I., Ljusutina, A.M., Parunov, J.: Natural vibration analysis of tensioned risers by segmentation method. Oil Gas Sci. Technol. **61**, 647–659 (2006)
37. Sparks, C.P.: Transverse modal vibrations of vertical tensioned risers. Oil Gas Sci. Technol. **57**, 71–86 (2002)