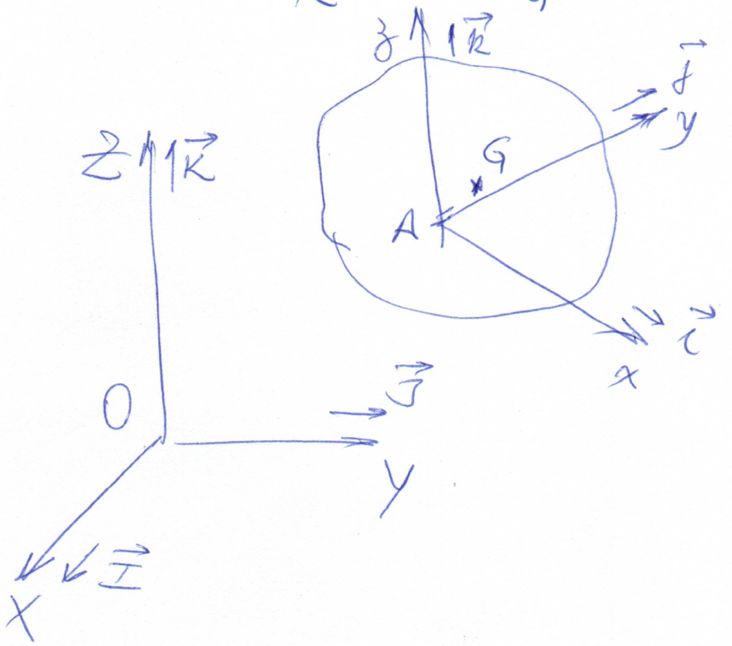
DINÂMICA DO C.R.

TMB

$$\vec{R} = m\vec{a}_G$$



$$\vec{H}_O = \int_C d\vec{H}_O = \int_C (P-O)\wedge\vec{v}\,dm \qquad (1)$$

$$\vec{v} = \vec{v}_A + \vec{\omega}\wedge(P-A) \qquad (2)$$

$$\vec{r} = (P-A)$$

Segue:

$$\vec{H}_O = (G-A)\wedge m\vec{v}_A + m(A-O)\wedge(\vec{\omega}\wedge(G-A)) + \underbrace{+\int_{CR} \vec{r}\wedge(\vec{\omega}\wedge\vec{r})\,dm}_{\mathbb{I}_A \underset{\sim}{\omega}} \qquad (3)$$

e/ou

$$\vec{H}_O = (G-A)\wedge m\vec{v}_A + \mathbb{I}_A \underset{\sim}{\omega} + (A-O)\wedge m\vec{v}_G \qquad (4)$$

Derivando-se (1) e (4) no tempo segue:

$$(G-A)\wedge m\vec{a}_A + \frac{d}{dt}(\mathbb{I}_A \underset{\sim}{\omega}) + (A-O)\wedge m\vec{a}_G = \vec{M}_O^{ext} \qquad (5)$$

Mas

$$\frac{d}{dt}\left(\mathbb{I}_A \underset{\sim}{\omega}\right) = \vec{\omega} \wedge \mathbb{I}_A \underset{\sim}{\omega} + \frac{d}{dt}\left(\mathbb{I}_A \underset{\sim}{\omega}\right)_{mov} \qquad (6)$$

mas no ref. do corpo $\mathbb{I}_A$ é invariante:

$$\frac{d}{dt}\left(\mathbb{I}_A \underset{\sim}{\omega}\right) = \vec{\omega} \wedge \mathbb{I}_A \underset{\sim}{\omega} + \mathbb{I}_A \underset{\sim}{\mathring{\omega}} \qquad (7)$$

Assim:

$$\vec{\omega} \wedge \mathbb{I}_A \underset{\sim}{\omega} + \mathbb{I}_A \underset{\sim}{\mathring{\omega}} = \underbrace{M_O^{ext} + (O-A)\wedge \vec{R}}_{\vec{M}_A^{ext}} + (A-G)\wedge m\vec{a}_A \qquad (8)$$

então:

$$\vec{\omega} \wedge \mathbb{I}_A \underset{\sim}{\omega} + \mathbb{I}_A \underset{\sim}{\mathring{\omega}} = \vec{M}_A^{ext} + (A-G)\wedge m\vec{a}_A \qquad (9)$$

se $A \equiv G$

$$\vec{\omega} \wedge \mathbb{I}_G \underset{\sim}{\omega} + \mathbb{I}_G \underset{\sim}{\mathring{\omega}} = \vec{M}_G^{ext} \qquad (10)$$

NOTAÇÃO ALTERNATIVA MATRICIAL

seja $\underset{\sim}{p} = [p_x \;\; p_y \;\; p_z]^t$, $\underset{\sim}{q} = [q_x \;\; q_y \;\; q_z]^t$

Então, definindo

$$\mathbb{P} = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix} \qquad (11)$$

podemos escrever:

$$\vec{p} \wedge \vec{q} \equiv \mathbb{P}\, \underset{\sim}{q} \qquad (12)$$

Assim:

$$\Omega\, \mathbb{I}_A\, \underset{\sim}{\omega} + \mathbb{I}_A\, \underset{\sim}{\mathring{\omega}} = \underset{\sim}{M_A} - m\, R_G\, \underset{\sim}{\vec{a}_A} \qquad (13)$$

com

$$\Omega = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \qquad (14)$$

e

$$R_G = \begin{bmatrix} 0 & -z_G & y_G \\ z_G & 0 & -x_G \\ -y_G & x_G & 0 \end{bmatrix} \qquad (15)$$

Se $A \equiv G$:

$$\Omega\, \mathbb{I}_G\, \underset{\sim}{\omega} + \mathbb{I}_G\, \underset{\sim}{\mathring{\omega}} = \underset{\sim}{M_G} \qquad (16)$$

ENERGIA CINÉTICA DE UM CR

$$T = \int_C \frac{1}{2} dm\, v^2 \qquad (17)$$

Para um C.R. :

$$T = \frac{1}{2} m\, \underset{\sim}{v_A}^2 + m\, \vec{v}_A \cdot \vec{\omega} \wedge (G - A) + \frac{1}{2} \underset{\sim}{\omega}^t\, \mathbb{I}_A\, \underset{\sim}{\omega} \qquad (18)$$

Se $A \equiv G$

$$T = \frac{1}{2} m\, \underset{\sim}{v_G}^2 + \frac{1}{2} \underset{\sim}{\omega}^t\, \mathbb{I}_G\, \underset{\sim}{\omega} \qquad (19)$$

Na forma matricial

$$T = \frac{1}{2} m \underset{\sim}{v_A}^t \underset{\sim}{v_A} + m \underset{\sim}{v_A}^t \mathcal{R} \underset{\sim}{r_G} + \frac{1}{2} \underset{\sim}{\omega}^t \mathbb{I}_A \underset{\sim}{\omega} \qquad (20)$$

e se $A \equiv G$

$$T = \frac{1}{2} m \underset{\sim}{v_G}^t \underset{\sim}{v_G} + \frac{1}{2} \underset{\sim}{\omega}^t \mathbb{I}_G \underset{\sim}{\omega} \qquad (21)$$

Usando a permutação cíclica do produto misto:

$$T = \frac{1}{2} m \underset{\sim}{v_A}^t \underset{\sim}{v_A} + m . \underset{\sim}{\omega}^t R_G \underset{\sim}{v_A} + \frac{1}{2} \underset{\sim}{\omega}^t . \mathbb{I}_A \underset{\sim}{\omega} =$$

$$= \frac{1}{2} m \underset{\sim}{v_A}^t \underset{\sim}{v_A} + m \underset{\sim}{r_G}^t \mathbb{V}_A \underset{\sim}{\omega} + \frac{1}{2} \underset{\sim}{\omega}^t \mathbb{I}_A \underset{\sim}{\omega} \qquad (22)$$

Note também que

$$\frac{\partial T}{\partial \underset{\sim}{\omega}} = \mathbb{I}_A \underset{\sim}{\omega} = \underset{\sim}{H_A} \qquad (23)$$

e

$$\frac{\partial T}{\partial \underset{\sim}{v_A}} = m \underset{\sim}{v_A} + m \mathcal{R} \underset{\sim}{r_G} \qquad (24)$$

Se $A \equiv G$

$$\frac{\partial T}{\partial \underset{\sim}{v_G}} = m \underset{\sim}{v_G} \qquad (25)$$

Que são os "momenta generalizados".

MECÂNICA ANALÍTICA

Equações de Euler-Lagrange

Seja $\underset{\sim}{q} = [q_j]^t \quad , j = 1, \ldots, n \qquad (26)$

o vetor de coordenadas generalizadas

Do PTV aplicado ao Princípio de D'Alembert seguem as Equações de Euler-Lagrange:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\underset{\sim}{q}}}\right) - \frac{\partial T}{\partial \underset{\sim}{q}} = \underset{\sim}{Q} \qquad (27)$$

onde

$\underset{\sim}{Q} = [Q_j]^t$ vetor de forças generalizadas

$$Q_j = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \qquad (28)$$

com $\dfrac{\partial \vec{r}_i}{\partial q_j}$: direções generalizadas

A equação (27) é válida para sistemas com vínculos holônomos

Se separamos

$$\underset{\sim}{Q} = \underset{\sim}{Q}^{c} + \underset{\sim}{Q}^{nc} \tag{29}$$

com c: forças conservativas
nc: forças não-conservativas

tal que

$$\underset{\sim}{Q}^{c} = - \frac{\partial V}{\partial \underset{\sim}{q}} \tag{30}$$

com

$$V = V(\underset{\sim}{q}, t) \tag{31}$$

a função potencial de forças, segue que

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\underset{\sim}{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \underset{\sim}{q}} = \underset{\sim}{Q}^{nc} \tag{32}$$

com

$$\mathcal{L}(\underset{\sim}{q}, \dot{\underset{\sim}{q}}, t) = T(\underset{\sim}{q}, \dot{\underset{\sim}{q}}, t) - V(\underset{\sim}{q}, t) \tag{33}$$

a função Lagrangiana do sistema.

Seja $\underset{\sim}{q} = [\xi_j]^t \quad j = 1, \ldots, n \qquad (34)$

$$\begin{array}{lll} \xi_1 = X_A & \xi_4 & \\ \xi_2 = Y_A & \xi_5 & \text{ângulos de Euler} \\ \xi_3 = Z_A & \xi_6 & \end{array}$$

A matriz de mudança de base, do CR para a base fixa é:

$$B = \begin{bmatrix} c5.c6 & c5.s5.s4 - s6.c4 & c6.s5.c4 + s6.s4 \\ c5.s6 & s5.s6.s4 + c6.c4 & s5.s6.c4 - c6.s4 \\ -s5 & s4.c5 & c5.c4 \end{bmatrix}$$

com $\cos \xi_j = c_j$

$\operatorname{sen} \xi_j = s_j \quad , \; j = 4, 5, 6 \qquad (35)$

O vetor de rotação $\underset{\sim}{\omega}$ no ref. do corpo é:

$$\underset{\sim}{\omega} = \begin{bmatrix} \dot{\xi}_4 - \dot{\xi}_6 \operatorname{sen} \xi_5 \\ \dot{\xi}_5 \cos \xi_4 - \dot{\xi}_6 \operatorname{sen} \xi_4 \operatorname{sen} \xi_5 \\ \dot{\xi}_6 \cos \xi_4 \cos \xi_5 + \dot{\xi}_5 \operatorname{sen} \xi_4 \end{bmatrix} \qquad (36)$$

Em primeira ordem, para pequenas rotações ξ_j:

$$\mathbb{B}_1 = \begin{bmatrix} 1 & -\xi_6 & \xi_5 \\ \xi_6 & 1 & -\xi_4 \\ -\xi_5 & \xi_4 & 1 \end{bmatrix} \qquad (37)$$

e

$$\underset{\sim}{\omega}_1 = \begin{bmatrix} \mathring{\xi}_4 & \mathring{\xi}_5 & \mathring{\xi}_6 \end{bmatrix}^t \qquad (38)$$

Já, para movimentos no plano OXY, tal que $\xi_4 = \xi_5 \equiv 0$, $\xi_6 = \psi$

$$\mathbb{B} = \begin{bmatrix} \cos\psi & -\operatorname{sen}\psi & 0 \\ \operatorname{sen}\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (39)$$

e

$$\underset{\sim}{\omega} = \begin{bmatrix} 0 & 0 & \mathring{\psi}_6 \end{bmatrix}^t \qquad (40)$$

De fato:

$$\begin{aligned} \vec{I} &= \cos\psi\,\vec{i} - \operatorname{sen}\psi\,\vec{j} \\ \vec{J} &= \operatorname{sen}\psi\,\vec{i} + \cos\psi\,\vec{j} \\ \vec{K} &= \vec{k} \end{aligned} \qquad (41)$$

No problema de navios-plataformas amarrados, trabalharemos com movimentos no plano horizontal.

DINÂMICA DE UM CORPO RÍGIDO FLUTUANTE NO PLANO HORIZONTAL

Neste caso, com

$$\underset{\sim}{r}_G = [\, x_G \quad y_G \quad z_G \,]^t \qquad (42)$$

e

$$\underset{\sim}{\omega} = [\, 0 \quad 0 \quad \dot{\psi} \,]^t \qquad (43)$$

Vem, de (20), que a energia cinética do corpo pode ser escrita como

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} J_A \dot{\psi}^2 + $$
$$+ m (\dot{y} \cos\psi - \dot{x} \,\mathrm{sen}\, \psi) \dot{\psi} x_G \quad + $$
$$- m (\dot{x} \cos\psi + \dot{y} \,\mathrm{sen}\, \psi) \dot{\psi} y_G \qquad (44)$$

pois,

$$\vec{\omega} \wedge \vec{r}_G = \Omega \underset{\sim}{r}_G = \dot{\psi} x_G \vec{j} - \dot{\psi} y_G \vec{i} \qquad (45)$$

e

$$\vec{v}_A = \dot{x} \vec{I} + \dot{y} \vec{J} = (\dot{x} \cos\psi + \dot{y} \,\mathrm{sen}\, \psi) \vec{i} + $$
$$+ (\dot{y} \cos\psi - \dot{x} \,\mathrm{sen}\, \psi) \vec{j} \qquad (46)$$

cf. (41).

Assim, com $\underset{\sim}{q} = [X \ Y \ \psi]^t$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\underset{\sim}{q}}}\right) - \frac{\partial T}{\partial \underset{\sim}{q}} = \underset{\sim}{Q} \qquad (47)$$

portanto:

$$\frac{\partial T}{\partial X} = \frac{\partial T}{\partial Y} = 0 \qquad (48)$$

$$\frac{\partial T}{\partial \psi} = -m\,(\dot{Y}\,\mathrm{sen}\,\psi + \dot{X}\cos\psi)\,\dot{\psi}\,x_G +$$
$$+\, m\,(\dot{X}\,\mathrm{sen}\,\psi - \dot{Y}\cos\psi)\,\dot{\psi}\,y_G$$

$$= m\,\dot{\psi}\,(\dot{X}\,y_G - \dot{Y}\,x_G)\,\mathrm{sen}\,\psi +$$
$$- m\,\dot{\psi}\,(\dot{X}\,x_G - \dot{Y}\,y_G)\cos\psi \qquad (49)$$

$$\frac{\partial T}{\partial \dot{X}} = m\dot{X} - m\dot{\psi}\left[x_G\,\mathrm{sen}\,\psi + y_G\cos\psi\right] \qquad (50)$$

$$\frac{\partial T}{\partial \dot{Y}} = m\dot{Y} + m\dot{\psi}\left[x_G\cos\psi - y_G\,\mathrm{sen}\,\psi\right] \qquad (51)$$

$$\frac{\partial T}{\partial \dot{\psi}} = J_\Delta\,\dot{\psi} + m\,x_G\,(\dot{Y}\cos\psi - \dot{X}\,\mathrm{sen}\,\psi) +$$
$$- m\,y_G\,(\dot{X}\cos\psi + \dot{Y}\,\mathrm{sen}\,\psi) \qquad (52)$$

e então,