$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x} - m\ddot{\varphi}\left[x_G \operatorname{sen}\varphi + y_G \cos\varphi\right] + \\ - m\dot{\varphi}^2\left[x_G\cos\varphi - y_G \operatorname{sen}\varphi\right] \qquad (53)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = m\ddot{y} + m\ddot{\varphi}\left[x_G\cos\varphi - y_G\operatorname{sen}\varphi\right] + \\ - m\dot{\varphi}^2\left[x_G\operatorname{sen}\varphi + y_G\cos\varphi\right] \qquad (54)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) = J_\Delta\ddot{\varphi} + m x_G\left(\ddot{y}\cos\varphi - \ddot{x}\operatorname{sen}\varphi\right) + \\ - m y_G\left(\ddot{x}\cos\varphi + \ddot{y}\operatorname{sen}\varphi\right) + \\ - m x_G\dot{\varphi}\left(\dot{y}\operatorname{sen}\varphi + \dot{x}\cos\varphi\right) + \\ + m y_G\dot{\varphi}\left(\dot{x}\operatorname{sen}\varphi - \dot{y}\cos\varphi\right) \qquad (55)$$

E as equações de movimento ficam:

em X

$$m\ddot{x} - m\ddot{\varphi}\left[x_G\operatorname{sen}\varphi + y_G\cos\varphi\right] - m\dot{\varphi}^2\left[x_G\cos\varphi - y_G\operatorname{sen}\varphi\right] = \\ = Q_x \qquad (56)$$

em Y

$$m\ddot{y} + m\ddot{\varphi}\left[x_G\cos\varphi - y_G\operatorname{sen}\varphi\right] - m\dot{\varphi}^2\left[x_G\operatorname{sen}\varphi + y_G\cos\varphi\right] = \\ = Q_y \qquad (57)$$

e, em φ,

$$J_\Delta \ddot{\varphi} - m\ddot{x}(x_G \operatorname{sen}\varphi + y_G \cos\varphi) + m\ddot{y}(x_G \cos\varphi - y_G \operatorname{sen}\varphi) +$$

$$+ m\dot{\varphi}(\dot{x} y_G - \dot{y} x_G)\operatorname{sen}\varphi - m\dot{\varphi}(\dot{x} x_G - \dot{y} y_G)\cos\varphi +$$

$$- m\dot{\varphi}(\dot{x} y_G - \dot{y} x_G)\operatorname{sen}\varphi + m\dot{\varphi}(\dot{x} x_G - \dot{y} y_G)\cos\varphi =$$

$$= Q_\varphi \tag{58}$$

então:

$$J_\Delta \ddot{\varphi} - m\ddot{x}(x_G \operatorname{sen}\varphi + y_G \cos\varphi) + m\ddot{y}(x_G \cos\varphi - y_G \operatorname{sen}\varphi) =$$

$$= Q_\varphi \tag{59}$$

Resumindo:

$$\begin{cases} m\ddot{x} - m\ddot{\varphi}\left[x_G \operatorname{sen}\varphi + y_G \cos\varphi\right] - m\dot{\varphi}^2\left[x_G \cos\varphi - y_G \operatorname{sen}\varphi\right] = \\ \qquad = Q_x \\ \\ m\ddot{y} + m\ddot{\varphi}\left[x_G \cos\varphi - y_G \operatorname{sen}\varphi\right] - m\dot{\varphi}^2\left[x_G \operatorname{sen}\varphi + y_G \cos\varphi\right] = \\ \qquad = Q_y \\ \\ J_\Delta \ddot{\varphi} - m\ddot{x}(x_G \operatorname{sen}\varphi + y_G \cos\varphi) + m\ddot{y}(x_G \cos\varphi - y_G \operatorname{sen}\varphi) = \\ \qquad = Q_\varphi \end{cases} \tag{60}$$

A equação (60) pode ser posta na forma matricial, como:

$$\mathbb{M}(\underset{\sim}{q})\,\underset{\sim}{\ddot{q}} + \underset{\sim}{G}(\underset{\sim}{q},\underset{\sim}{\dot{q}}) = \underset{\sim}{Q} \qquad (61)$$

com

$$\mathbb{M}(\underset{\sim}{q}) = \begin{bmatrix} m & 0 & -m(x_G \operatorname{sen}\psi + y_G \cos\psi) \\ & m & +m(x_G \cos\psi - y_G \operatorname{sen}\psi) \\ \underline{\underline{\text{SIMET.}}} & & J_A \end{bmatrix} \qquad (62)$$

e

$$\underset{\sim}{G}(\underset{\sim}{q},\underset{\sim}{\dot{q}}) = -m\dot{\psi}^2 \begin{bmatrix} x_G \cos\psi - y_G \operatorname{sen}\psi \\ \\ x_G \operatorname{sen}\psi + y_G \cos\psi \end{bmatrix} \qquad (63)$$

Note que

$$\underset{\sim}{Q} = \underset{\sim}{Q}(\underset{\sim}{q}, \underset{\sim}{\dot{q}}, \underset{\sim}{\ddot{q}}, t) \qquad (64)$$

que pode ser escrita

$$\underset{\sim}{Q} = \underset{\sim}{Q}^c(\underset{\sim}{q}) + \underset{\sim}{Q}^{nc}(\underset{\sim}{q},\underset{\sim}{\dot{q}},t) + \underset{\sim}{Q}^I(\underset{\sim}{\ddot{q}}) \qquad (65)$$

Energia Cinética e Massa Adicional

A energia cinética do meio fluido pode ser escrita como

$$T_f = \frac{1}{2} \dot{\underline{q}}^t \, \mathbb{M}_a \, \dot{\underline{q}} \qquad (66)$$

onde

$\mathbb{M}_a$ é a matriz de massa adicional no referencial OXYZ. No entanto, usualmente conhece-se $\hat{\mathbb{M}}_a$ no sistema Axyz, fixo ao C.R. Mas, sabemos que:

$$\mathbb{M}_a = \mathbb{B} \, \hat{\mathbb{M}}_a \, \mathbb{B}^t \qquad (67)$$

com

$$\mathbb{B} = \begin{bmatrix} \cos\psi & -\operatorname{sen}\psi & 0 \\ \operatorname{sen}\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (68)$$

Assim, conhecendo-se $\hat{\mathbb{M}}_a$ as equações de movimento podem ser devidamente modificadas.

De fato

$$\frac{d}{dt}\left(\frac{\partial T_f}{\partial \dot{q}}\right) - \frac{\partial T_f}{\partial q} = -\underset{\sim}{Q}^{I} \qquad (69)$$

Note-se que

$$\frac{\partial T_f}{\partial q_j} = \frac{1}{2}\,\dot{\underset{\sim}{q}}^{t}\,\frac{\partial \mathbb{M}a}{\partial q_j}\,\dot{\underset{\sim}{q}} \qquad (70)$$

e

$$\frac{\partial T_f}{\partial \dot{\underset{\sim}{q}}} = \mathbb{M}a\,\dot{\underset{\sim}{q}} \qquad (71)$$

logo

$$\frac{d}{dt}\left(\frac{\partial T_f}{\partial \dot{\underset{\sim}{q}}}\right) = \frac{d\mathbb{M}a}{dt}\,\dot{\underset{\sim}{q}} + \mathbb{M}a\,\ddot{\underset{\sim}{q}} \qquad (72)$$

Vejamos como ficam essas equações no caso de movimento no plano

Suponha um corpo flutuante com simetria em torno de Ax. Teremos

$$\hat{\mathbb{M}}a = \begin{bmatrix} m_{xx} & 0 & 0 \\ 0 & m_{yy} & m_{y\psi} \\ 0 & m_{y\psi} & m_{\psi\psi} \end{bmatrix} \qquad (73)$$

Então:

$$\mathbb{M}_a = \mathbb{B}\,\hat{\mathbb{M}}_a\,\mathbb{B}^t = \begin{bmatrix} m_{xx}c^2 + m_{yy}s^2 & (m_{xx}-m_{yy})sc & -m_{y\psi}s \\ (m_{xx}-m_{yy})sc & m_{xx}s^2 + m_{yy}c^2 & m_{y\psi}c \\ -m_{y\psi}s & m_{y\psi}c & m_{\psi\psi} \end{bmatrix} \tag{74}$$

e as operações podem ser realizadas.

Outra forma de escrever consiste em representar o vetor $\vec{v}_A$ na base do corpo e lá determinar a energia cinética em função de $(\dot{x}, \dot{y}, \dot{\psi}) = \dot{\underset{\sim}{q}}$.

As equações de Lagrange então seguem, de forma análoga.

Então, vejamos

$$T_f = \frac{1}{2}\overset{\circ}{\underset{\sim}{q}}{}^t \mathbb{M}_a \overset{\circ}{\underset{\sim}{q}} = \frac{1}{2}\overset{\circ}{\underset{\sim}{q}}{}^t \mathbb{B}\hat{\mathbb{M}}_a \mathbb{B}^t \overset{\circ}{\underset{\sim}{q}} =$$

$$= \frac{1}{2}\begin{bmatrix}\overset{\circ}{x} & \overset{\circ}{y} & \overset{\circ}{\varphi}\end{bmatrix}\begin{bmatrix} m_{xx}c^2+m_{yy}s^2 & (m_{xx}-m_{yy})sc & -m_{y\varphi}s \\ (m_{xx}-m_{yy})sc & m_{xx}s^2+m_{yy}c^2 & m_{y\varphi}c \\ -m_{y\varphi}s & m_{y\varphi}c & m_{\varphi\varphi}\end{bmatrix}\begin{bmatrix}\overset{\circ}{x} \\ \overset{\circ}{y} \\ \overset{\circ}{\varphi}\end{bmatrix}$$

(75)

e assim prossegue-se...

Voltando agora às equações (60) podemos combiná-las de forma a projetá-las na base do C.R. De fato, com

$$(60.a)\cos\varphi + (60.b)\operatorname{sen}\varphi = F_x$$

$$-(60.a)\operatorname{sen}\varphi + (60.b)\cos\varphi = F_y$$

teremos; com $\overset{\circ\circ}{x} = \ddot{X}\cos\varphi + \ddot{Y}\operatorname{sen}\varphi$ e $\overset{\circ\circ}{y} = -\ddot{X}\operatorname{sen}\varphi + \ddot{Y}\cos\varphi$; (*)

$$\begin{cases} m\overset{\circ\circ}{x} - my_G\overset{\circ\circ}{\varphi} - mx_G\overset{\circ}{\varphi}{}^2 = Q_x \\ m\overset{\circ\circ}{y} + mx_G\overset{\circ\circ}{\varphi} - my_G\overset{\circ}{\varphi}{}^2 = Q_y \\ J_A\overset{\circ\circ}{\varphi} + mx_G\overset{\circ\circ}{y} - my_G\overset{\circ\circ}{x} = Q_\varphi \end{cases} \quad (76)$$

Com

$$Q_x = Q_x^{c} + Q_x^{nc} + Q_x^{I}$$
$$Q_y = Q_y^{c} + Q_y^{nc} + Q_y^{I} \qquad (77)$$
$$Q_\psi = Q_\psi^{c} + Q_\psi^{nc} + Q_\psi^{I}$$

Neste caso; com simetria em Ax,

$$Q_x^{I} = -m_{xx}\ddot{x}$$
$$Q_y^{I} = -m_{yy}\ddot{y} - m_{y\psi}\ddot{\psi} \qquad (78)$$
$$Q_\psi^{I} = -m_{\psi\psi}\ddot{\psi} - m_{y\psi}\ddot{y}$$

E as equações de movimento podem ser escritas:

$$\begin{cases} (m+m_{xx})\,\ddot{x} - m y_G\,\ddot{\psi} - m x_G\,\dot{\psi}^2 = Q_x^{c} + Q_x^{nc} \\ (m+m_{yy})\,\ddot{y} + (m x_G + m_{y\psi})\,\ddot{\psi} - m y_G\,\dot{\psi}^2 = Q_y^{c} + Q_y^{nc} \\ (J_A + m_{\psi\psi})\,\ddot{\psi} + (m x_G + m_{y\psi})\,\ddot{y} - m y_G\,\ddot{x} = Q_\psi^{c} + Q_\psi^{nc} \end{cases} \qquad (79)$$

ou na forma matricial:

$$\hat{\mathbb{M}}\,\ddot{\hat{\underset{\sim}{q}}} + \hat{\underset{\sim}{G}}(\hat{\underset{\sim}{q}}, \dot{\hat{\underset{\sim}{q}}}) = \hat{\underset{\sim}{Q}}^{c} + \hat{\underset{\sim}{Q}}^{nc}$$

com

$$\hat{\mathbb{M}} = \begin{bmatrix} (m+m_{xx}) & 0 & -my_G \\ 0 & (m+m_{yy}) & (mx_G+m_{y\psi}) \\ -my_G & (mx_G+m_{y\psi}) & (J_\Delta+m_{\psi\psi}) \end{bmatrix} \quad (80)$$

e

$$\hat{\underset{\sim}{G}}(\hat{\underset{\sim}{q}}, \dot{\hat{\underset{\sim}{q}}}) = \begin{bmatrix} -mx_G\dot{\psi}^2 & -my_G\dot{\psi}^2 & 0 \end{bmatrix}^t \quad (81)$$

As equações (76) podem ser transformadas novamente para a base $(\vec{I}, \vec{J}, \vec{K})$ segundo as coordenadas (X, Y, ψ). Basta fazer

$$(76.a)\cos\psi - (76.b)\,\mathrm{sen}\,\psi = Q_X$$

$$(76.a)\,\mathrm{sen}\,\psi + (76.b)\cos\psi = Q_Y$$

e usar (*), novamente, inclusive em (76.c)

Isso equivale a usar a transformação:

$$\mathbb{B}\hat{\mathbb{M}}\mathbb{B}^t\mathbb{B}\ddot{\hat{\underset{\sim}{q}}} + \mathbb{B}\hat{\underset{\sim}{G}} = \mathbb{B}\hat{\underset{\sim}{Q}}^c + \mathbb{B}\hat{\underset{\sim}{Q}}^{nc} \quad (82)$$

ou

$$\mathbb{M}\ddot{\underset{\sim}{q}} + \underset{\sim}{G}(\underset{\sim}{q}, \dot{\underset{\sim}{q}}) = \underset{\sim}{Q}^c + \underset{\sim}{Q}^{nc} \quad (83)$$

com

$$\mathbb{M} = \mathbb{B}\hat{\mathbb{M}}\mathbb{B}^t \qquad \underset{\sim}{Q}^c = \mathbb{B}\hat{\underset{\sim}{Q}}^c$$

$$\underset{\sim}{q} = \mathbb{B}\hat{\underset{\sim}{q}} \qquad \underset{\sim}{Q}^{nc} = \mathbb{B}\hat{\underset{\sim}{Q}}^{nc}$$

$$\underset{\sim}{G} = \mathbb{B}\hat{\underset{\sim}{G}}$$