Presented at the 34th Annual Convention of the American Institute of Electrical Engineers, Atlantic City, N. J., June 28, 1918.

METHOD OF SYMMETRICAL CO-ORDINATES APPLIED TO THE SOLUTION OF POLYPHASE NETWORKS

BY C. L. FORTESCUE

ABSTRACT OF PAPER

In the introduction a general discussion of unsymmetrical systems of co-planar vectors leads to the conclusion that they may be represented by symmetrical systems of the same number of vectors, the number of symmetrical systems required to define the given system being equal to its degrees of freedom. A few trigonometrical theorems which are to be used in the paper are called to mind. The paper is subdivided into three parts, an abstract of which follows. It is recommended that only that part of Part I up to formula (33) and the portion dealing with star-delta transformations be read before proceeding with Part II.

Part I deals with the resolution of unsymmetrical groups of numbers into symmetrical groups. These numbers may represent rotating vectors of systems of operators. A new operator termed the sequence operator is introduced which simplifies the manipulation. Formulas are derived for three-phase circuits. Star-delta transformations for symmetrical co-ordinates are given and expressions for power deduced. A short discussion of harmonics in three-phase systems is given.

Part II deals with the practical application of this method to symmetrical rotating machines operating on unsymmetrical circuits. General formulas are derived and such special cases, as the single-phase induction motor, synchronous motor-generator, phase converters of various types, are discussed.

INTRODUCTION

IN THE latter part of 1913 the writer had occasion to investigate mathematically the operation of induction motors under unbalanced conditions. The work was first carried out, having particularly in mind the determination of the operating characteristics of phase converters which may be considered as a particular case of unbalanced motor operation, but the scope of the subject broadened out very quickly and the writer undertook this paper in the belief that the subject would be of interest to many.

The most striking thing about the results obtained was their symmetry; the solution always reduced to the sum of two or more symmetrical solutions. The writer was then led to inquire if there were no general principles by which the solution of unbalanced polyphase systems could be reduced to the solution of two or more balanced cases. The present paper is an endeavor to present a general method of solving polyphase network which has peculiar advantages when applied to the type of polyphase networks which include rotating machines.

In physical investigations success depends often on a happy choice of co-ordinates. An electrical network being a dynamic system should also be aided by the selection of a suitable system of co-ordinates. The co-ordinates of a system are quantities which when given, completely define the system. Thus a system of three co-planar concurrent vectors are defined when their magnitude and their angular position with respect to some fixed direction are given. Such a system may be said to have six degrees of freedom, for each vector may vary in magnitude and phase position without regard to the others. If, however, we impose the condition that the vector sum of these vectors shall be zero, we find that with the direction of one vector given, the other two vectors are completely defined when their magnitude alone is given, the system has therefore lost two degrees of freedom by imposing the above condition which in dynamical theory is termed a "constraint". If we impose a further condition that the vectors be symmetrically disposed about their common origin this system will now have but two degrees of freedom.

It is evident from the above definition that a system of n coplanar concurrent vectors may have $2n$ degrees of freedom and that a system of n symmetrically spaced vectors of equal magnitude has but two degrees of freedom. It should be possible then by a simple transformation to define the system of n arbitrary congruent vectors by n other systems of concurrent vectors which are symmetrical and have a common point. The n symmetr cal systems so obtained are the symmetrical co-ordinates of the given system of vectors and completely define it.

This method of representing polyphase systems has been employed in the past to a limited extent, but up to the present time there has been as far as the author is aware no systematic presentation of the method. The writer hopes by this paper to interest others in the application of the method, which will be found to be a valuable instrument for the solution of certain classes of polyphase networks.

In dealing with alternating currents in this paper, use is made of the complex variable which in its most general form may be represented as a vector of variable length rotating about a given point at variable angular velocity or better as the resultant of a number of vectors each of constant length rotating at different angular velocities in the same direction about a given point. This vector is represented in the text by $\check{I}$, $\check{E}$, etc., and the conjugate vector which rotates at the same speed in the opposite direction is represented by $\hat{I}$, $\hat{E}$, etc. The effective value of the vector is represented by the symbol without the distinguishing mark as I, E, etc. The impedances Z_a, Z_b, Z_{ab}, etc., are generalized expressions for the self and mutual impedances. For a circuit A the self-impedance operator will be denoted by Z_{aa} or Z_a. In the case of two circuits A and B the self impedance operators would be $Z_{aa}\, Z_{bb}$ and the mutual impedance operator Z_{ab}. The subletters denote the circuits to which the operators apply. These operators are generally functions of the operator, $D = \dfrac{d}{dt}$ and the characteristics of the circuit; these characteristics are constants only when there is no physical motion. It will therefore be necessary to carefully distinguish between $Z_a\, \check{I}_a$ and $\check{I}_a\, Z_a$ when Z_a has the form of a differential operator. In the first case a differential operation is carried out on the time variable $\check{I}_a$ in the second case the differential operator is merely multiplied by $\check{I}_a$.

The most general expression for a simple harmonic quantity e is

$$e = A \cos pt - B \sin pt$$

in exponential form this becomes

$$e = \frac{A + j\,B}{2}\,\epsilon^{j\,pt} + \frac{A - j\,B}{2}\,\epsilon^{-j\,pt}$$

$(A + j\,B)\,\epsilon^{j\,pt}$ represents a vector of length $\sqrt{A^2 + B^2}$ rotating in the positive direction with angular velocity p while $(A - j\,B)$ $\epsilon^{-j\,pt}$ is the conjugate vector rotating at the same angular velocity in the opposite direction. Since $\epsilon^{j\,pt}$ is equal to $\cos pt + j \sin pt$, the positively rotating vector $\check{E} = (A + j\,B)\,\epsilon^{j\,pt}$ will be

$$\check{E} = A \cos pt - B \sin pt + j\,(A \sin pt + B \cos pt)$$

or the real part of $\check{E}$ which is its projection on a given axis is equal to e and therefore $\check{E}$ may be taken to represent e in phase and magnitude. It should be noted that the conjugate vector $\hat{E}$ is equally available, but it is not so convenient since the operation $\frac{d}{dt}\,\epsilon^{-j\,pt}$ gives $-j\,p\,\epsilon^{-j\,pt}$ and the imaginary part of the impedance operator will have a negative sign.

The complex roots of unity will be referred to from time to time in the paper. Thus the complete solution of the equation $x^n - 1 = 0$ requires n different values of x, only one of which is real when n is an odd integer. To obtain the other roots we have the relation

$$\begin{aligned} 1 &= \cos 2\,\pi\,r + j \sin 2\,\pi\,r \\ &= \epsilon^{j\,2\,\pi\,r} \end{aligned}$$

Where r is any integer. We have therefore

$$1^{\frac{1}{n}} = \epsilon^{j\frac{2\,\pi\,r}{n}}$$

and by giving successive integral values to r from 1 to n, all the n roots of $x^n - 1 = 0$ are obtained namely,

$$\begin{aligned} a_1 &= \epsilon^{j\frac{2\,\pi}{n}} = \cos\frac{2\,\pi}{n} + j\sin\frac{2\,\pi}{n} \\ a_2 &= \epsilon^{j\frac{4\,\pi}{n}} = \cos\frac{4\,\pi}{n} + j\sin\frac{4\,\pi}{n} \\ a_3 &= \epsilon^{j\frac{6\,\pi}{n}} = \cos\frac{6\,\pi}{n} + j\sin\frac{6\,\pi}{n} \\ a_n &= \epsilon^{j\,2\,\pi} = 1 \end{aligned}$$

It will be observed that $a_2\,a_3 \ldots a_n$ are respectively equal to $a_1^2\,a_1^3 \ldots a_1^{(n-1)}$.

When there is relative motion between the different parts of a circuit as for example in rotating machinery, the mutual inductances enter into the equation as time variables and when the motion is angular the quantities ϵ^{jwt} and ϵ^{-jwt} will appear in the operators. In this case we do not reject the portion of the operator having ϵ^{-jwt} as a factor, because the equations require that each vector shall be operated on by the operator as a whole which when it takes the form of a harmonic time function will contain terms with ϵ^{jwt} and ϵ^{-jwt} in conjugate relation. In some cases as a result of this, solutions will appear with indices of ϵ which are negative time variables; in such cases in the final statement the vectors with negative index should be replaced by their conjugates which rotate in the positive direction.

This paper is subdivided as follows:

Part I.—"The Method of Symmetrical Co-ordinates." Deals with the theory of the method, and its application to simple polyphase circuits.

Part II.—Application to Symmetrical Machines on Unbalanced Polyphase Circuits. Takes up Induction Motors, Generator and Synchronous Motor, Phase Balancers and Phase Convertors.

Part III. Application to Machines having Unsymmetrical Windings.

In the Appendix the mathematical representation of field forms and the derivation of the constants of different forms of networks is taken up.

The portions of Part I dealing with unsymmetrical windings are not required for the applications taken up in Part II and may be deferred to a later reading. The greater part of Part I is taken up in deriving formulas for special cases from the general formulas (**30**) and (**33**), and the reading of the text following these equations may be confined to the special cases of immediate interest.

I wish to express my appreciation of the valuable help and suggestions that have been given me in the preparation of this paper by Prof. Karapetoff who suggested that the subject be presented in a mathematical paper and by Dr. J. Slepian to whom I am indebted for the idea of sequence operators and by others who have been interested in the paper.

PART I

Method of Symmetrical Generalized Co-ordinates

Resolution of Unbalanced Systems of Vectors and Operators

The complex time function $\check{E}$ may be used instead of the harmonic time function e in any equation algebraic or differential in which it appears linearly. The reason of this is because if any linear operation is performed on $\check{E}$ the same operation performed on its conjugate $\hat{E}$ will give a result which is conjugate to that obtained from $\check{E}$, and the sum of the two results obtained is a solution of the same operation performed on $\check{E} + \hat{E}$, or $2\,e$.

It is customary to interpret $\check{E}$ and $\hat{E}$ as coplanar vectors, rotating about a common point and e as the projection of either vector on a given line, $\check{E}$ being a positively rotating vector and

$\hat{E}$ being a negatively rotating vector, and their projection on the given line being

$$e = \frac{\check{E} + \hat{E}}{2} \tag{1}$$

Obviously if this interpretation is accepted one of the two vectors becomes superfluous and the positively rotating vector $\check{E}$ may be taken to represent the variable "e" and we may define "e" by saying that "e" is the projection of the vector $\check{E}$ on a given line or else by saying that "e" is the real part of the complex variable $\check{E}$.

If (1), a, $a^2 \ldots . a^{n-1}$ are the n roots of the equation $x^n - 1 = 0$ a symmetrical polyphase system of n phases may be represented by

$$\left.\begin{aligned} \check{E}_{11} &= \check{E}_{11} \\ \check{E}_{21} &= a\,\check{E}_{11} \\ \check{E}_{31} &= a^2\,\check{E}_{11} \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ \check{E}_{n1} &= a^{n-1}\,\check{E}_{11} \end{aligned}\right\} \tag{2}$$

Another n phase system may be obtained by taking

$$\left.\begin{aligned} \check{E}_{12} &= \check{E}_{12} \\ \check{E}_{22} &= a^2\,\check{E}_{12} \\ \check{E}_{32} &= a^4\,\check{E}_{12} \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ \check{E}_{n2} &= a^{2\,(n-1)}\check{E}_{12} \end{aligned}\right\} \tag{3}$$

and this also is symmetrical, although it is entirely different from (**2**).

Since $1 + a + a^2 + a^{n-1} = 0$, the sum of all the vectors of a symmetrical polyphase system is zero.

If $\check{E}_1\ \check{E}_2\ \check{E}_3 \ldots . \check{E}_n$ be a system of n vectors, the following identities may be proved by inspection:

$$
\left.\begin{aligned}
\check{E}_1 \equiv\ & \frac{\check{E}_1 + \check{E}_2 + \check{E}_3 + \ldots \check{E}_n}{n} \\
& + \frac{\check{E}_1 + a\,\check{E}_2 + a^2\,\check{E}_3 + \ldots a^{n-1}\,\check{E}_n}{n} \\
& + \frac{\check{E}_1 + a^2\,\check{E}_2 + a^4\check{E}_3 + \ldots a^{2(n-1)}\,\check{E}_n}{n} \\
& + \frac{\check{E}_1 + a^{r-1}\,\check{E}_2 + a^{2\,(r-1)}\,\check{E}_3 + \ldots a^{(n-1)(r-1)}\,\check{E}_n}{n} \\
& + \ldots \frac{\check{E}_1 + a^{-1}\,\check{E}_2 + a^{-2}\,\check{E}_3 + \ldots a^{-(n-1)}\,\check{E}_n}{n} \\
\check{E}_2 \equiv\ & \frac{\check{E}_1 + \check{E}_2 + \check{E}_3 + \ldots \check{E}_n}{n} \\
& + a^{-1}\,\frac{\check{E}_1 + a\,\check{E}_2 + a^2\,\check{E}_3 + \ldots a^{n-1}\,\check{E}_n}{n} \\
& + a^{-2}\,\frac{\check{E}_1 + a^2\,\check{E}_2 + a^4\,\check{E}_3 + \ldots a^{2(n-1)}\,\check{E}_n}{n} \\
& + a^{-(r-1)}\,\frac{\check{E}_1 + a^{r-1}\,\check{E}_2 + a^{2(r-1)}\,\check{E}_3 + a^{(n-1)(r-1)}\,\check{E}_n}{n} \\
& + a^{-(n-1)}\,\frac{\check{E}_1 + a^{-1}\,\check{E}_2 + a^{-2}\,\check{E}_3 + \ldots a^{-(n-1)}\,\check{E}_n}{n} \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\check{E}_n \equiv\ & \frac{\check{E}_1 + \check{E}_2 + \check{E}_3 + \ldots \check{E}_n}{n} \\
& + a^{-(n-1)}\,\frac{\check{E}_1 + a\,\check{E}_2 + a^2\,\check{E}_3 + \ldots a^{n-1}\,\check{E}_n}{n} \\
& + a^{-2\,(n-1)}\,\frac{\check{E}_1 + a^2\,\check{E}_2 + a^4\,\check{E}_4 + \ldots a^{2\,(n-1)}\,\check{E}_n}{n} \\
& + a^{-(n-1)(r-1)}\,\frac{\check{E}_1 + a^{r-1}\,\check{E}_2 + \ldots a^{(n-1)(r-1)}\,\check{E}_n}{n} \\
& + a^{-1}\,\frac{\check{E}_1 + a^{-1}\,\check{E}_2 + a^{-2}\,\check{E}_3 + \ldots a^{-(n-1)}\,\check{E}_n}{n}
\end{aligned}\right\} \quad (4)
$$

It will be noted that in the expression for $\check{E}_1$ in the above formulas if the first term of each component is taken the result is $n\,\dfrac{\check{E}_1}{n}$ or $\check{E}_1$. If the succeeding terms of each component involving

$\check{E}_2\,\check{E}_3\ldots\check{E}_n$ respectively, are taken separately they add up to expressions of the form $\frac{\check{E}_r}{n}\,(1+a+a^2+\ldots a^{n-1})$ which are all equal to zero since $(1+a+a^2+\ldots a^{n-1})$ is equal to zero. In like manner in the expression for $\check{E}_2\,\check{E}_3\ldots\check{E}_n$ respectively, all the terms of the components involving each of the quantities $\check{E}_1\,\check{E}_2\,\check{E}_3\ldots$ etc. excepting the terms involving that one of which the components are to be determined add up to expressions of the form $\frac{\check{E}_r}{n}$ $(1+a+a^2+\ldots a^{n-1})$ all of which are equal to zero, the remaining terms add up to $\check{E}_2\;\check{E}_3\ldots E_n$ respectively. It will now be apparent that (**4**), is true whatever may be the nature of $\check{E}_1\;\check{E}_2$ etc., and therefore it is true of all numbers, real complex or imaginary, whatever they may represent and therefore similar relations may be obtained for current vectors and they may be extended to include not only vectors but also the operators.

In order to simplify the expressions which become unwieldy when applied to the general n-phase system, let us consider a three-phase system of vectors $\check{E}_a\;\check{E}_b\;\check{E}_c$. Then we have the following identities:

$$\left.\begin{aligned}
\check{E}_a &\equiv \frac{\check{E}_a+\check{E}_b+\check{E}_c}{3} + \frac{E_a + a\,\check{E}_b + a^2\,E_c}{3} + \frac{\check{E}_a + a^2\,\check{E}_b + a\,\check{E}_c}{3} \\
\check{E}_b &\equiv \frac{\check{E}_a+\check{E}_b+\check{E}_c}{3} + a^2\,\frac{\check{E}_a + a\,\check{E}_b + a^2\,E_c}{3} + a\,\frac{\check{E}_a + a^2\,\check{E}_b + a\,\check{E}_c}{3} \\
\check{E}_c &\equiv \frac{\check{E}_a+\check{E}_b+\check{E}_c}{3} + a\,\frac{\check{E}_a + a\,\check{E}_b + a^2\,\check{E}_c}{3} + a^2\,\frac{\check{E}_a + a^2\,\check{E}_b + a\,\check{E}_c}{3}
\end{aligned}\right\}\quad(\mathbf{5})$$

(**4**) states the law that a system of n vectors or quantities may be resolved when n is prime into n different symmetrical groups or systems, one of which consists of n equal vectors and the remaining $(n-1)$ systems consist of n equi-spaced vectors which with the first mentioned groups of equal vectors forms an equal number of symmetrical n-phase systems. When n is not prime some of the n-phase systems degenerate into repetitions of systems having numbers of phases corresponding to the factors of n.

Equation (**5**) states that any three vectors $\check{E}_a$ $\check{E}_b$ $\check{E}_c$ may be resolved into a system of three equal vectors $\check{E}_{a0}$ $\check{E}_{a0}$ $\check{E}_{a0}$ and two symmetrical three-phase systems $\check{E}_{a1}$, $a^2\, \check{E}_{a1}$, $a\, \check{E}_{a1}$, and $\check{E}_{a2}$,

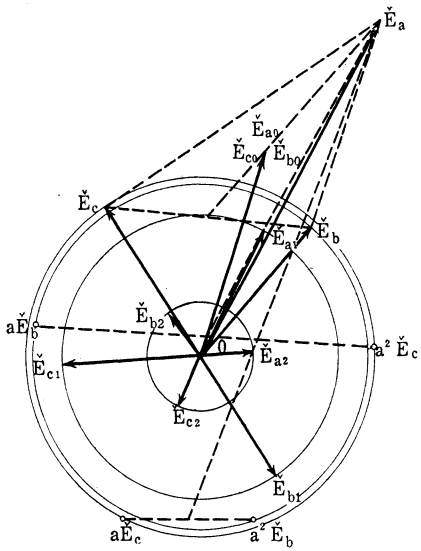

Fig. 1—Graphical Representation of Equation 5.

$a\, \check{E}_{a2}$, $a^2\, \check{E}_{a2}$, the first of which is of positive phase sequence and the second of negative phase sequence, or

$$\left.\begin{aligned}\check{E}_a &= \check{E}_{a0} + \check{E}_{a1} + \check{E}_{a2} \\ \check{E}_b &= \check{E}_{a0} + a^2\, \check{E}_{a1} + a\, \check{E}_{a2} \\ \check{E}_c &= \check{E}_{a0} + a\, \check{E}_{a1} + a^2\, \check{E}_{a2}\end{aligned}\right\} \qquad (6)$$

Similarly

$$\left.\begin{aligned}\check{I}_a &= \check{I}_{a0} + \check{I}_{a1} + \check{I}_{a2} \\ \check{I}_b &= \check{I}_{a0} + a_2\, \check{I}_{a1} + a\, \check{I}_{a2} \\ \check{I}_c &= \check{I}_{a0} + a\, \check{I}_{a1} + a^2\, \check{I}_{a2}\end{aligned}\right\} \qquad (7)$$

Figs. (1) and (2) show a graphical method of resolving three vectors $\check{E}_a$ $\check{E}_b$ and $\check{E}_c$ into their symmetrical three-phase components corresponding to equations (**5**). The construction is as follows:—$\check{E}_{a0}$ $\check{E}_{b0}$, $\check{E}_{c0}$ are obtained by drawing a line from 0 to the centroid of the triangle $\check{E}_a\ \check{E}_b\ \check{E}_c$. $\check{E}_{a1}\ \check{E}_{b1}$, $\check{E}_{c1}$ are obtained by rotating $\check{E}_b$ positively through an angle $\frac{2\pi}{3}$, and $\check{E}_c$ negatively through the same angle giving the points $a\ \check{E}_b$ and $a^2\ \check{E}_c$ respectively. $\check{E}_{a1}$ is the vector obtained by a line drawn from 0 to the centroid of the triangle $\check{E}_a$, $a\ \check{E}_b$, $a^2\ \check{E}_c$; and $\check{E}_{b1}$ and $\check{E}_{c1}$ *lag* this vector by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ respectively. To obtain $\check{E}_{a2}\ \check{E}_{b2}$ $\check{E}_{c2}$, $\check{E}_b$ is rotated negatively and $\check{E}_c$ positively through the angle $\frac{2\pi}{3}$ giving the points $a_2\ \check{E}_b$ and $a\ \check{E}_c$ respectively; the line drawn from 0 to the centroid of the triangle $\check{E}_a$, $a^2\ \check{E}_b$, $a\ \check{E}_c$ is the

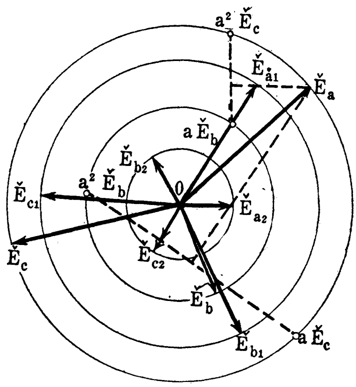

FIG. 2—GRAPHICAL REPRESENTATION OF EQUATION 5.

vector $\check{E}_{a2}$, $\check{E}_{b2}$ and $\check{E}_{c2}$ *lead* this vector by the angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ respectively.

The system of operators $Z_{aa}\ Z_{bb}\ Z_{cc}\ Z_{ab}\ Z_{bc}\ Z_{ca}$ may be resolved in a similar manner into symmetrical groups,

$$\left.\begin{aligned} Z_{aa} &= Z_{aa0} + Z_{aa1} + Z_{aa2} \\ Z_{bb} &= Z_{aa0} + a^2\ Z_{aa1} + a\ Z_{aa2} \\ Z_{cc} &= Z_{aa0} + a\ Z_{aa1} + a^2\ Z_{aa2} \end{aligned}\right\} \quad (8)$$

$$\left.\begin{aligned} Z_{ab} &= Z_{ab0} + Z_{ab1} + Z_{ab2} \\ Z_{bc} &= Z_{ab0} + a^2\ Z_{ab1} + a\ Z_{ab2} \\ Z_{ca} &= Z_{ab0} + a\ Z_{ab1} + a^2\ Z_{ab2} \end{aligned}\right\} \quad (9)$$

There are similar relations for *n*-phase systems.

EXPLANATION OF THEORY AND USE OF SEQUENCE OPERATOR

Let us define the symmetrical sequences of nth roots of unity in the following manner:

$$\left.\begin{aligned}
S^0 &= 1, \quad 1, \quad 1 \ldots 1 \\
S^1 &= 1, \quad a^{-1}, \quad a^{-2} \ldots a^{-(n-1)} \\
S^2 &= 1, \quad a^{-2}, \quad a^{-4} \ldots a^{-2(n-1)} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
S^r &= 1, \quad a^{-r}, \quad a^{-2r} \ldots a^{-(n-1)r} \\
S^{(r+1)} &= 1, \ a^{-(r+1)}, \ a^{-2(r+1)} \ldots a^{-(n-1)(r+1)} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
S^{(n-1)} &= 1, \ a^{-(n-1)}, \ a^{-2(n-1)} \ldots a^{-(n-1)^2}
\end{aligned}\right\} \tag{10}$$

Consider the sequence obtained by the products of similar terms of S^r and S^1. It will be

$$S^{(r+1)} = 1, \ a^{-(r+1)}, \quad a^{-2(r+1)} \ldots a^{-(n-1)(r+1)} \tag{11}$$

Similarly

$$S^k = 1, \quad a^{-k}, \quad a^{-2k} \ldots a^{-(n-1)k} \tag{12}$$

and the sequence obtained by products of like terms of this sequence and S^r is

$$S^{(r+k)} = 1, \ a^{-(r+k)}, \quad a^{-2(r+k)} \ldots a^{-(n-1)(r+k)} \tag{13}$$

We may therefore apply the law of indices to the products of sequences to obtain the resulting sequence.

In the case of the three-phase system we shall have the following sequences only to consider, viz.:

$$\left.\begin{aligned}
S^0 &= 1, \quad 1, \quad 1 \\
S^1 &= 1, \quad a^2, \quad a \\
S^2 &= 1, \quad a, \quad a^2
\end{aligned}\right\} \tag{14}$$

The complete system of currents $\check{I}_a \ \check{I}_b \ \check{I}_c$ are defined by

$$S\ (\check{I}_a) = S^0 \, \check{I}_{a0} + S^1 \, \check{I}_{a1} + S^2 \, \check{I}_{a2} \tag{15}$$

Similarly the impedances $Z_{aa} \ Z_{bb} \ Z_{cc}$ may be expressed in symmetrical form

$$S\ (Z_{aa}) \equiv S^0 \, Z_{aa0} + S^1 \, Z_{aa1} + S^2 \, Z_{aa2} \tag{16}$$

and the mutual impedances Z_{ab}, Z_{bc}, Z_{ca} are expressed by

$$S\ (Z_{ab}) \equiv S^0 \, Z_{ab0} + S^1 \, Z_{ab1} + S^2 \, Z_{ab2} \tag{17}$$

Attention is called to the importance of preserving the cyclic order of self and mutual impedances, otherwise the rule for the sequence operator will not hold. Thus, Z_{ab}, Z_{bc} and Z_{ca} are in proper sequence as also are Z_{ca}, Z_{ab}, Z_{bc}.

When it is desired to change the first term in the sequence of polyphase vectors the resulting expression will be

$$\left.\begin{aligned} S\,(\check{I}_b) &= S^0\,\check{I}_{a0} + S^1\,a^2\,\check{I}_{a1} + S^2\,a\,\check{I}_{a2} \\ S\,(\check{I}_c) &= S^0\,\check{I}_{a0} + S^1\,a\,\check{I}_{a1} + S^2\,a^2\,\check{I}_{a2} \end{aligned}\right\} \quad (18)$$

Similarly in the case of the operators $S\,(Z_{ab})$ we have

$$\left.\begin{aligned} S\,(Z_{bc}) &= S^0\,Z_{ab0} + S^1\,a^2\,Z_{ab1} + S^2\,a\,Z_{ab2} \\ S\,(Z_{ca}) &= S^0\,Z_{ab0} + S^1\,a\,Z_{ab1} + S^2\,a^2\,Z_{ab2} \end{aligned}\right\} \quad (19)$$

Similar rules apply to the e.m.fs. $E_a\ E_b\ E_c$

$$\left.\begin{aligned} S\,(\check{E}_a) &= S^0\,\check{E}_{a0} + S^1\,\check{E}_{a1} + S^2\,\check{E}_{a2} \\ S\,(\check{E}_b) &= S^0\,\check{E}_{a0} + S^1\,a^2\,\check{E}_{a1} + S^2\,a\,\check{E}_{a2} \\ S\,(\check{E}_c) &= S^0\,\check{E}_{a0} + S^1\,a\,E_{a1} + S^2\,a^2\,\check{E}_{a2} \end{aligned}\right\} \quad (20)$$

It should be kept in mind that any one of the several expressions $S\,(\check{I}_a)$ $S\,(\check{I}_b)$ $S\,(\check{I}_c)$, etc., completely specifies the system, and each of the members of the groups of equations given above is a complete statement of the system of vectors or operators and their relation.

Application to Self and Mutual Impedance Operations

We may now proceed with the current systems $S\,(\check{I}_a)$, $S\,(\check{I}_b)$, $S\,(\check{I}_c)$ and the operating groups $S\,(Z_{aa})$ $S\,(Z_{bb})$ $S\,(Z_{cc})$ etc. and the electromotive forces in exactly the same manner as for simple a-c. circuits. Thus,

$$\begin{aligned} S\,(\check{E}_a) &= S\,(Z_{aa})\,S\,(\check{I}_a) + S\,(Z_{ab})\,S\,(\check{I}_b) + S\,(Z_{ca})\,S\,(\check{I}_c) \qquad (21) \\ &= (S^0\,Z_{aa0} + S^1\,Z_{aa1} + S^2\,Z_{aa2})\,(S^0\,\check{I}_{a0} + S^1\,\check{I}_{a1} + S^2\,I_{a2}) \\ &\quad + (S^0\,Z_{ab0} + S^1\,Z_{ab1} + S^2\,Z_{ab2}) \\ &\qquad\qquad (S^0\,\check{I}_{a0} + S^1\,a^2\,\check{I}_a + S^2\,a\,\check{I}_{a2}) \\ &\quad + (S^0\,Z_{ab0} + S^1\,a\,Z_{ab1} + S^2\,a^2\,Z_{ab2}) \\ &\qquad\qquad (S^0\,\check{I}_{a0} + S^1\,a\,\check{I}_{a1} + S^2\,a^2\,\check{I}_{a2}) \\ &= S^0\,(Z_{aa0} + 2\,Z_{ab0})\,\check{I}_{a0} + S^0\,\{Z_{aa2} + (1 + a^2)\,Z_{ab2}\}\,\check{I}_{a1} \end{aligned}$$

$$+ S^0 \{Z_{aa1} + (1 + a) Z_{ab1}\} \check{I}_{a2}$$
$$+ S^1 \{Z_{aa1} + (1 + a) Z_{ab1}\} \check{I}_{a0}$$
$$+ S^1 \{Z_{aa0} + (a + a^2) Z_{ab0}\} \check{I}_{a1}$$
$$+ S^1 \{Z_{aa2} + 2\, a\, Z_{ab2}\} \check{I}_{a2}$$
$$+ S^2 \{Z_{aa2} + (1 + a^2) Z_{ab2}\} \check{I}_{a0}$$
$$+ S^2 \{Z_{aa1} + 2\, a^2 Z_{ab1}\} \check{I}_{a1}$$
$$+ S^2 \{Z_{aa0} + (a + a^2) Z_{ab0}\} \check{I}_{a2} \qquad \textbf{(22)}$$

Or since $1 + a + a^2 = 0$, $1 + a = - a^2$, $1 + a^2 = - a$ and $a + a^2 = - 1$

$$S(\check{E}_a) = S^0 (Z_{aa0} + 2\, Z_{ab0}) \check{I}_{a0} + S^0 (Z_{aa2} - a\, Z_{ab2}) \check{I}_{a1}$$
$$+ S^0 (Z_{aa1} - a^2 Z_{ab1}) \check{I}_{a2} + S^1 (Z_{aa1} - a^2 Z_{ab1}) \check{I}_{a0}$$
$$+ S^1 (Z_{aa0} - Z_{ab0}) \check{I}_{a1} + S^1 (Z_{aa2} + 2\, a\, Z_{ab2}) \check{I}_{a2}$$
$$+ S^2 (Z_{aa2} - a\, Z_{ab2}) \check{I}_{a0} + S^2 (Z_{aa1} + 2\, a^2 Z_{ab1}) \check{I}_{a1}$$
$$+ S^2 (Z_{aa0} - Z_{ab0}) \check{I}_{a2} \qquad \textbf{(23)}$$

Or since

$$S(Z_{bc}) = S^0 Z_{bc0} + S^1 Z_{bc1} + S^2 Z_{bc2}$$
$$= S^0 Z_{ab0} + S^1 a^2 Z_{ab1} + S^2 a\, Z_{ab2}$$

we may write (**23**) in the form

$$S(\check{E}_a) = S^0 (Z_{aa0} + 2\, Z_{bc0}) \check{I}_{a0} + S^0 (Z_{aa2} - Z_{bc2}) \check{I}_{a1}$$
$$+ S^0 (Z_{aa1} - Z_{bc1}) \check{I}_{a2} + S^1 (Z_{aa1} - Z_{bc1}) \check{I}_{a0}$$
$$+ S^1 (Z_{aa0} - Z_{bc0}) \check{I}_{a1} + S^1 (Z_{aa2} + 2\, Z_{bc2}) \check{I}_{a2}$$
$$+ S^2 (Z_{aa2} - Z_{bc2}) \check{I}_{a0} + S^2 (Z_{aa1} + 2\, Z_{bc1}) \check{I}_{a1}$$
$$+ S^2 (Z_{aa0} - Z_{bc0}) \check{I}_{a2} \qquad \textbf{(24)}$$

which is the more symmetrical form. We have therefore from (**24**) by expressing $S(\check{E}_a)$ in terms of symmetrical co-ordinates the three symmetrical equations

$$\left.\begin{aligned} S^0 \check{E}_{a0} &= S^0 \{(Z_{aa0} + 2\, Z_{bc0}) \check{I}_{a0} + (Z_{aa2} - Z_{bc2}) \check{I}_{a1} \\ &\qquad + (Z_{aa1} - Z_{bc1}) \check{I}_{a2}\} \\ S^1 E_{a1} &= S^1 \{(Z_{aa1} - Z_{bc1}) \check{I}_{a0} + (Z_{aa0} - Z_{bc0}) \check{I}_{a1} \\ &\qquad + (Z_{aa2} + 2\, Z_{bc2}) \check{I}_{a2}\} \\ S^2 E_{a2} &= S^2 \{(Z_{aa2} - Z_{bc2}) \check{I}_{a0} + (Z_{aa1} + 2\, Z_{bc1}) \check{I}_{a1} \\ &\qquad + (Z_{aa0} - Z_{bc0}) \check{I}_{a2}\} \end{aligned}\right\} \qquad \textbf{(25)}$$

An important case to which we must next give consideration is that of mutual inductance between a primary polyphase circuit and a secondary polyphase circuit. The mutual impedances may be arranged in three sets. Let the currents in the secondary windings be I_u I_v and I_w, we may then express the generalized mutual impedances as follows:

$$\left.\begin{array}{r} \text{(I)}\ Z_{au}\ Z_{bv}\ Z_{cw} \\ \text{(II)}\ Z_{bw}\ Z_{cu}\ Z_{av} \\ \text{(III)}\ Z_{cv}\ Z_{aw}\ Z_{bu} \end{array}\right\} \quad \textbf{(26)}$$

Each set may be resolved into three symmetrical groups, so that

$$\left.\begin{array}{l} S\ (Z_{au}) = S^0\ Z_{au0} + S^1\ Z_{au1} + S^2\ Z_{au2} \\ S\ (Z_{bw}) = S^0\ Z_{bw0} + S^1\ Z_{bw1} + S^2\ Z_{bw2} \\ S\ (Z_{cv}) = S^0\ Z_{cv0} + S^1\ Z_{cv1} + S^2\ Z_{cv2} \end{array}\right\} \quad \textbf{(27)}$$

and we have for $S\ (\check{E}_a)$ the primary induced e.m.f. due to the secondary currents $S\ (\check{I}_u)$

$$S\ (\check{E}_a) = S\ (Z_{au})\ S\ (\check{I}_u) + S\ (Z_{av})\ S\ (\check{I}_v) + S\ (Z_{aw})\ S\ (\check{I}_w) \quad \textbf{(28)}$$

Substituting for $S\ (\check{I}_u)$, $S\ (\check{I}_v)$ and $S\ (\check{I}_w)$ and $S\ (Z_{au})$, $S\ (Z_{av})$ $S\ (Z_{aw})$ their symmetrical equivalents we have

$$\begin{aligned} S\ (\check{E}_a) = {} & S^0\ (Z_{au0} + Z_{bw0} + Z_{cv0})\ I_{uo} \\ & + S^0\ (Z_{au2} + a\ Z_{bw2} + a^2\ Z_{cv2})\ \check{I}_{u1} \\ & + S^0\ (Z_{au1} + a^2\ Z_{bw1} + a\ Z_{cv1})\ \check{I}_{u2} \\ & + S^1\ (Z_{au1} + a\ Z_{bw1} + a^2\ Z_{cv1})\ \check{I}_{u0} \\ & + S^1\ (Z_{au0} + a^2\ Z_{bw0} + a\ Z_{cv0})\ \check{I}_{u1} \\ & + S^1\ (Z_{au2} + Z_{bw2} + Z_{cv2})\ \check{I}_{u2} \\ & + S^2\ (Z_{au2} + a^2\ Z_{bw2} + a\ Z_{v2})\ \check{I}_{u0} \\ & + S^2\ (Z_{au1} + Z_{bw1} + Z_{cv1})\ \check{I}_{u1} \\ & + S^2\ (Z_{au0} + a\ Z_{bw0} + a^2\ Z_{cv0})\ \check{I}_{u2} \end{aligned} \quad \textbf{(29)}$$

On expressing $S\ (\check{E}_a)$ in symmetrical form we have the following three symmetrical equations

$$\left.\begin{aligned} S^0\ \check{E}_{a0} = S^0\ \{ & (Z_{au0} + Z_{bw0} + Z_{cv0})\ \check{I}_{u0} \\ & + (Z_{au2} + a\ Z_{bw2} + a^2\ Z_{cv2})\ \check{I}_{u1} \\ & + (Z_{au1} + a^2\ Z_{bw1} + a\ Z_{cv1})\ \check{I}_{u2} \} \\ S^1\ \check{E}_{a1} = S^1\ \{ & (Z_{au1} + a\ Z_{bw1} + a^2\ Z_{cv1})\ \check{I}_{u0} \\ & + (Z_{au0} + a^2\ Z_{bw0} + a\ Z_{cv0})\ \check{I}_{u1} \\ & + (Z_{au2} + Z_{bw2} + Z_{cv2})\ \check{I}_{u2} \} \\ S^2\ E_{a2} = S^2\ \{ & (Z_{au2} + a^2\ Z_{bw2} + a\ Z_{cv2})\ \check{I}_u{}^0 \\ & + (Z_{au1} + Z_{bw1} + Z_{cv1})\ \check{I}_{u1} \\ & + (Z_{au0} + a\ Z_{bw0} + a^2\ Z_{cv0})\ \check{I}_{u2} \} \end{aligned}\right\} \quad (\mathbf{30})$$

For the e.m.f. $S\ (\check{E}_u)$ induced in the secondary by the primary currents $S\ (\check{I}_a)$ we have

$$S\ (\check{E}_u) = S\ (Z_{au})\ S\ (\check{I}_a) + S\ (Z_{bu})\ S\ (\check{I}_b) + S\ (Z_{cu})\ S\ (\check{I}_c) \quad (\mathbf{31})$$

Since $S\ (Z_{bu})$ bears the same relation to $S\ (Z_{cv})$ as $S\ (Z_{av})$ does to $S\ (Z_{bw})$ and $S\ (Z_{au})$ bears the same relation to $S\ (Z_{bw})$ as $S\ (Z_{aw})$ does to $S\ (Z_{cv})$ to obtain $S\ (\check{E}_u)$ all that will be necessary will be to interchange Z_{bw} and Z_{cv} in (**29**) and change $\check{I}_{u0}\ \check{I}_{u1}\ \check{I}_{u2}$ to $\check{I}_{a0}\ I_{a1}$ and $\check{I}_{a2}$ respectively, this gives

$$\begin{aligned} S\ (\check{E}_u) = {} & S^0\ (Z_{au0} + Z_{bw0} + Z_{cv0})\ \check{I}_{a0} \\ & + S^0\ (Z_{au2} + a^2\ Z_{bw2} + a\ Z_{cv2})\ \check{I}_{a1} \\ & + S^0\ (Z_{au1} + a\ Z_{bw1} + a^2\ Z_{cv1})\ \check{I}_{a2} \\ & + S^1\ (Z_{au1} + a^2\ Z_{bw1} + a\ Z_{cv1})\ \check{I}_{a0} \\ & + S^1\ (Z_{au0} + a\ Z_{bw0} + a^2\ Z_{cv0})\ \check{I}_{a1} \\ & + S^1\ (Z_{au2} + Z_{bw2} + Z_{cv2})\ \check{I}_{a2} \\ & + S^2\ (Z_{au2} + a\ Z_{bw2} + a^2\ Z_{cv2})\ \check{I}_{a0} \\ & + S^2\ (Z_{au1} + Z_{bw1} + Z_{cv1})\ \check{I}_{a1} \\ & + S^2\ (Z_{au0} + Z_{bw0} + Z_{cv0})\ \check{I}_{a2} \end{aligned} \quad (\mathbf{32})$$

and the three symmetrical equations will be

$$\left.\begin{aligned}
S^0\,\check{E}_{u0} &= S^0\,\{(Z_{au0} + Z_{bw0} + Z_{cv0})\,\check{I}_{a0} \\
&\quad + (Z_{au2} + a^2\,Z_{bw2} + a\,Z_{bw2} + a\,Z_{cv2})\,\check{I}_{a1} \\
&\quad + (Z_{au1} + a\,Z_{bw1} + a^2\,Z_{cv1})\,\check{I}_{a2}\} \\
S^1\,\check{E}_{a1} &= S^1\,\{(Z_{au1} + a^2\,Z_{bw1} + a\,Z_{cv1})\,\check{I}_{a0} \\
&\quad + (Z_{au0} + a\,Z_{bw0} + a^2\,Z_{cv0})\,\check{I}_{a1} \\
&\quad + (Z_{au2} + Z_{bw2} + Z_{cv2})\,\check{I}_{a2}\} \\
S^2\,\check{E}_{u2} &= S^2\,\{(Z_{au2} + a\,Z_{bw2} + a^2\,Z_{cv2})\,\check{I}_{a0} \\
&\quad + (Z_{au1} + Z_{bw1} + Z_{cv1})\,\check{I}_{a1} \\
&\quad + (Z_{au0} + a^2\,Z_{bw0} + a\,Z_{cv0})\,\check{I}_{a2}\}
\end{aligned}\right\} \quad \textbf{(33)}$$

The same methods may be applied to polyphase systems of any number of phases. When the number of phases is not prime the system may sometimes be dealt with as a number of polyphase systems having mutual inductance between them:—For example, a nine-phase system may be treated as three three-phase systems, a twelve-phase system as three four-phase or four three-phase systems. In certain forms of dissymmetry this method is of great practical value, and its application will be taken up later.

For the present part of the paper we shall confine ourselves to the three-phase system, and dissymmetries of several different kinds.

The operators Z_{au} Z_{aa}, etc., must be interpreted in the broadest sense. They may be simple complex quantities or they may be functions of the differential operator $\frac{d}{dt}$. For if

$$i = \Sigma\,(A_n \cos n\,w\,t + B_n \sin n\,w\,t)$$

it may be expressed in the form

$$\left.\begin{aligned}
i &= \Sigma\left(\frac{A_n - j\,B_n}{2}\,\epsilon^{jnwt} + \frac{A_n + j\,B_n}{2}\,\epsilon^{-jnwt}\right) \\
&= \frac{\check{I}}{2} + \frac{\hat{I}}{2} \\
&= \text{real part of } \check{I}
\end{aligned}\right\} \quad \textbf{(34)}$$

and any linear algebraic operation performed on $\check{I}/2$ will give a result which will be conjugate to that obtained by carrying out the same operation on $\hat{I}/2$ and since the true solution is the sum of these results, it may also be obtained by taking the real part of the result of performing the operation on $\check{I}$.

Modification of the General Case Met With in Practical Networks

Several symmetrical arrangements of the operator Z_{au} etc.' are frequently met with in practical networks which result in a much simpler system of equations than those obtained for the general case as in equations (**29**) to (**33**). Thus for example if all the operators in (**26**) are equal, all the operators in (**27**), except $S^0\, Z_{au0}$ $S^0\, Z_{bw0}$ and $S^0\, Z_{cv0}$ are equal to zero, and these three quantities are also equal to one another so that equation (**30**) becomes

$$\left.\begin{aligned} S^0\, \check{E}_{a0} &= S^0\, (Z_{au0} + Z_{bw0} + Z_{cv0})\, \check{I}_{u0} \\ S^1\, \check{E}_{a1} &= 0 \\ S^2\, E_{a2} &= 0 \end{aligned}\right\} \qquad (\mathbf{35})$$

and equation (**33**)

$$\left.\begin{aligned} S^0\, \check{E}_{u0} &= S^0\, (Z_{au0} + Z_{bw0} + Z_{cv0})\, \check{I}_{a0} \\ S^1\, E_{u1} &= 0 \\ S^2\, E_{u2} &= 0 \end{aligned}\right\} \qquad (\mathbf{36})$$

This is the statement in symmetrical co-ordinates that a symmetrically disposed polyphase transmission line will produce no electromagnetic induction in a second similar polyphase system so disposed with respect to the first that mutual inductions between all phases of the two are equal except that due to single-phase currents passing through the conductors.

If in (**26**) the quantities in each group only are equal, equations (**30**) and (**33**) become

$$\left.\begin{aligned} S^0\, \check{E}_{a0} &= S^0\, (Z_{au0} + Z_{bw0} + Z_{cv0})\, \check{I}_{u0} \\ S^1\, \check{E}_{a1} &= S^1\, (Z_{au0} + a^2\, Z_{bw0} + a\, Z_{cv0})\, \check{I}_{u1} \\ S^2\, \check{E}_{a2} &= S^2\, (Z_{au0} + a\, Z_{bw0} + a^2\, Z_{cv0})\, \check{I}_{u2} \end{aligned}\right\} \qquad (\mathbf{37})$$

$$\left.\begin{aligned} S^0\,\check{E}_{u0} &= S^0\,(Z_{au0} + Z_{bw0} + Z_{cv0})\,\check{I}_{a0} \\ S^1\,\check{E}_{a1} &= S^1\,(Z_{au0} + a\,Z_{bw0} + a^2\,Z_{cv0})\,\check{I}_{a1} \\ S^2\,\check{E}_{a2} &= S^1\,(Z_{au0} + a^2\,Z_{bw0} + a\,Z_{cv0})\,\check{I}_{a2} \end{aligned}\right\} \quad \textbf{(38)}$$

Symmetrical Forms of Common Occurrence

A symmetrical form which is of importance because it is of frequent occurrence in practical polyphase networks has the terms in group (I) equation (**26**) all equal and those in group (II) $\cos\frac{2\pi}{3}$ times those in group (I) and those in group (III) $\cos\frac{4\pi}{3}$ times those in group (I).

Since $\cos\frac{2\pi}{3} = \frac{a + a^2}{2} = \cos\frac{4\pi}{3}$ we have on substituting the values of the impedances in this case,

$$\left.\begin{aligned} S^0\,\check{E}_{a0} &= S^0\,\{Z_{au0}\,(1 + a + a^2)\}\,\check{I}_{u0} = 0 \\ S^1\,\check{E}_{a1} &= S^1\,1\tfrac{1}{2}\,Z_{au0}\,\check{I}_{u1} \\ S^2\,E_{a2} &= S^2\,1\tfrac{1}{2}\,Z_{au0}\,\check{I}_{u2} \end{aligned}\right\} \quad \textbf{(39)}$$

$$\left.\begin{aligned} S^0\,E_{u0} &= S^0\,\{Z_{au0}\,(1 + a + a^2)\}\,\check{I}_{a0} = 0 \\ S^1\,\check{E}_{u1} &= S^1\,1\tfrac{1}{2}\,Z_{au0}\,\check{I}_{a1} \\ S^2\,\check{E}_{u2} &= S^2\,1\tfrac{1}{2}\,Z_{au0}\,\check{I}_{a2} \end{aligned}\right\} \quad \textbf{(40)}$$

The elements in group I may be unequal but groups II and III may be obtained from group I by multiplying by $\cos\frac{4\pi}{3}$ and $\cos\frac{2\pi}{3}$ respectively.

The members of the three groups will then be related as follows, the same sequence being used as before,

$$\left.\begin{array}{ll} \text{(I)} & Z_{au},\quad Z_{bv},\quad Z_{cw} \\ \text{(II)} & \frac{a + a^2}{2}\,Z_{cw},\quad \frac{a + a^2}{2}\,Z_{au},\quad \frac{a + a^2}{2}\,Z_{bv} \\ \text{(III)} & \frac{a + a^2}{2}\,Z_{bv},\quad \frac{a + a^2}{2}\,Z_{cw},\quad \frac{a + a^2}{2}\,Z_{au} \end{array}\right\} \quad \textbf{(41)}$$

Consequently the following relations are true:

$$\left.\begin{aligned}
S^0 Z_{bw0} &= \frac{a + a^2}{2} S^0 Z_{au0} \\
S^0 Z_{cv0} &= \frac{a + a^2}{2} S^0 Z_{au0} \\
S^1 Z_{bw1} &= \frac{1 + a^2}{2} S^1 Z_{au1} \\
S^2 Z_{bw2} &= \frac{1 + a}{2} S^2 Z_{au2} \\
S^1 Z_{cv1} &= \frac{1 + a}{2} S^1 Z_{au1} \\
S^2 Z_{cv2} &= \frac{1 + a^2}{2} S^2 Z_{au2}
\end{aligned}\right\} \qquad (42)$$

Substituting these relations in (**30**) and (**33**) we have for this system of mutual impedances

$$\left.\begin{aligned}
Z_{au0} + Z_{bw0} + Z_{cv0} &= 0 \\
Z_{au0} + a\, Z_{bw0} + a^2\, Z_{cv0} &= 1\tfrac{1}{2}\, Z_{cv0} \\
Z_{au0} + a^2\, Z_{bw0} + a\, Z_{cv0} &= 1\tfrac{1}{2}\, Z_{aw0}
\end{aligned}\right\} \qquad (43)$$

$$\left.\begin{aligned}
Z_{au1} + Z_{bw1} + Z_{cv1} &= 1\tfrac{1}{2}\, Z_{au1} \\
Z_{au1} + a\, Z_{bw1} + a^2\, Z_{cv1} &= 1\tfrac{1}{2}\, Z_{au1} \\
Z_{au1} + a^2\, Z_{bw1} + a\, Z_{cv1} &= 0
\end{aligned}\right\} \qquad (44)$$

$$\left.\begin{aligned}
Z_{au2} + Z_{bw2} + Z_{cv2} &= 1\tfrac{1}{2}\, Z_{au2} \\
Z_{au2} + a\, Z_{bw2} + a^2\, Z_{cv2} &= 0 \\
Z_{au2} + a^2\, Z_{bw2} + a\, Z_{cv2} &= 1\tfrac{1}{2}\, Z_{au2}
\end{aligned}\right\} \qquad (45)$$

which on substitution in (**30**) and (**33**) gives

$$\left.\begin{aligned}
S^0\, \check{E}_{a0} &= 0 \\
S^1\, \check{E}_{a1} &= S^1 \left\{1\tfrac{1}{2}\, Z_{au1}\, \check{I}_{u0} + 1\tfrac{1}{2}\, Z_{au0}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au2}\, \check{I}_{u2}\right\} \\
S^2\, \check{E}_{a2} &= S^2 \left\{1\tfrac{1}{2}\, Z_{au2}\, \check{I}_{u0} + 1\tfrac{1}{2}\, Z_{au1}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au0}\, \check{I}_{u2}\right\}
\end{aligned}\right\} \qquad (46)$$

$$\left.\begin{aligned}
S^0\, \check{E}_{u0} &= S^0 \left\{1\tfrac{1}{2}\, Z_{au2}\, \check{I}_{a1} + 1\tfrac{1}{2}\, Z_{au1}\, \check{I}_{a2}\right\} \\
S^1\, \check{E}_{u1} &= S^1 \left\{1\tfrac{1}{2}\, Z_{au0}\, \check{I}_{a1} + 1\tfrac{1}{2}\, Z_{au2}\, \check{I}_{a2}\right\} \\
S^2\, \check{E}_{u2} &= S^2 \left\{1\tfrac{1}{2}\, Z_{au1}\, \check{I}_{a1} + 1\tfrac{1}{2}\, Z_{au0}\, \check{I}_{a2}\right\}
\end{aligned}\right\} \qquad (47)$$

The above symmetrical forms in which the factors $\cos \frac{2\pi}{3}$ and $\cos \frac{4\pi}{3}$ occur apply particularly to electromagnetic induction between windings distributed over the surfaces of co-axial cylinders; where if the plane of symmetry of one winding be taken as the datum plane, the mutual impedance between this winding and any other is a harmonic function of the angle between its plane of symmetry and the datum plane. In other words, the mutual impedances are functions of position on the circumference of a circle and may therefore be expanded by Fourier's theorem in a series of integral harmonics of the angle made by the planes of symmetry with the datum plane. Since the same procedure applies to all the terms of the expansion it is necessary only to consider the simple harmonic case. In the partially symmetrical cases of mutual induction, such as that taken up in the preceding discussion, there will be a difference between two possible cases, viz:—Symmetrical primary, unsymmetrical secondary, which is the case just considered, and unsymmetrical primary and symmetrical secondary in which the impedances of (**26**) will have the following values

$$\left.\begin{array}{ll} \text{(I)} & Z_{au}, \quad Z_{bc}, \quad Z_{cw} \\ \text{(II)} & \frac{a+a^2}{2} Z_{bc}, \quad \frac{a+a^2}{2} Z_{cw}, \quad \frac{a+a^2}{2} Z_{au} \\ \text{(III)} & \frac{a+a^2}{2} Z_{cw}, \quad \frac{a+a^2}{2} Z_{au}, \quad \frac{a+a^2}{2} Z_{bc} \end{array}\right\} \quad \textbf{(48)}$$

The results may be immediately set down by symmetry from equations (**46**) and (**47**), but the difference between the two cases will be better appreciated by setting down the component symmetrical impedances, thus we have

$$\left.\begin{array}{l} S^0 Z_{bw0} = \frac{a+a^2}{2} S^0 Z_{au0} \\ S^0 Z_{cv0} = \frac{a+a^2}{2} S^0 Z_{au0} \\ S^1 Z_{bw1} = \frac{1+a}{2} S^1 Z_{au1} \\ S^2 Z_{bw2} = \frac{1+a^2}{2} S^2 Z_{au2} \\ S^1 Z_{cv1} = \frac{1+a^2}{2} S^1 Z_{cu1} \\ S^2 Z_{cv2} = \frac{1+a}{2} S^2 Z_{au2} \end{array}\right\} \quad \textbf{(49)}$$

Substituting these relations in the impedances used in (**30**) and (**33**) they become

$$\left.\begin{aligned} Z_{au0} + Z_{bw0} + Z_{cv0} &= 0 \\ Z_{au0} + a\, Z_{bw0} + a^2\, Z_{cv0} &= 1\tfrac{1}{2}\, Z_{au0} \\ Z_{au0} + a^2\, Z_{bw0} + a\, Z_{cv0} &= 1\tfrac{1}{2}\, Z_{au0} \end{aligned}\right\} \quad (\mathbf{50})$$

$$\left.\begin{aligned} Z_{au1} + Z_{bw1} + Z_{cv1} &= 1\tfrac{1}{2}\, Z_{au1} \\ Z_{au1} + a\, Z_{bw1} + a^2\, Z_{cv1} &= 0 \\ Z_{au1} + a^2\, Z_{bw1} + a\, Z_{cv1} &= 1\tfrac{1}{2}\, Z_{au1} \end{aligned}\right\} \quad (\mathbf{51})$$

$$\left.\begin{aligned} Z_{au2} + Z_{bw2} + Z_{cv2} &= 1\tfrac{1}{2}\, Z_{au2} \\ Z_{au2} + a\, Z_{bw2} + a^2\, Z_{cv2} &= 1\tfrac{1}{2}\, Z_{au2} \\ Z_{au2} + a^2\, Z_{bw2} + a\, Z_{cv2} &= 0 \end{aligned}\right\} \quad (\mathbf{52})$$

And we have from (**30**) and (**33**), or by symmetry

$$\left.\begin{aligned} S^0\, \check{E}_{a0} &= S^0\, \{1\tfrac{1}{2}\, Z_{au2}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au1}\, \check{I}_{u2}\} \\ S^1\, \check{E}_{a1} &= S^1\, \{1\tfrac{1}{2}\, Z_{au0}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au2}\, \check{I}_{u2}\} \\ S^2\, \check{E}_{a2} &= S^2\, \{1\tfrac{1}{2}\, Z_{au1}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au0}\, \check{I}_{u2} \end{aligned}\right\} \quad (\mathbf{53})$$

$$\left.\begin{aligned} S^0\, \check{E}_{u0} &= 0 \\ S^1\, \check{E}_{u1} &= S^1\, \{1\tfrac{1}{2}\, Z_{au1}\, \check{I}_{a0} + 1\tfrac{1}{2}\, Z_{au0}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au2}\, \check{I}_{u2}\} \\ S^2\, \check{E}_{u2} &= S^2\, \{1\tfrac{1}{2}\, Z_{au2}\, \check{I}_{a0} + 1\tfrac{1}{2}\, Z_{au1}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au0}\, \check{I}_{u2}\} \end{aligned}\right\} \quad (\mathbf{54})$$

If the angle between the planes of symmetry of the coils and the datum plane are subject to changes, $\cos \dfrac{2\pi}{3}$ and $\cos \dfrac{4\pi}{3}$ in the preceding discussion must be replaced by

$$\left.\begin{aligned} \cos\left(\frac{2\pi}{3} + \theta\right) &= \frac{a}{2}\, \epsilon^{j\theta} + \frac{a^2}{2}\, \epsilon^{-j\theta} \\ \cos\left(\frac{4\pi}{3} + \theta\right) &= \frac{a}{2}\, \epsilon^{-j\theta} + \frac{a^2}{2}\, \epsilon^{j\theta} \end{aligned}\right\} \quad (\mathbf{55})$$

where θ is measured from the datum plane

In the strictly symmetrical case of co-axial cylindrical surface windings in which the members of each group of mutual impedances are equal, the result of substituting (**55**) in the equations for induced e.m.f. will be

$$\left.\begin{aligned} S^0\,\check{E}_{a0} &= 0 \\ S^1\,\check{E}_{a1} &= S^1\,(1\tfrac{1}{2}\,Z_{au0}\,\epsilon^{j\theta}\;\check{I}_{u1}) \\ S^2\,\check{E}_{a2} &= S^2\,(1\tfrac{1}{2}\,Z_{au0}\,\epsilon^{-j\theta}\,\check{I}_{u2}) \end{aligned}\right\} \quad (56)$$

$$\left.\begin{aligned} S^0\,\check{E}_{u0} &= 0 \\ S^1\,\check{E}_{u1} &= S^1\,(1\tfrac{1}{2}\,Z_{au0}\,\epsilon^{-j\theta}\,\check{I}_{a1}) \\ S^2\,\check{E}_{u2} &= S^2\,(1\tfrac{1}{2}\,Z_{au0}\,\epsilon^{j\theta}\;\check{I}_{a2}) \end{aligned}\right\} \quad (57)$$

In the case having symmetrical primary and unsymmetrical secondary in which members of each group are different, but in which there are harmonic relations between corresponding members of the different groups, the impedances are

$$\left.\begin{aligned} &(\mathrm{I})\; Z_{au},\quad Z_{bv},\quad Z_{cw} \\ &(\mathrm{II})\left(\frac{a}{2}\,\epsilon^{j\theta} + \frac{a^2}{2}\,\epsilon^{-j\theta}\right) Z_{cw} \\ &\left(\frac{a}{2}\,\epsilon^{j\theta} + \frac{a^2}{2}\,\epsilon^{-j\theta}\right) Z_{au},\; \left(\frac{a}{2}\,\epsilon^{j\theta} + \frac{a^2}{2}\,\epsilon^{-j\theta}\right) Z_{bv} \\ &(\mathrm{III})\left(\frac{a^2}{2}\,\epsilon^{j\theta} + \frac{a}{2}\,\epsilon^{-j\theta}\right) Z_{bv}, \\ &\left(\frac{a^2}{2}\,\epsilon^{j\theta} + \frac{a}{2}\,\epsilon^{-j\theta}\right) Z_{cw},\; \left(\frac{a^2}{2}\,\epsilon^{j\theta} + \frac{a}{2}\,\epsilon^{-j\theta}\right) Z_{au} \end{aligned}\right\} \quad (58)$$

The symmetrical component mutual impedances will have the following values in terms of $Z_{au0}\;Z_{au1}\;Z_{au2}$

$$\left.\begin{aligned} S^0\,Z_{bw0} &= \left(\frac{a}{2}\,\epsilon^{j\theta} + \frac{a^2}{2}\,\epsilon^{-j\theta}\right) S^0\,Z_{au0} \\ S^0\,Z_{cv0} &= \left(\frac{a^2}{2}\,\epsilon^{j\theta} + \frac{a}{2}\,\epsilon^{-j\theta}\right) S^0\,Z_{au0} \\ S^1\,Z_{bw1} &= \left(\frac{a^2}{2}\,\epsilon^{j\theta} + \frac{\epsilon^{-j\theta}}{2}\right) S^1\,Z_{au1} \\ S^2\,Z_{bw2} &= \left(\frac{\epsilon^{j\theta}}{2} + \frac{a}{2}\,\epsilon^{-j\theta}\right) S^2\,Z_{au2} \\ S^1\,Z_{cv1} &= \left(\frac{a}{2}\,\epsilon^{j\theta} + \frac{\epsilon^{-j\theta}}{2}\right) S^1\,Z_{au1} \\ S^2\,Z_{cv2} &= \left(\frac{\epsilon^{j\theta}}{2} + \frac{a^2}{2}\,\epsilon^{-j\theta}\right) S^2\,Z_{au2} \end{aligned}\right\} \quad (59)$$

Substituting these relations in the impedances of equations (**30**) and (**33**) they become

$$\left.\begin{aligned} Z_{au0} + Z_{bw0} + Z_{cv0} &= 0 \\ Z_{au0} + a\, Z_{bw0} + a^2\, Z_{cv0} &= 1\tfrac{1}{2}\, Z_{au0}\, \epsilon^{-j\theta} \\ Z_{au0} + a^2\, Z_{bw0} + a\, Z_{cv0} &= 1\tfrac{1}{2}\, Z_{au0}\, \epsilon^{j\theta} \end{aligned}\right\} \quad (60)$$

$$\left.\begin{aligned} Z_{au1} + Z_{bw1} + Z_{cv1} &= 1\tfrac{1}{2}\, Z_{au1}\, \epsilon^{-j\theta} \\ Z_{au1} + a\, Z_{bw1} + a^2\, Z_{cv1} &= 1\tfrac{1}{2}\, Z_{au1}\, \epsilon^{j\theta} \\ Z_{au1} + a^2\, Z_{bw1} + a\, Z_{cv1} &= 0 \end{aligned}\right\} \quad (61)$$

$$\left.\begin{aligned} Z_{au2} + Z_{bw2} + Z_{cv2} &= 1\tfrac{1}{2}\, Z_{au2}\, \epsilon^{j\theta} \\ Z_{au2} + a\, Z_{bw2} + a^2\, Z_{cv2} &= 0 \\ Z_{au2} + a^2\, Z_{bw2} + a\, Z_{cv2} &= 1\tfrac{1}{2}\, Z_{au}\, \epsilon^{-j\theta} \end{aligned}\right\} \quad (62)$$

which on substitution in (**30**) and (**33**) give

$$\left.\begin{aligned} S^0\, \check{E}_{a0} &= 0 \\ S^1\, \check{E}_{a1} &= S^1\, \{1\tfrac{1}{2}\, Z_{au1}\, \epsilon^{j\theta}\, \check{I}_{u0} + 1\tfrac{1}{2}\, Z_{au0}\, \epsilon^{j\theta}\, \check{I}_{u1} \\ &\qquad + 1\tfrac{1}{2}\, Z_{au2}\, \epsilon^{j\theta}\, \check{I}_{u2}\} \\ S^2\, \check{E}_{a2} &= S^2\, \{1\tfrac{1}{2}\, Z_{au2}\, \epsilon^{-j\theta}\, \check{I}_{u0} + 1\tfrac{1}{2}\, Z_{au1}\, \epsilon^{-j\theta}\, \check{I}_{u1} \\ &\qquad + 1\tfrac{1}{2}\, Z_{au0}\, \epsilon^{-j\theta}\, \check{I}_{u2}\} \end{aligned}\right\} \quad (63)$$

$$\left.\begin{aligned} S^0\, \check{E}_{u0} &= S^0\, \{1\tfrac{1}{2}\, Z_{au2}\, \epsilon^{-j\theta}\, \check{I}_{a1} + 1\tfrac{1}{2}\, Z_{au1}\, \epsilon^{j\theta}\, \check{I}_{a2}\} \\ S^1\, \check{E}_{u1} &= S^1\, \{1\tfrac{1}{2}\, Z_{au0}\, \epsilon^{-j\theta}\, \check{I}_{a1} + 1\tfrac{1}{2}\, Z_{au2}\, \epsilon^{j\theta}\, \check{I}_{a2}\} \\ S^2\, \check{E}_{u2} &= S^2\, \{1\tfrac{1}{2}\, Z_{au1}\, \epsilon^{-j\theta}\, \check{I}_{a1} + 1\tfrac{1}{2}\, Z_{au0}\, \epsilon^{j\theta}\, \check{I}_{a2}\} \end{aligned}\right\} \quad (64)$$

In the case of unsymmetrical primary and symmetrical secondary, we have for the value of the impedance in terms of $Z_{au0}\ Z_{au1}$ and Z_{au2}

$$\left.\begin{aligned} &(\mathrm{I})\ Z_{au},\quad Z_{bv},\quad Z_{cw} \\ &(\mathrm{II})\ \left(\frac{a}{2}\, \epsilon^{j\theta} + \frac{a^2}{2}\, \epsilon^{-j\theta}\right) Z_{bc}, \\ &\left(\frac{a}{2}\, \epsilon^{j\theta} + \frac{a^2}{2}\, \epsilon^{-j\theta}\right) Z_{cw},\ \left(\frac{a}{2}\, \epsilon^{j\theta} + \frac{a^2}{2}\, \epsilon^{-j\theta}\right) Z_{au} \\ &(\mathrm{III})\ \left(\frac{a^2}{2}\, \epsilon^{j\theta} + \frac{a}{2}\, \epsilon^{-j\theta}\right) Z_{cw}, \\ &\left(\frac{a^2}{2}\, \epsilon^{j\theta} + \frac{a}{2}\, \epsilon^{-j\theta}\right) Z_{au},\ \left(\frac{a^2}{2}\, \epsilon^{j\theta} + \frac{a}{2}\, \epsilon^{-j\theta}\right) Z_{bc} \end{aligned}\right\} \quad (65)$$

The symmetrical component mutual impedances in terms of Z_{au0}, Z_{au1}, Z_{au2} are

$$\left.\begin{aligned}
S^0\, Z_{bw0} &= \left(\frac{a}{2}\,\epsilon^{j\theta} + \frac{a^2}{2}\,\epsilon^{-j\theta}\right) S^0\, Z_{au0} \\
S^0\, Z_{cv0} &= \left(\frac{a^2}{2}\,\epsilon^{j\theta} + \frac{a}{2}\,\epsilon^{-j\theta}\right) S^0\, Z_{au0} \\
S^1\, Z_{bw1} &= \left(\frac{\epsilon^{j\theta}}{2} + \frac{a}{2}\,\epsilon^{-j\theta}\right) S^1\, Z_{au1} \\
S^2\, Z_{bw2} &= \left(\frac{a^2}{2}\,\epsilon^{j\theta} + \frac{\epsilon^{-j\theta}}{2}\right) S^2\, Z_{au2} \\
S^1\, Z_{cv1} &= \left(\frac{\epsilon^{j\theta}}{2} + \frac{a^2}{2}\,\epsilon^{-j\theta}\right) S^1\, Z_{au1} \\
S^2\, Z_{cv2} &= \left(\frac{a}{2}\,\epsilon^{j\theta} + \frac{\epsilon^{-j\theta}}{2}\right) S^2\, Z_{au2}
\end{aligned}\right\} \qquad (66)$$

And the impedances of equations (**30**) and (**33**) become

$$\left.\begin{aligned}
Z_{au0} + Z_{bw0} + Z_{cv0} &= 0 \\
Z_{au0} + a\, Z_{bw0} + a^2\, Z_{cv0} &= 1\tfrac{1}{2}\, Z_{au0}\, \epsilon^{-j\theta} \\
Z_{au0} + a^2\, Z_{bw0} + a\, Z_{cv0} &= 1\tfrac{1}{2}\, Z_{au0}\, \epsilon^{j\theta}
\end{aligned}\right\} \qquad (67)$$

$$\left.\begin{aligned}
Z_{au1} + Z_{bw1} + Z_{cv1} &= 1\tfrac{1}{2}\, Z_{au1}\, \epsilon^{j\theta} \\
Z_{au1} + a\, Z_{bw1} + a^2\, Z_{cv1} &= 0 \\
Z_{au1} + a^2\, Z_{bw1} + a\, Z_{cv1} &= 1\tfrac{1}{2}\, Z_{au1}\, \epsilon^{-j\theta}
\end{aligned}\right\} \qquad (68)$$

$$\left.\begin{aligned}
Z_{au2} + Z_{bw2} + Z_{cv2} &= 1\tfrac{1}{2}\, Z_{au2}\, \epsilon^{-j\theta} \\
Z_{au2} + a\, Z_{bw2} + a^2\, Z_{cv2} &= 1\tfrac{1}{2}\, Z_{au2}\, \epsilon^{j\theta} \\
Z_{au2} + a^2\, Z_{bw} + a\, Z_{cv2} &= 0
\end{aligned}\right\} \qquad (69)$$

And on substitution in (**30**) and (**33**), or by symmetry from (**63**) and (**64**), we have

$$\left.\begin{aligned}
S^0\, \check{E}_{a0} &= S^0\, \{1\tfrac{1}{2}\, Z_{au2}\, \epsilon^{j\theta}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au1}\, \epsilon^{-j\theta}\, \check{I}_{u2}\} \\
S^1\, \check{E}_{a1} &= S^1\, \{1\tfrac{1}{2}\, Z_{au0}\, \epsilon^{j\theta}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au2}\, \epsilon^{-j\theta}\, \check{I}_{u2}\} \\
S^2\, \check{E}_{a2} &= S^2\, \{1\tfrac{1}{2}\, Z_{au1}\, \epsilon^{j\theta}\, \check{I}_{u1} + 1\tfrac{1}{2}\, Z_{au0}\, \epsilon^{-j\theta}\, \check{I}_{u2}
\end{aligned}\right\} \qquad (70)$$

$$\left.\begin{aligned}
S^0\,\check{E}_{u0} &= 0\\
S^1\,\check{E}_{u1} &= S^1\,\{1\tfrac{1}{2}\,Z_{au1}\,\epsilon^{-j\theta}\,\check{I}_{a0} + 1\tfrac{1}{2}\,Z_{au0}\,\epsilon^{-j\theta}\,\check{I}_{u1}\\
&\qquad + 1\tfrac{1}{2}\,Z_{au2}\,\epsilon^{-j\theta}\,\check{I}_{a2}\}\\
S^2\,\check{E}_{u2} &= S^2\,\{1\tfrac{1}{2}\,Z_{au2}\,\epsilon^{j\theta}\,\check{I}_{a0} + 1\tfrac{1}{2}\,Z_{au1}\,\epsilon^{j\theta}\,\check{I}_{a1}\\
&\qquad + 1\tfrac{1}{2}\,Z_{au0}\,\epsilon^{j\theta}\,\check{I}_{a2}\}
\end{aligned}\right\} \quad (71)$$

A fuller discussion of self and mutual impedances of co-axial cylindrical windings will be found in the Appendix. It will be sufficient to note here that in the case of self inductance and mutual inductance of stationary windings symmetrically disposed if they are equal

$$\left.\begin{aligned}
M_{ab} = M_{bc} = M_{ca} &= \Sigma\left(A_n \cos\frac{2\,n\,\pi}{3}\right)\\
L_{aa} = L_{bb} = L_{cc} = M_{aa} = M_{bb} = M_{cc} &= \Sigma\,A_n
\end{aligned}\right\} \quad (72)$$

If the windings are symmetrically disposed but have different number of turns

$$\left.\begin{aligned}
L_{aa} &= M_{aa} = \Sigma\,A_n\\
L_{bb} &= M_{bb} = \Sigma\,B_n\\
L_{cc} &= M_{cc} = \Sigma\,C_n
\end{aligned}\right\} \quad (73)$$

$$\left.\begin{aligned}
M_{ab} &= \Sigma\left(\sqrt{A_n\,B_n}\cos\frac{2\,n\,\pi}{3}\right)\\
M_{bc} &= \Sigma\left(\sqrt{B_n\,C_n}\cos\frac{2\,n\,\pi}{3}\right)\\
M_{ca} &= \Sigma\left(\sqrt{C_n\,A_n}\cos\frac{2\,n\,\pi}{3}\right)
\end{aligned}\right\} \quad (74)$$

If the coils are alike but unsymmetrically spaced $L_{aa}\,L_{bb}\,L_{cc}$ have the same values, namely $\Sigma\,A_n$ and

$$\left.\begin{aligned}
M_{ab} &= \Sigma\,\Big\{(A_n \cos n\,\theta_1)\cos\frac{2\,n\,\pi}{3}\\
&\qquad + (A_n \sin n\,\theta_1)\sin\frac{2\,n\,\pi}{3}\Big\}\\
M_{bc} &= \Sigma\,\Big\{(A_n \cos n\,\theta_2)\cos\frac{2\,n\,\pi}{3}\\
&\qquad + (A_n \sin n\,\theta_2)\sin\frac{2\,n\,\pi}{3}\Big\}\\
M_{ca} &= \Sigma\,\Big\{(A_n \cos n\,\theta_3)\cos\frac{2\,n\,\pi}{3}\\
&\qquad + (A_n \sin n\,\theta_3)\sin\frac{2\,n\,\pi}{3}\Big\}
\end{aligned}\right\} \quad (75)$$

If they are unequal as well as unsymmetrically disposed but are otherwise similar $L_{aa}\ L_{bb}\ L_{cc}$ have values as in (**64**) and

$$\left.\begin{aligned} M_{ab} &= \Sigma \left\{ (\sqrt{A_n\, B_n} \cos n\, \theta_1) \cos \frac{2\, n\, \pi}{3} \right. \\ &\qquad \left. + (\sqrt{A_n\, B_n} \sin n\, \theta_1) \sin \frac{2\, n\, \pi}{3} \right\} \\ M_{bc} &= \Sigma \left\{ (\sqrt{B_n\, C_n} \cos n\, \theta_2) \cos \frac{2\, n\, \pi}{3} \right. \\ &\qquad \left. + (\sqrt{B_n\, C_n} \sin n\, \theta_2) \sin \frac{2\, n\, \pi}{3} \right\} \\ M_{ca} &= \Sigma \left\{ (\sqrt{C_n\, A_n} \cos n\, \theta_1) \cos \frac{2\, n\, \pi}{3} \right. \\ &\qquad \left. + (\sqrt{C_n\, A_n} \sin n\, \theta_3) \sin \frac{2\, n\, \pi}{3} \right\} \end{aligned}\right\} \qquad (\mathbf{76})$$

Where the windings are dissimilar in every respect the expressions become more complicated. A short outline of this subject is given in the Appendix.

In the case of mutual inductance between two coaxial cylindrical systems, one of which A, B, C is the primary and the other U, V, W the secondary, the following conventions should be followed:

(a) All angles are measured, taking the primary planes of symmetry as data in a positive direction.

(b) The datum plane for all windings is the plane of symmetry of the primary A phase.

(c) All mechanical motions unless otherwise stated shall be considered as positive rotations of the secondary cylinder about its axis.

(d) The conventional disposition of the phases and the direction of rotation of the secondary winding are indicated in Fig. 3.

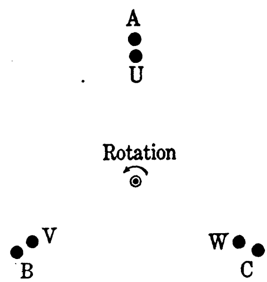

FIG. 3—CONVENTIONAL DISPOSITION OF PHASES AND DIRECTION OF ROTATION.

We shall consider five cases; Case 1 being the completely symmetrical case and the rest being symmetrical in one winding, the other winding being unsymmetrical in magnitude and phase, or both, but all windings having the same form and distribution of coils.

Case I. All Windings Symmetrical.

$$\left.\begin{aligned} M_{au} &= M_{bv} = M_{cw} = \Sigma A_n \cos n\,\theta \\ M_{bw} &= M_{cu} = M_{av} = \Sigma A_n \cos n\left(\frac{2\,\pi}{3} + \theta\right) \\ M_{cv} &= M_{aw} = M_{bu} = \Sigma A_n \cos n\left(\frac{4\,\pi}{3} + \theta\right) \end{aligned}\right\} \quad (77)$$

Case II. Primary Windings equal and Symmetrical, Secondary Windings unequal but otherwise Symmetrical.

$$\left.\begin{aligned} M_{au} &= \Sigma A_n \cos n\,\theta,\ M_{bv} = \Sigma B_n \cos n\,\theta, \\ &\qquad M_{cw} = \Sigma C_u \cos n\,\theta \\ M_{bw} &= \Sigma C_n \cos n\left(\frac{2\,\pi}{3} + \theta\right), \\ M_{cu} &= \Sigma A_n \cos n\left(\frac{2\,\pi}{3} + \theta\right), \\ M_{av} &= \Sigma B_n \cos n\left(\frac{2\,\pi}{3} + \theta\right) \\ M_{cv} &= \Sigma B_n \cos n\left(\frac{4\,\pi}{3} + \theta\right), \\ M_{aw} &= \Sigma C_n \cos n\left(\frac{4\,\pi}{3} + \theta\right), \\ M_{bu} &= \Sigma A_n \cos n\left(\frac{4\,\pi}{3} + \theta\right) \end{aligned}\right\} \quad (78)$$

Case III. Primary Windings Unequal but Otherwise Symmetrical, Secondary Winding Equal and Symmetrical.

$$\left.\begin{aligned} M_{au} &= \Sigma A_n \cos n\,\theta, M_{bv} = \Sigma B_n \cos n\,\theta,\ M_{cw} = \Sigma C_n \cos n\theta \\ M_{bw} &= \Sigma B_n \cos n\left(\frac{2\,\pi}{3} + \theta\right), \\ M_{cu} &= \Sigma C_n \cos n\left(\frac{2\,\pi}{3} + \theta\right) \\ M_{av} &= \Sigma A_n \cos n\left(\frac{2\,\pi}{3} + \theta\right) \\ M_{cv} &= \Sigma C_n \cos n\left(\frac{4\,\pi}{3} + \theta\right), \\ M_{aw} &= \Sigma A_n \cos n\left(\frac{4\,\pi}{3} + \theta\right), \\ M_{bu} &= \Sigma B_n \cos n\left(\frac{4\,\pi}{3} + \theta\right) \end{aligned}\right\} \quad (79)$$

Case IV. Same as Case II except in addition to inequality Secondary Windings are Displaced from Symmetry by angles α_1 α_2 and α_3 whose sum is zero.

$$\left.\begin{aligned}
M_{au} &= \Sigma\,(A_n \cos\alpha_1 \cos n\,\theta + A_n \sin\alpha_1 \sin n\,\theta) \\
M_{bv} &= \Sigma\,(B_n \cos\alpha_2 \cos n\,\theta + B_n \sin\alpha_2 \sin n\,\theta) \\
M_{cw} &= \Sigma\,(C_n \cos\alpha_3 \cos n\,\theta + C_n \sin\alpha_3 \sin n\,\theta) \\
M_{bw} &= \Sigma\left\{C_n \cos\alpha_3 \cos n\left(\frac{2\,\pi}{3} + \theta\right) + C_n \sin\alpha_3 \sin n\left(\frac{2\,\pi}{3} + \theta\right)\right\} \\
M_{cu} &= \Sigma\left\{A_n \cos\alpha_1 \cos n\left(\frac{2\,\pi}{3} + \theta\right) + A_n \sin\alpha_1 \sin n\left(\frac{2\,\pi}{3} + \theta\right)\right\} \\
M_{av} &= \Sigma\left\{B_n \cos\alpha_2 \cos n\left(\frac{2\,\pi}{3} + \theta\right) + B_n \sin\alpha_2 \sin n\left(\frac{2\,\pi}{3} + \theta\right)\right\} \\
M_{cv} &= \Sigma\left\{B_n \cos\alpha_2 \cos n\left(\frac{4\,\pi}{3} + \theta\right) + B_n \sin\alpha_2 \sin n\left(\frac{4\,\pi}{3} + \theta\right)\right\} \\
M_{aw} &= \Sigma\left\{C_n \cos\alpha_3 \cos n\left(\frac{4\,\pi}{3} + \theta\right) + C_n \sin\alpha_3 \sin n\left(\frac{4\,\pi}{3} + \theta\right)\right\} \\
M_{bu} &= \Sigma\left\{A_n \cos\alpha_1 \cos n\left(\frac{4\,\pi}{3} + \theta\right) + A_n \sin\alpha_1 \sin n\left(\frac{4\,\pi}{3} + \theta\right)\right\}
\end{aligned}\right\} \quad (80)$$

Case V. Same as Case III except that the Primary Windings are Unsymmetrically disposed with respect to one another as well as being unequal.

$$
\left.
\begin{aligned}
M_{au} &= \Sigma\,(A_n \cos\alpha_1 \cos n\,\theta + A_n \sin\alpha_1 \sin n\,\theta)\\
M_{bv} &= \Sigma\,(B_n \cos\alpha_2 \cos n\,\theta + B_n \sin\alpha_2 \sin n\,\theta)\\
M_{cw} &= \Sigma\,(C_n \cos\alpha_3 \cos n\,\theta + C_n \sin\alpha_3 \sin n\,\theta)\\
M_{bw} &= \Sigma\,\left\{B_n \cos\alpha_2 \cos n\left(\frac{2\,\pi}{3} + \theta\right)\right.\\
&\qquad \left. + B_n \sin\alpha_2 \sin n\left(\frac{2\,\pi}{3} + \theta\right)\right\}\\
M_{cu} &= \Sigma\,\left\{A_n \cos\alpha_1 \cos n\left(\frac{2\,\pi}{3} + \theta\right)\right.\\
&\qquad \left. + A_n \sin\alpha_1 \sin n\left(\frac{2\,\pi}{3} + \theta\right)\right\}\\
M_{av} &= \Sigma\,\left\{C_n \cos\alpha_3 \cos n\left(\frac{2\,\pi}{3} + \theta\right)\right.\\
&\qquad \left. + C_u \sin\alpha_3 \sin n\left(\frac{2\,\pi}{3} + \theta\right)\right\}\\
M_{cv} &= \Sigma\,\left\{C_n \cos\alpha_3 \cos n\left(\frac{4\,\pi}{3} + \theta\right)\right.\\
&\qquad \left. + C_n \sin\alpha_3 \sin n\left(\frac{4\,\pi}{3} + \theta\right)\right\}\\
M_{aw} &= \Sigma\,\left\{A_n \cos\alpha_1 \cos n\left(\frac{4\,\pi}{3} + \theta\right)\right.\\
&\qquad \left. + A_n \sin\alpha_1 \sin n\left(\frac{4\,\pi}{3} + \theta\right)\right\}\\
M_{bu} &= \Sigma\,\left\{B_n \cos\alpha_2 \cos n\left(\frac{4\,\pi}{3} + \theta\right)\right.\\
&\qquad \left. + B_n \sin\alpha_2 \sin n\left(\frac{4\,\pi}{3} + \theta\right)\right\}
\end{aligned}
\right\} \quad (81)
$$

The expressions for dissymmetry in both windings and for unsymmetrically wound coils, etc., are more complicated and will be dealt with in the Appendix.

The impedances $Z_{aa}\ Z_{bb}$, etc., $Z_{au}\ Z_{bv}$, etc., are functions of $M_{aa}\ M_{bb}$, etc., $M_{au}\ M_{bv}$, etc., and the resistances of the system. The component of e. m. f. proportional to the current due to mutual impedance is so small that it may generally be neglected so that Z_{au} becomes $\frac{d}{dt} M_{au}$, $Z_{bv} = \frac{d}{dt} M_{bv}$ and so forth.

If the secondary winding is rotating at an angular velocity α, θ in equation (**55**) becomes $\alpha\, t$ and the operators Z_{aa}, etc. operate on such products as $\epsilon^{j\alpha t}\, \check{I}_{u1}$ $\epsilon^{j\alpha t}\, \check{I}_{u2}$ where $\check{I}_{u1}$ and $\check{I}_{u2}$ are the variables.

The following relations will be found useful in the application of the method in actual examples.

If D denotes the operator $\frac{d}{d\,x}$ and $\varphi\,(Z)$ is a rational algebraic function of Z

$$\left.\begin{aligned} &\varphi\,(D)\; e^{ax} = \varphi\,(a)\; e^{ax} \\ &\varphi\,(D)\;\{e^{ax}\;\; X\} = e^{ax}\;\; \varphi\,(D + a)\; X \\ &\varphi\,(D)\; Y = e^{ax}\, \varphi\,(D + a)\; Y\, e^{-ax} \end{aligned}\right\} \qquad (\mathbf{82})$$

Where X and Y may be any function of x.

Star and Delta e.m.fs. and Currents in Terms of Symmetrical Components

It has been shown in the preceding portion of this paper that the e. m. fs. $\check{E}_a$ $\check{E}_b$ and $\check{E}_c$ and the currents $\check{I}_a$ $\check{I}_b$ and $\check{I}_c$ whatever their distortion, may be represented by the sum of symmetrical systems of e. m. fs. or currents so that the two expressions

$$\left.\begin{aligned} S\,(\check{E}_a) &= S^0\, \check{E}_{a0} + S^1\, \check{E}_{a1} + S^2\, \check{E}_{a2} \\ S\,(\check{I}_a) &= S^0\, \check{I}_{a0} + S^1\, \check{I}_{a1} + S^2\, \check{I}_{a2} \end{aligned}\right\} \qquad (\mathbf{83})$$

completely define these two systems.

If we take the delta e. m. fs. and currents corresponding to $S^0\, \check{E}_{a0}$, $S^1\, \check{E}_{a1}$ and $S^2\, \check{E}_{a2}$, $S^1\, \check{I}_{a1}$, $S^2\, \check{I}_{a2}$, we have, since $\check{E}_{bc1}$ leads $\check{E}_{a1}$ by $\frac{\pi}{2}$ and $\check{E}_{bc2}$ lags behind $\check{E}_{a2}$ by the same angle

$$\left.\begin{aligned} S^0\, \check{E}_{bc0} &= 0 \\ S^1\, \check{E}_{bc1} &= j\, \sqrt{3}\, S^1\, \check{E}_{a1} \\ S^2\, \check{E}_{bc2} &= -\, j\, \sqrt{3}\, S^2\, \check{E}_{a2} \\ S^0\, \check{I}_{bc0} &= \text{indeterminate from } S\,(\check{I}_a) \\ S^1\, \check{I}_{bc1} &= \;\; j\, \frac{1}{\sqrt{3}}\, S^1\, \check{I}_{a1} \\ S^2\, \check{I}_{bc2} &= -j\, \frac{1}{\sqrt{3}}\, S^2\, \check{I}_{a2} \end{aligned}\right\} \qquad (\mathbf{84})$$

And therefore if we take $\check{E}_{ab}$ as the principal vector

$$\left.\begin{aligned} S^0\,\check{E}_{ab0} &= 0 \\ S^1\,E_{ab1} &= \quad j\,a\,s^1\,\sqrt{3}\,\check{E}_{a1} \\ S^2\,\check{E}_{ab2} &= -j\,a^2\,s^2\,\sqrt{3}\,\check{E}_{a2} \\ S\,(\check{E}_{ab}) &= S^1\,\check{E}_{ab1} + S^2\,\check{E}_{ab2} \end{aligned}\right\} \quad (\mathbf{85})$$

The last equation of group (**85**) when expanded gives

$$\left.\begin{aligned} \check{E}_{ab} &= j\,\sqrt{3}\,(a\,\check{E}_{a1} - a^2\,\check{E}_{a2}) \\ \check{E}_{bc} &= j\,\sqrt{3}\,(\check{E}_{a1} - \check{E}_{a2}) \\ \check{E}_{ca} &= j\,\sqrt{3}\,(a^2\,\check{E}_{a1} - a\,\check{E}_{a2}) \end{aligned}\right\} \quad (\mathbf{86})$$

which may also be obtained direct from (**83**) by means of the relations

$$\begin{aligned} \check{E}_{ab} &= \check{E}_b - \check{E}_a \\ \check{E}_{bc} &= \check{E}_c - \check{E}_b \\ \check{E}_{ca} &= \check{E}_a - \check{E}_0 \end{aligned}$$

Similarly

$$\begin{aligned} S^0\,\check{I}_{ab} &= \text{indeterminate from } S\,(\check{I}_a) \\ S^1\,\check{I}_{ab1} &= j\,a\,\frac{1}{\sqrt{3}}\,\check{I}_{a1} \\ S^2\,\check{I}_{ab2} &= j\,a^2\,\frac{1}{\sqrt{3}}\,\check{I}_{a2} \qquad (\mathbf{87}) \\ S\,(\check{I}_{ab}) &= S^0\,\check{I}_{ab0} + S^1\,I_{ab1} + S^2\,\check{I}_{ab2} \end{aligned}$$

with similar expression for $\check{I}_{ab}$ $\check{I}_{bc}$ and $\check{I}_{ea}$ which may be verified by means of the relations

$$\begin{aligned} \check{I}_a &= \check{I}_{ca} - \check{I}_{ab} + \check{I}_{a0} \\ \check{I}_b &= \check{I}_{ab} - \check{I}_{bc} + \check{I}_{a0} \\ \check{I}_c &= \check{I}_{bc} - \check{I}_{ca} + \check{I}_{a0} \end{aligned}$$

Conversely to (**84**) we have the following relations

$$\left.\begin{aligned}
S^0\,\check{E}_{a0} &= \text{indeterminate from } S\,(\check{E}_{ab}) \\
S^1\,\check{E}_{a1} &= -\,j\frac{1}{\sqrt{3}}\,S^1\,\check{E}_{bc1} = -j\,\frac{a^2}{\sqrt{3}}\,S^1\,\check{E}_{ab1} \\
S^2\,\check{E}_{a2} &= \quad j\,\frac{1}{\sqrt{3}}\,S^2\,E_{bc2} = \quad j\,\frac{a}{\sqrt{3}}\,S^2\,E_{ab2} \\
S^0\,\check{I}_{a0} &= \text{indeterminate from } S\,(\check{I}_{ab}) \\
S^1\,\check{I}_{a1} &= -\,j\,\sqrt{3}\,S^1\,I_{bc1} = -\,j\,a^2\,\sqrt{3}\,S^1\,\check{I}_{ab1} \\
S^2\,\check{I}_{a2} &= \quad j\,\sqrt{3}\,S^2\,\check{I}_{bc} = \quad j\,a\,\sqrt{3}\,S^2\,\check{I}_{ab2}
\end{aligned}\right\}\qquad \textbf{(88)}$$

It will be sufficient in order to illustrate the application of the principle of symmetrical coordinates to simple circuits to apply it to a few simple cases of transformer connections before proceeding to its application to rotating polyphase systems to which it is particularly adapted.

Unsymmetrical Bank of Delta-Delta Transformers Operating on a Symmetrical Circuit Supplying a Balanced System

Let the transformer effective impedances be $Z_{AB}\ Z_{BC}\ Z_{CA}$ and let the secondary load currents be $\check{I}_U\ \check{I}_V$ and $\check{I}_W$ and let the star load impedance be Z. One to one ratio of transformation will be assumed, and the effect of the magnetizing current will be neglected. The symmetrical equations are

$$\left.\begin{aligned}
O &= S^0\,(Z_{AB0}\,\check{I}_{ab0} + Z_{AB2}\,\check{I}_{ab1} + Z_{AB1}\,\check{I}_{ab2}) \\
S^1\,\check{E}_{uv1} &= S^1\,\check{E}_{ab1} - S^1\,(Z_{AB1}\,\check{I}_{ab0} + Z_{AB0}\,\check{I}_{ab1} + Z_{AB2}\,\check{I}_{ab2}) \\
S^2\,\check{E}_{uv2} &= 0 - S^2\,(Z_{AB2}\,\check{I}_{ab0} + Z_{AB1}\,\check{I}_{ab1} + Z_{AB0}\,\check{I}_{ab2}) \\
S^0\,\check{I}_{u0} &= 0 \\
S^1\,Z\,\check{I}_{u1} &= \check{E}_{u1} \\
S^2\,Z\,\check{I}_{u2} &= \check{E}_{u2}
\end{aligned}\right\}\qquad \textbf{(89)}$$

Since the transformation ratio is unity and the effects of magnetizing currents are negligible $S^1\,\check{I}_{ab1} = S^1\,\check{I}_{UV1}$, $S^2\,\check{I}_{ab2} = S^2\,\check{I}_{UV2}$. And therefore by means of the relations (**85**), the last two equations may be expressed

$$\left.\begin{aligned} S^1\,\check{E}_{uv1} &= S^1\,3\,Z\,\check{I}_{ab1} \\ S^2\,\check{E}_{uv2} &= S^2\,3\,Z\,\check{I}_{ab2} \end{aligned}\right\} \quad (90)$$

in other words, the symmetrical components appear in the secondary as independent systems, $3\,Z$ being the delta load impedance equivalent to the star impedance Z.

Substituting from (**90**) in the second and third equation and eliminating $\check{I}_{ab0}$ by means of the first equation, and we have

$$\left.\begin{aligned} S^1\,\check{E}_{ab1} &= S^1\left\{\left(3\,Z + Z_{AB0} - \frac{Z_{AB1}\,Z_{AB2}}{Z_{AB0}}\right)\check{I}_{ab1} + \left(Z_{AB2} - \frac{Z^2_{AB1}}{Z_{AB0}}\right)\check{I}_{ab2}\right\} \\ S^2\,O &= S^2\left\{\left(Z_{AB1} - \frac{Z^2_{AB2}}{Z_{AB0}}\right)\check{I}_{ab1} + \left(3\,Z + Z_{AB0} - \frac{Z_{AB1}\,Z_{AB2}}{Z_{AB0}}\right)\check{I}_{ab2}\right\} \end{aligned}\right\} \quad (91)$$

which, when S^1 and S^2 are removed, give two simultaneous equations in $\check{I}_{ab1}$ and $\check{I}_{ab2}$.

A modification of the problem may occur even when the load impedances are symmetrical, as they may have symmetrical but unequal impedances Z_1 and Z_2, to the two components $\check{I}_{U1}$ and $\check{I}_{U2}$ respectively, as in the case of a load consisting of a symmetrical rotating machine. The equations corresponding to (**89**), (**90**) and (**91**) then become

$$\left.\begin{aligned} O &= S^0\,(Z_{AB0}\,\check{I}_{ab0} + Z_{AB2}\,\check{I}_{ab1} + Z_{AB1}\,\check{I}_{ab2}) \\ S^1\,\check{E}_{uv1} &= S^1\,\check{E}_{ab1} - S^1\,(Z_{AB1}\,\check{I}_{ab0} + Z_{AB0}\,\check{I}_{ab1} + Z_{AB2}\,\check{I}_{ab2}) \\ S^2\,E_{uv2} &= O - S^2\,(Z_{AB2}\,\check{I}_{ab0} + Z_{AB1}\,\check{I}_{ab1} + Z_{AB0}\,\check{I}_{ab2}) \\ S^0\,\check{I}_{u0} &= O \\ S^1\,Z_1\,\check{I}_{u1} &= \check{E}_{u1} \\ S^2\,Z_2\,\check{I}_{u2} &= \check{E}_{u2} \end{aligned}\right\} \quad (92)$$

$$\left.\begin{aligned} S^1\,\check{E}_{uv1} &= S^1\,3\,Z_1\,\check{I}_{ab1} \\ S^2\,\check{E}_{uv2} &= S^2\,3\,Z_2\,\check{I}_{ab2} \end{aligned}\right\} \quad (93)$$

$$\left.\begin{aligned} S^1 \check{E}_{ab1} &= S^1 \left\{ \left(3 Z_1 + Z_{AB0} - \frac{Z_{AB1}\, Z_{AB2}}{Z_{AB0}} \right) \check{I}_{ab1} + \right. \\ &\quad \left. \left(Z_{AB2} - \frac{Z^2_{AB1}}{Z_{AB0}} \right) \check{I}_{ab2} \right\} \\ S^2\, O &= S^2 \left\{ \left(Z_{AB1} - \frac{Z^2_{AB1}}{Z_{AB0}} \right) \check{I}_{ab1} + \right. \\ &\quad \left. \left(3\, Z_2 + Z_{AB0} - \frac{Z_{AB1}\, Z_{AB2}}{Z_{AB0}} \right) \check{I}_{ab2} \right\} \end{aligned} \right\} \quad (94)$$

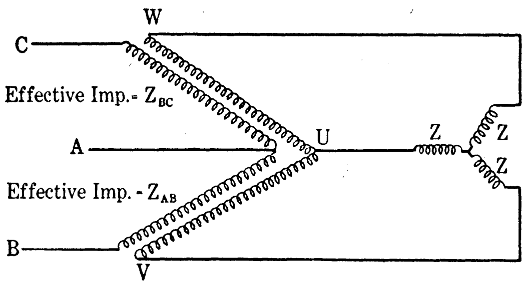

FIG. 4—OPEN DELTA OR V CONNECTION.

In an open delta system $Z_{AB1} = Z_{AB2} = Z_{AB0} - Z_{AB}$ the transformers in this case being both the same. Equation (**91**) becomes in this particular case where Z_{AB0} is infinite

$$\left.\begin{aligned} S^1 \check{E}_{ab1} &= S^1 \left\{ (3\, Z + 2\, Z_{AB})\, \check{I}_{ab} + Z_{AB}\, \check{I}_{ab2} \right\} \\ S^2\, O &= S^2 \left\{ Z_{AB}\, \check{I}_{ab1} + (3\, Z + 2\, Z_{AB})\, \check{I}_{ab2} \right\} \end{aligned} \right\} \quad (95)$$

and we have

$$\check{I}_{ab0} = -\check{I}_{ab1} - I_{ab2} \quad (96)$$

Similarly, instead of (**94**) we have

$$\left.\begin{aligned} S^1 \check{E}_{ab1} &= S \left\{ (3\, Z_1 + 2\, Z_{AB})\, \check{I}_{ab1} + Z_{AB}\, \check{I}_{ab2} \right\} \\ S^2\, O &= S^2 \left\{ Z_{AB}\, \check{I}_{ab1} + (3\, Z_2 + 2\, Z_{AB})\, \check{I}_{ab2} \right\} \end{aligned} \right\} \quad (97)$$

The secondary voltages are obtained from (**90**) and (**93**) for this latter case.

The solution of (**95**) gives

$$\left.\begin{aligned}\check{I}_{ab1} &= \frac{3\,Z_1 + 2\,Z_{AB}}{(3\,Z_1 + 3\,Z_{AB})\;(3\,Z_1 + Z_{AB})}\;\check{E}_{ab}\\ \check{I}_{ab2} &= -\;\frac{Z_{AB}}{(3\,Z_1 + 3\,Z_{AB})\;(3\,Z_1 + Z_{AB})}\;\check{E}_{ab}\\ \check{I}_{ab0} &= -\;\frac{1}{3\,Z_1 + 3\,Z_{AB}}\;\check{E}_{ab}\end{aligned}\right\}\qquad(\mathbf{98})$$

And we have

$$\left.\begin{aligned}S^1\,\check{I}_{a1} &= S^1\;\frac{3\,Z_1 + 2\,Z_{AB}}{3\,(Z_1 + Z_{AB})\left(Z_1 + \frac{Z_{AB}}{3}\right)}\;\check{E}_a\\ S^2\,\check{I}_{a2} &= S^2\;\frac{Z_{AB}}{3\,(Z_1 + Z_{AB})\left(Z_1 + \frac{Z_{AB}}{3}\right)}\;\check{E}_b\end{aligned}\right\}\qquad(\mathbf{99})$$

And therefore

$$\left.\begin{aligned}\check{I}_a &= \frac{\check{E}_a}{Z_1 + \frac{Z_{AB}}{3}} + \frac{\frac{1}{3}\,Z_{AB}}{(Z_1 + Z_{AB})\left(Z_1 + \frac{Z_{AB}}{3}\right)}\check{E}_{ab}\\ \check{I}_b &= \frac{\check{E}_b}{Z_1 + \frac{Z_{AB}}{3}} - \frac{\frac{1}{3}\,Z_{AB}}{(Z_1 + Z_{AB})\left(Z_1 + \frac{Z_{AB}}{3}\right)}\check{E}_{ab}\\ \check{I}_c &= \frac{\check{E}_c}{Z_1 + \frac{Z_{AB}}{3}}\end{aligned}\right\}\qquad(\mathbf{100})$$

Three-Phase System with Symmetrical Waves Having Harmonics

We may express $\check{E}_a$ in the following form:

$$\left.\begin{aligned}\check{E}_a &= E_1\,\epsilon^{jwt} + E_2\,\epsilon^{j2wt} + E_3\,\epsilon^{j\,3wt} + \;.\;.\\ &= \Sigma\,E_n\,\epsilon^{j\,nwt}\end{aligned}\right\}\qquad(\mathbf{101})$$

where E_n is in general a complex number.

If the system is symmetrical three-phase $\check{E}_b$ is obtained by displacing the complete wave by the angle $-\dfrac{2\,\pi}{3}$ or

$$\check{E}_b = \epsilon^{-j\frac{2\pi}{3}} E_1\, \epsilon^{jwt} + \epsilon^{-j\frac{4\pi}{3}} E_2\, \epsilon^{j2wt} + \epsilon^{-j\frac{6\pi}{3}} E_3\, \epsilon^{j3wt} + ..$$

$$E_c = \epsilon^{j\frac{2\pi}{3}} E_1\, \epsilon^{jwt} + \epsilon^{j\frac{4\pi}{3}} E_2\, \epsilon^{j2wt}\ \ \epsilon^{j\frac{6\pi}{3}} E_3\, \epsilon^{j3wt} + ..$$

or since $\epsilon^{-j\frac{2\pi}{3}} = a^2$, $\epsilon^{j\frac{2\pi}{3}} = a$ etc.

$$\left.\begin{aligned} \check{E}_a &= E_1\, \epsilon^{jwt} + E_2\, \epsilon^{j2wt} + E_3\, \epsilon^{j3wt} + \ \ldots \\ \check{E}_b &= a^2 E_1\, \epsilon^{jwt} + a\, E_2\, \epsilon^{j2wt} + E_3\, \epsilon^{j3wt} + \ \ldots \\ \check{E}_c &= a\, E_1\, \epsilon^{jwt} + a^2 E_2\, \epsilon^{j2wt} + E_3\, \epsilon^{j3wt} + \ \ldots \end{aligned}\right\} \quad (102)$$

or

$$\left.\begin{aligned} S(\check{E}_a) = S^0 &\{E_3\, \epsilon^{j3wt} + E_6 \epsilon^{j6wt} + E_9\, \epsilon^{j9wt} + \ \ldots \} \\ + S^1 &\{E_1\, \epsilon^{jwt} + E_4\, \epsilon^{j4wt} + E_7\, \epsilon^{j7wt} + \ \ldots\} \\ + S^2 &\{E_2\, \epsilon^{j2wt} + E_5\, \epsilon^{j5wt} + E_8\, \epsilon^{j8wt} + \ \ldots \} \end{aligned}\right\} \quad (103)$$

$$S(\check{E}_a) = S^0 \Sigma\, (E_{3n}\, \epsilon^{j3nwt}) + S^1 \Sigma\, (E_{3n-2}\, \epsilon^{j(3n-2)wt}) + S^2 \Sigma\, (E_{3n-1}\, \epsilon^{j(3n-1)wt}) \quad (104)$$

This shows that a symmetrical three-phase system having harmonics is made up of positive and negative phase sequence harmonic systems and others of zero phase sequence, that is to say of the same phase in all windings, which comprise the group of third harmonics. These facts are not generally appreciated though they are factors that may have an appreciable influence in the performance of commercial machines. It should be particularly noted that in three-phase generators provided with dampers the fifth, eleventh, seventeenth, and twenty-third harmonics produce currents in the damper windings.

In dealing with the complex variable it will be convenient to use for the amplitude the root mean square value for each harmonic. When instantaneous values are required, the real part of the complex variable should be multiplied by $\sqrt{2}$. In the remainder of this paper this convention will be adopted.

Power Representation in Symmetrical Co-ordinates

Since the power in an alternating-current system is also a harmonically varying scalar quantity, it may therefore be represented in the same manner as the current or electromotive force, that is to say by a complex variable which we shall denote by $(P + j\,Q) + (P_H + j\,Q_H)$, $P + j\,Q$ being the mean value, is the term of the complex variable of zero frequency, P representing the real power and Q the wattless power, $\sqrt{P^2 + Q^2}$ will be the volt-amperes.

The value of the complex variable $(P + j\,Q) + (P_H + j\,Q_H)$ may be taken as

$$(P + j\,Q) + (P_H + j\,Q_H) = \check{E}\,\hat{I} + \check{E}\,\check{I} \tag{105}$$

with the provision that for all terms having negative indices the conjugate terms must be substituted, these terms being present in the product $\hat{E}\,\check{I} + \hat{E}\,\hat{I}$, which is the conjugate of the product (**105**). A similar rule holds good for the symmetrical vector system

$$\left.\begin{aligned} S(\check{E}_a) &= S^0\,\check{E}_{a0} + S^1\,\check{E}_{a1} + \quad . \quad . \quad . \quad + S^{n-1}\,\check{E}_{a(n-1)} \\ S(\check{I}_a) &= S^0\,\check{I}_{a0} + S^1\,\check{I}_{a1} + \quad . \quad . \quad . \quad + S^{n-1}\,\check{I}_{a(n-1)} \end{aligned}\right\} \tag{106}$$

The conjugate of $S\,\check{I}_a$ is

$$S(\hat{I}_a) = S^0\,\hat{I}_{a0} + S^{(n-1)}\,\hat{I}_{a1} + \quad . \quad . \quad . \quad + S^1\,\hat{I}_{a(n-1)} \tag{107}$$

and the power is represented by

$$(P + P_b) + j\,(Q + Q_b) = \Sigma\,\{S(\check{E}_a)\,S(\hat{I}_a) + S(\check{E}_a)\,S(\check{I}_a)\} \tag{108}$$

with the same provision for terms having negative indices. The sign Σ signifies that all the products in each sequence are added together.

$$\left.\begin{aligned} \Sigma\,\{S(\hat{I}_a)\,S(\check{E}_a)\} = \Sigma\,S^0\,\{&\hat{I}_{a0}\,\check{E}_{a0} + \hat{I}_{a1}\,\check{E}_{a1} + \quad . \quad . \\ &+ \hat{I}_{a(n-1)}\,\check{E}_{a(n-1)}\} \\ + \Sigma\,S^1\,\{&\hat{I}_{a0}\,\check{E}_{a1} + \hat{I}_{a1}\,\check{E}_{a2} + \hat{I}_{a2}\,\check{E}_{a3} + \quad . \quad . \quad . \\ &+ \hat{I}_{a(n-1)}\,\check{E}_{a0}\} \\ + \Sigma\,S^2\,\{&\hat{I}_{a0}\,\check{E}_{a2} + \hat{I}_{a1}\,\check{E}_{a3} + \hat{I}_{a2}\,\check{E}_{4} + \quad . \quad . \quad . \\ &+ \check{I}_{a(n-1)}\,\check{E}_{a1}\} \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad &. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ + \Sigma\,S^{(n-1)}\,\{&\hat{I}_{a0}\,\check{E}_{a(n-1)} + \hat{I}_{a1}\,\check{E}_{a0} + \quad . \quad . \quad . \\ &+ \hat{I}_{a(n-1)}\,\check{E}_{a(n-2)}\} \end{aligned}\right\} \tag{109}$$

The terms prefixed by S^1, S^2, S^3 . . . $S^{(n-1)}$ all become zero and since S^0 becomes n

$$\Sigma\, S(\hat{I}_a)\, S(\check{E}_a) = n\,\{\hat{I}_{a0}\,\check{E}_{a0} + \hat{I}_{a1}\,\check{E}_{a1} + \ldots + \hat{I}_{a(n-1)}\,\check{E}_{a(n-1)}\} \quad (110)$$

In a similar manner it may be shown that

$$\Sigma\, S(\check{I}_a)\, S(\check{E}_a) = n\,\{\check{I}_{a0}\,\check{E}_{a0} + \check{I}_{a1}\,\check{E}_{a(n-1)} + \check{I}_{a2}\,\check{E}_{a(n-2)} + \ldots + \check{I}_{a(n-1)}\,\check{E}_{a1}\} \quad (111)$$

and therefore

$$\begin{aligned}(P + j\,Q) + (P_H + j\,Q_H) = {} & n\,\{\hat{I}_{a0}\,\check{E}_{a0} + \hat{I}_{a1}\,\check{E}_{a1} + \ldots + \hat{I}_{a(n-1)}\,\check{E}_{a(n-1)}\} \\ & + n\,\{\check{I}_{a0}\,\check{E}_{a0} + \check{I}_{a1}\,\check{E}_{a(n-1)} + \ldots + \check{I}_{a(n-1)}\check{E}_{a1}\}\end{aligned} \quad (112)$$

For a three-phase system the expression reduces to

$$\begin{aligned}(P + j\,Q) + (P_H + j\,Q_b) = {} & 3\,(\hat{I}_{a0}\,\check{E}_{a0} + \hat{I}_{a1}\,\check{E}_{a1} + \hat{I}_{a2}\,\check{E}_{a2}) \\ & + 3\,(\check{I}_{a0}\,\check{E}_{a0} + \check{I}_{a1}\,\check{E}_{a2} + \check{I}_{a2}\,\check{E}_{a1})\end{aligned} \quad (113)$$

In the above expression $P + P_H$ is the value of the instantaneous power on the system, P being the mean value and P_H the harmonic portion. When the currents are simple sine waves, Q may be interpreted to be the mean wattless power of the circuit or the sum of the wattless volt-amperes of each circuit. In rotating machinery since the coefficients of mutual induction may be complex harmonic functions of the angular velocity, this is not strictly true for all cases; but if the effective impedances to the various frequencies of the component currents be used, it will be found to be equal to the mean wattless volt-amperes of the system with each harmonic considered independent.

In a balanced polyphase system P_H and Q_H both become zero.

The instantaneous power is a quantity of great importance in polyphase systems because the instantaneous torque is proportional to it and this quantity enters into the problem of vibrations which is at times a matter of great importance, especially when caused by unbalanced e. m. fs. A system of currents and e. m. fs. may be transformed to balanced polyphase by means of transformers alone, provided that the value of P_H is zero, while on the other hand polyphase power cannot be supplied from a pulsating power system without means for supplying the necessary storage to make a continuous flow of energy.

PART II

Application of the Method to Rotating Polyphase Networks

The methods of determining the constants Z_a Z_u, M, etc., of co-axial cylindrical networks is taken up in Appendix I of this paper. It will be assumed that the reader has familiarized himself with these quantities and understands their significance. We shall first consider the case of symmetrically wound machines taking up the simple cases first and proceeding to more complex ones.

Symmetrically Wound Induction Motor Operating on Unsymmetrical Polyphase Circuit

Denoting the pole-pitch angle by π let the synchronous angular velocity be ω_0 and let the angular slip velocity be ω_1. And let $S^1\,E_{a1}$ $S^2\,E_{a2}$ be the symmetrical components of impressed polyphase e. m. f. Let R_a be the primary resistance and R_u the secondary resistance. The primary self-inductance being M_{aa}, that of the secondary being M_{uu} and corresponding symbols being used to denote the mutual inductances between the different pairs of windings. Then by means of (**39**), (**40**), (**56**) and (**57**)

$$\left.\begin{aligned}
S^1\,\check{E}_{a1} &= S^1\left\{R_a\,\check{I}_{a1} + 1\tfrac{1}{2}\,M_{aa}\,\frac{d}{d\,t}\,\check{I}_{a1}\right.\\
&\qquad\left. + 1\tfrac{1}{2}\,M_{au}\,\frac{d}{d\,t}\epsilon^{j(w_0 - w_1)t}\,\check{I}_{u1}\right\}\\
S^2\,\check{E}_{a2} &= S^2\left\{R_a\,\check{I}_{a2} + 1\tfrac{1}{2}\,M_{aa}\,\frac{d}{d\,t}\,\check{I}_{a2}\right.\\
&\qquad\left. + 1\tfrac{1}{2}\,M_{au}\,\frac{d}{d\,t}\epsilon^{-j(w_0 - w_1)t}\,\check{I}_{u2}\right\}\\
S^1\,\check{E}_{u1} = O &= S^1\left\{R_u\,\check{I}_{u1} + 1\tfrac{1}{2}\,M_{uu}\,\frac{d}{d\,t}\,\check{I}_{u1}\right.\\
&\qquad\left. + 1\tfrac{1}{2}\,M_{au}\,\frac{d}{d\,t}\epsilon^{-j(w_0 - w_1)t}\,\check{I}_{a1}\right\}\\
S^2\,\check{E}_{u2} = O &= S^2\left\{R_u\,\check{I}_{u2} + 1\tfrac{1}{2}\,M_{uu}\,\frac{d}{d\,t}\,\check{I}_{u2}\right.\\
&\qquad\left. + 1\tfrac{1}{2}\,M_{au}\,\frac{d}{d\,t}\,\epsilon^{j(w_0 - w_1)t}\,\check{I}_{a2}\right\}
\end{aligned}\right\} \quad (\mathbf{114})$$

denote $1\frac{1}{2}\ M_{aa}$ by L_a and $1\frac{1}{2}\ M_{uu}$ by L_u, $1\frac{1}{2}\ M_{au}$ by M, the equations (**1**) become

$$\left.\begin{aligned}
S^1\,\check{E}_{a1} &= S^1\left\{\left(R_a + L_a\frac{d}{d\,t}\right)\check{I}_{a1}\right.\\
&\qquad\left. + M\frac{d}{d\,t}\,\epsilon^{j\,(w_0 - w_1)\,t}\,\check{I}_{u1}\right\}\\
S^2\,\check{E}_{a2} &= S^2\left\{\left(R_a + L_a\frac{d}{d\,t}\right)\check{I}_{a2}\right.\\
&\qquad\left. + M\frac{d}{d\,t}\,\epsilon^{-j\,(w_0 - w_1)\,t}\,\check{I}_{u2}\right\}\\
S^1\,\check{E}_{u1} = O &= S^1\left\{\left(R_u + L_u\frac{d}{d\,t}\right)\check{I}_{u1}\right.\\
&\qquad\left. + M\frac{d}{d\,t}\,\epsilon^{-j\,(w\ - w_1)\,t}\,\check{I}_{a1}\right\}\\
S^2\,\check{E}_{u2} = O &= S^2\left\{\left(R_u + L_u\frac{d}{d\,t}\right)\check{I}_{u2}\right.\\
&\qquad\left. + M\frac{d}{d\,t}\,\epsilon^{j\,(w_0 - w_1)\,t}\,\check{I}_{a2}\right\}
\end{aligned}\right\} \qquad (\mathbf{115})$$

From the last two equations we have

$$\check{I}_{u1} = -\frac{M\,\frac{d}{d\,t}}{R_u + L_u\,\frac{d}{d\,t}}\,\epsilon^{-j\,(w_0 - w_1)\,t}\,\check{I}_{a1} \qquad (\mathbf{116})$$

$$\check{I}_{u2} = -\frac{M\,\frac{d}{d\,t}}{R_u + L_u\,\frac{d}{d\,t}}\,\epsilon^{j\,(w_0 - w_1)\,t}\,\check{I}_{a2} \qquad (\mathbf{117})$$

Substituting these in the first two equations of (**115**) we obtain

$$S'\,\check{E}_{a1} = S^1\left[\left(R_a + L_a\,\frac{d}{d\,t}\right) - \frac{M^2\,\frac{d}{d\,t}\left\{\frac{d}{d\,t} - j\,(w_0 - w_1)\right\}}{R_u + L_u\left\{\frac{d}{d\,t} - j\,(w_0 - w_1)\right\}}\right]\check{I}_{a1} \qquad (\mathbf{118})$$

$$S^2 \check{E}_{a2} = S^2 \left[\left(R_a + L_a \frac{d}{d\,t} \right) - \frac{M^2 \frac{d}{d\,t} \left\{ \frac{d}{d\,t} + (j\,w_0 - w_1) \right\}}{R_u + L_u \left\{ \frac{d}{d\,t} - j\,(w_0 - w_1) \right\}} \right] \check{I}_{a2} \qquad (119)$$

If $\check{E}_{a1} = E_{a1}\,\epsilon^{jwt}$ and $\check{E}_{a2} = E_{a2}\,\epsilon^{jwt}$ the solution for $\check{I}_{a1}$ and $\check{I}_{a2}$ will be

$$\check{I}_{a1} = \frac{\check{E}_{a1}}{Z_1} \qquad (120)$$

$$\check{I}_{a2} = \frac{\check{E}_{a2}}{Z_2} \qquad (121)$$

Where

$$Z_1 = R_a + j\,w_0\,L_a + \frac{w_0\,w_1\,M^2}{R_u^2 + w_1^2\,L_u^2}\,(R_u - j\,w_1\,L_u) \qquad (122)$$

$$Z_2 = R_a + j\,w_0\,L_a + \frac{w_0\,(2\,w_0 - w_1)\,M^2}{R_u^2 + (2\,w_0 - w_1)^2\,L_u^2}\,\{R_u - j\,(2\,w_0 - w_2)\,L_u\} \qquad (123)$$

The impedances Z_1 and Z_2 will be found more convenient to use in the form

$$Z_1 = (R_a + K_1^2\,R_u) + j\,w_0\,(L_a - K_1^2\,L_u) + \frac{w_0 - w_1}{w_1}\,K_1^2\,R_u \qquad (124)$$

$$Z_2 = (R_a + K_2^2\,R_u) + j\,w_0\,(L_a - K_2^2\,L_u) - \frac{w_0 - w_1}{Z\,w_0 - w_1}\,K_2^2\,R_u \qquad (125)$$

Where, as we will see later, K_1^2 and K_2^2 are the squares of the transformation ratios between primary and secondary currents of positive and negative phase sequence.

The last real term in each expression is the virtual resistance due to mechanical rotation and when combined with the mean square current represents mechanical work performed, the positive sign representing work performed and the negative sign work required.

Thus, for example, to enable the currents $S^2\,\check{I}_{a2}$ to flow, the mechanical work $3\,I_{a2}^2\,\frac{w_0 - w_1}{2\,w_0 - w_1}\,K_1^2\,R_u$ must be applied to the shaft of the motor.

The phase angles of the symmetrical systems $S^1 \check{I}_{a1}$ $S^2 \check{I}_{a2}$ with respect to their impressed e. m. f., $S^1 \check{E}_{a1}$ and $S^2 \check{E}_{a2}$ are given by these impedances so that the complete solution of the primary circuit is thus obtained.

The secondary currents are given by equations (**116**) and (**117**) and are

$$\check{I}_{u1} = - \frac{j\, w_1\, M}{R_u + j\, w_1\, L_u}\, I_{a1}\, \epsilon^{j w_1 t} = \check{K}_1\, I_{a1} \epsilon^{j w_1 t} \tag{126}$$

$$\check{I}_{u2} = - \frac{j\,(2\, w_0 - w_1)\, M}{R_u + j\,(2\, w_0 - w_1)\, L_u}\, I_{a2}\, \epsilon^{j(2 w_0 - w_1)t} = \check{K}_2\, I_{a2}\, \epsilon^{j(2 w_0 - w_1)t} \tag{127}$$

In the results just given, M is not the maximum value of mutual inductance between a pair of primary and secondary windings but is equal to the total mutual inductance due to a current passing through the two coils W and V through the coil

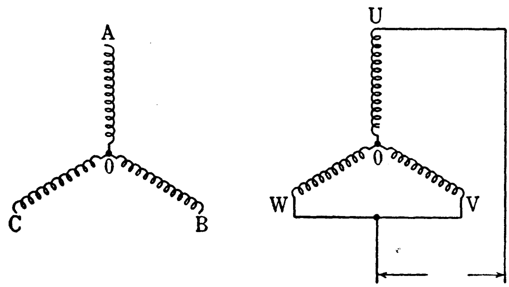

FIG. 5

U as shown in the sketch Fig 5 and the winding "A" when A and U have their planes of symmetry coincident.

Where the windings are symmetrical the induced e. m. f. is independent of the division of current between W and V, but this quantity must not be used in unsymmetrical windings, or with star windings having a neutral point connection so that $\check{I}_{a0}$ is not zero.

The appearance of M in this equation follows from the equation

$$\check{I}_u + \check{I}_v + \check{I}_w = O$$

so that

$$\check{I}_u = - (\check{I}_v + \check{I}_w)$$

The power delivered by the motor is

$$P_o = 3 \left\{ \frac{w_0 - w_1}{w_1}\, K_1^2\, I_{a1}^2\, R_u - \frac{w_0 - w_1}{2\, w_0 - w_1}\, K_2^2\, I_{a2}^2\, R_u \right\} \tag{128}$$

The copper losses are given by

$$P_L = 3\{I_{a1}^2 (R_p + K_1^2 R_u) + I_{a2}^2 (R_p + K_2^2 R_u)\} \quad \textbf{(129)}$$

The iron loss is independent of the copper loss and power output. The iron loss and windage may be taken as

$$P_F = \text{Iron loss and windage} \quad \textbf{(130)}$$

The power input as

$$P_I = P_O + P_L + P_F \quad \textbf{(131)}$$

The mechanical power output is P_O less friction and windage losses.

$$\text{Torque} = 3\left\{\frac{1}{w_1} K_1^2 I_{a1}^2 R_u - \frac{1}{2\,w_0 - w_1} K_2^2 I_{a2}^2 R_u\right\} \times 10^7 \text{ dyne-cm.} \quad \textbf{(132)}$$

The kv-a. at the terminals is

$$\sqrt{P_I^2 + Q_I^2} = \text{The effective value of } 3\,(E_{a1}\, I_{a1} + E_{a2}\, I_{a2}) \quad \textbf{(133)}$$

This last result may be arrived at in the following way

$$\left.\begin{aligned} S\,(\check{E}_a) &= S^1\,\check{E}_{a1} + S^2\,\check{E}_{a2}) \\ S\,(\hat{I}_a) &= S^2\,\hat{I}_{a1} + S^1\,\hat{I}_{a2} \end{aligned}\right\} \quad \textbf{(134)}$$

Since $S^2 \hat{I}_{a1}$ is conjugate to $S^1\,\check{I}_{a1}$, etc.

The product of $\check{E}_{sa}$ and $\hat{I}_{sa}$ is the power product of the two vectors, $S\,(\check{E}_a)$ and $S\,(\check{I}_a)$ and omits the harmonic variation as a double frequency quantity, the average wattless appears as an imaginary non-harmonic quantity.

$$P_I + j\,Q_I = \Sigma\,(S^0\,\check{E}_{a1}\,\hat{I}_{a1} + S^0\,\check{E}_{a2}\,\hat{I}_{a2} + S^1\,\check{E}_{a2}\,\hat{I}_{a1} + S^2\,\check{E}_{a1}\,\hat{I}_{a2}) \quad \textbf{(135)}$$

The S^1 and S^2 products have zero values, since the sum of the terms of each sequence is zero, hence—

$$P_I + j\,Q_I = 3\,(\check{E}_{a1}\,\hat{I}_{a1} + \check{E}_{a2}\,\hat{I}_{a2}) \quad \textbf{(136)}$$

$$\sqrt{P_I^2 + Q_I^2} = \text{The effective value of } 3\,(\check{E}_{a1}\,\hat{I}_{a1} + \check{E}_{a2}\,\hat{I}_{a2}) \quad \textbf{(137)}$$

The solution for the general case of symmetrical motor operating on an unsymmetrical circuit is not of as much interest as certain special cases depending thereon. Some of the most important of these will be taken up in the following paragraphs.

Case I. Single-phase e. m. f. impressed across one phase of three-phase motor.

Assuming the single-phase voltage to be $\check{E}_{bc}$ impressed across the terminals $B\ C$. The known data or constraints are

$$\left.\begin{aligned} \check{E}_{bc} &= j\sqrt{3}\,(\check{E}_{a1} - \check{E}_{a2}) \\ \check{I}_a &= O,\ \check{I}_b = -\check{I}_c \end{aligned}\right\} \quad (138)$$

and therefore

$$\check{I}_{a1} = -\check{I}_{a2} \quad (139)$$

$$\frac{\check{E}_{a1}}{Z_1} = -\frac{\check{E}_{a2}}{Z_2}$$

$$\check{E}_{a2} = -\frac{Z_2}{Z_1}\check{E}_{a1} \quad (140)$$

Substituting in (**138**)

$$\left.\begin{aligned} \check{E}_{a1} &= -j\frac{\check{E}_{bc}}{\sqrt{3}} \cdot \frac{Z_1}{Z_1 + Z_2} \\ \check{E}_{a2} &= j\frac{\check{E}_{bc}}{\sqrt{3}} \cdot \frac{Z_2}{Z_1 + Z_2} \end{aligned}\right\} \quad (141)$$

and therefore

$$\left.\begin{aligned} \check{I}_{a1} &= -j\frac{\check{E}_{bc}}{\sqrt{3}} \cdot \frac{1}{Z_1 + Z_2} \\ \check{I}_{a2} &= j\frac{\check{E}_{bc}}{\sqrt{3}} \cdot \frac{1}{Z_1 + Z_2} \end{aligned}\right\} \quad (142)$$

Since $\check{I}_b = \check{I}_{b1} + \check{I}_{b2} = a^2\,\check{I}_{a1} + a\,\check{I}_{a2}$

$$\check{I}_b = -I_c = -\frac{\check{E}_{bc}}{Z_1 + Z_2} \quad (143)$$

$$P_0 = \left(\frac{w_0 - w_1}{w_1}K_2^2 R_u - \frac{w_0 - w_1}{2\,w_0 - w_1}K_2^2 R_u\right)I_0^2 \quad (144)$$

$$P_1 + j\,Q_1 = I_b^2\,(Z_1 + Z_2) + P_F \quad (145)$$

The power factor is obtained from (**145**) by the formula

$$\cos\alpha = \frac{P_1}{\sqrt{P_1^2 + Q_1^2}} \quad (146)$$

Substituting from (**142**) in equation (**126**) and (**127**) of the general case we obtain for the secondary currents

$$\left.\begin{aligned} \check{I}_{u1} &= -j\,\check{K}_1 \frac{E_{bc}}{Z_1 + Z_2}\,\epsilon^{jw_1 t} \\ \check{I}_{u2} &= \quad j\,\check{K}_2 \frac{E_{bc}}{Z_1 + Z_2}\,\epsilon^{j(2w_0 - w_1)t} \end{aligned}\right\} \qquad (\mathbf{147})$$

Many unsymmetrical cases may be expressed in terms of the operation of coupled symmetrical motors operating on symmetrical systems. This is invariably the case with symmetrical polyphase motors operating on single-phase circuits. Since the physical interpretations are useful in impressing the facts on ones memory they will be given whenever they appear to be useful.

Equations (**141**) and (**142**) show that single-phase operation is exactly equivalent to operating two duplicate motors in series with a symmetrical polyphase e. m. f. $S^1\,E_{ab}$ impressed across one motor, the other being connected in series with the first but with phase sequence reversed, the two motors being directly coupled.

Case II. B and C connected together e. m. f. impressed across A B.

The data given by the conditions of constraint are

$$\left.\begin{aligned} \check{E}_{ab} &= -\,\check{E}_{ca} \\ \check{E}_{bc} &= O = j\,\sqrt{3}\,(\check{E}_{a1} - \check{E}_{a2}) \end{aligned}\right\} \qquad (\mathbf{148})$$

We therefore have

$$\check{E}_{a1} = \check{E}_{a2} = -\,\frac{\check{E}_{ab}}{3} \qquad (\mathbf{149})$$

and

$$\left.\begin{aligned} \check{I}_{a1} &= -\,\frac{\check{E}_{ab}}{3\,Z_1} \\ \check{I}_{a2} &= -\,\frac{\check{E}_{ab}}{3\,Z_2} \end{aligned}\right\} \qquad (\mathbf{150})$$

The remainder follows from the general solution and need not be repeated here.

(**150**) shows that a motor operated in this manner is the exact equivalent in all respects to two duplicate mechanically coupled polyphase motors, one of which has sequence reversed, operating in parallel on a balanced three-phase circuit of e. m. f. $S^1\,\dfrac{E_{ab}}{\sqrt{3}}$.

The secondary currents follow from substitution of (**150**) in equations (**126**) and (**127**) of the general case.

Case III. B and C connected together by the terminals of a balance coil, the impressed e. m. f. E_{AD} *applied between A and the middle point of the balance coil. Resistance and reactance of balance coil negligible.*

The data furnished by the connection in this case is

$$\check{I}_b = \check{I}_c = -\frac{\check{I}_a}{2} \qquad \textbf{(151)}$$

and therefore

$$\check{I}_{a1} = \frac{\check{I}_a - a\,\frac{\check{I}_a}{2} - a^2\,\frac{\check{I}_a}{2}}{3} = \frac{\check{I}_a}{2}$$

$$\check{I}_{a2} = \check{I}_{a1} = \frac{\check{I}_a}{2}$$

We therefore have

$$\left.\begin{aligned} \check{E}_{a1} &= \frac{Z_1\,\check{I}_a}{2} \\ \check{E}_{a2} &= \frac{Z_2\,\check{I}_a}{2} \end{aligned}\right\} \qquad \textbf{(152)}$$

we have

$$\check{E}_{ab} = j\,\sqrt{3}\,(a\,\check{E}_{a1} - a^2\,\check{E}_{a2})$$

$$= j\,\sqrt{3}\,\frac{\check{I}_a}{2}\,(a\,Z_1 - a^2\,Z_2)$$

$$\check{E}_{bc} = j\,\sqrt{3}\,\frac{\check{I}_a}{2}\,(Z_1 - Z_2)$$

$$\left.\begin{aligned} \check{E}_{ad} &= \left(\check{E}_{ab} + \frac{\check{E}_{bc}}{2}\right) \\ &= j\,\sqrt{3}\,\frac{\check{I}_a}{2}\,\{(a + \tfrac{1}{2})\,Z_1 - (a^2 + \tfrac{1}{2})\,Z_2\} \\ &= -\tfrac{3}{4}\,\check{I}_a\,(Z_1 + Z_2) \end{aligned}\right\} \qquad \textbf{(153)}$$

and therefore,

$$\check{I}_a = -1\tfrac{1}{3}\,\frac{\check{E}_{ad}}{Z_1 + Z_2} \qquad \textbf{(154)}$$

$$P_0 = \tfrac{3}{4} \left\{ \frac{w_0 - w_1}{w_1} K_1^2 - \frac{w_0 - w_1}{2 w_0 - w_1} K_2^2 \right\} I_a^2 R_s \qquad \textbf{(155)}$$

$$P_1 + j Q_1 = \tfrac{3}{4} I_a^2 (Z_1 + Z_2) + P_F \qquad \textbf{(156)}$$

$$\cos \alpha = \frac{P_1}{\sqrt{P_1^2 + Q_1^2}} \qquad \textbf{(157)}$$

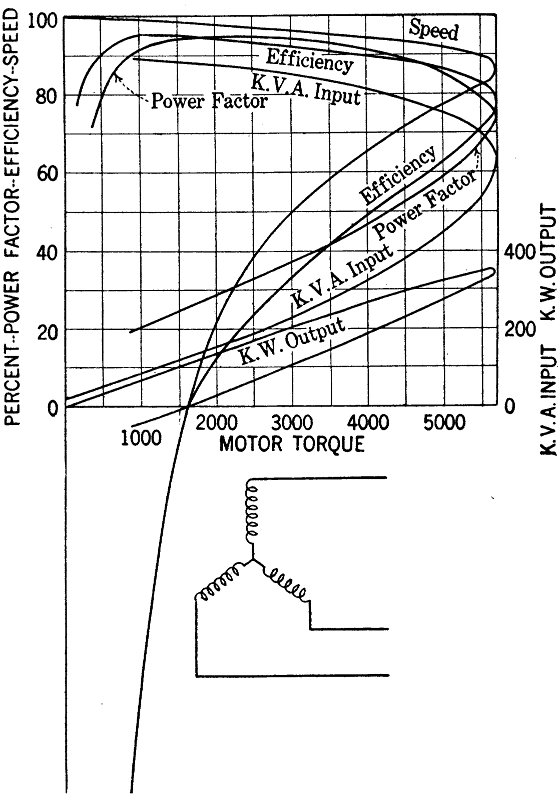

FIG. 6—CHARACTERISTICS OF THREE-PHASE INDUCTION MOTOR—BALANCED THREE-PHASE

Evidently (**155**), (**156**) and (**157**) are identical to (**144**), (**145**) and (**146**) if I_a is equal to $I_b \div \frac{\sqrt{3}}{2}$. This will be the case if the value of $E_{ad} = \frac{\sqrt{3}}{2}$ times that of E_{bc}. The total heating of the motors will be the same in each case but the heating in one phase for Case III will be one-third greater than for Case I.

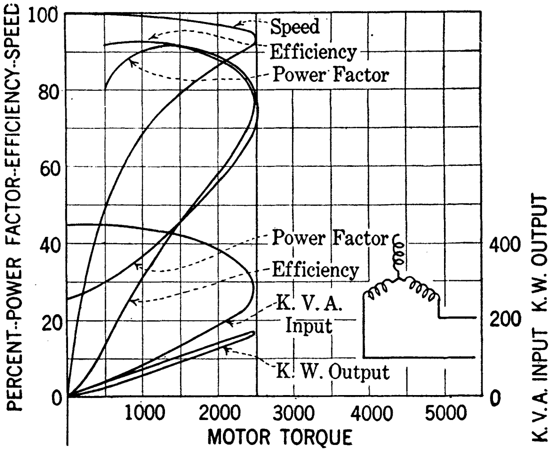

FIG. 7—CHARACTERISTICS OF THREE-PHASE INDUCTION MOTOR—SINGLE-PHASE OPERATION —ONE LEAD OPEN

This method of operation is therefore, as far as total losses, etc. are concerned, the exact counterpart of two polyphase

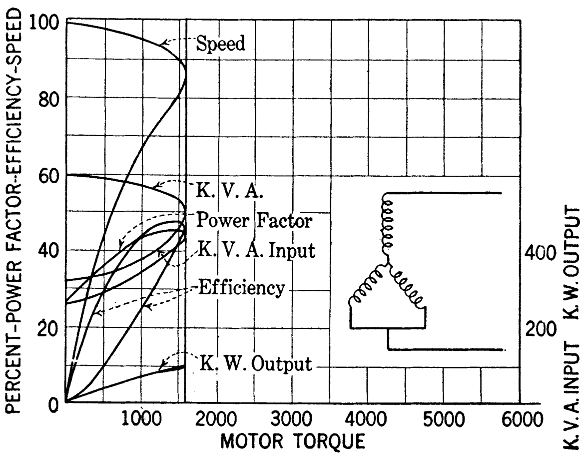

FIG. 8—CHARACTERISTICS OF THREE-PHASE INDUCTION MOTOR—SINGLE-PHASE OPERATION IN MANNER INDICATED

motors connected in series with shafts mechanically connected, one of which has its phase sequence reversed.

Figs. 6, 7 and 8 show characteristic curves of a three-phase induction motor operating respectively on a symmetrical circuit, according to Case I and according to Case II.

Synchronous Machinery

The Symmetrical Three-Phase Generator Operating on Unsymmetrically Loaded Circuit

The polyphase salient pole generator is not strictly a symmetrical machine, the exciting winding is not a symmetrical polyphase winding and it therefore sets up unsymmetrical trains of harmonics in exactly the same way as they are set up in an induction motor with unsymmetrical secondary winding. These cases will therefore be taken up later on. A three-phase generator may however be wound with a distributed polyphase winding to serve both as exciting and damper winding and if properly connected will be perfectly symmetrical. Such a machine will differ from an induction motor only in respect to the fact that it operates in synchronism and has internally generated symmetrical e. m. fs. which we will denote by $S^1 \check{E}_{a1}$, $S^2 \check{E}_{a2}$ the negative phase sequence component being zero; an e. m. f. $S^0 \check{E}_{a0}$ may exist but since in all the connections that will be considered there will be no neutral connection its value may be ignored. If the load impedances be Z_a', Z_b' and Z_c' they may be expressed by

$$Z_{sa}^1 = S^0 Z_{a0}^1 + S^1 Z_{a1}' + S^2 Z_{a2}'$$

and the equations of the generator will be

$$\left.\begin{aligned} S^1 \check{E}_{a1} &= S^1 \left\{ \left(R_a + L_a \frac{d}{d\,t}\right) \check{I}_{a1}' + Z_{a0}' I_{a1}' \right. \\ &\qquad \left. + Z_{a2}' I_{a2}' + M \frac{d}{d\,t} \epsilon^{j w_0 t} \check{I}_{u1}' \right\} \\ O &= S^2 \left\{ \left(R_a + L_a \frac{d}{d\,t}\right) \check{I}_{a2}' + Z_{a0}' I_{a2}' \right. \\ &\qquad \left. + Z_{a1}^1 \check{I}_{a1}' + M \frac{d}{d\,t} \epsilon^{-j w_0 t} \check{I}_{u2}' \right\} \\ O &= \left(R_u + L_u \frac{d}{d\,t}\right) \check{I}_{u1}' + M \frac{d}{d\,t} \epsilon^{-j w_0 t} \check{I}_{a1}' \\ O &= \left(R_u + L_u \frac{d}{d\,t}\right) \check{I}_{u2}' + M \frac{d}{d\,t} \epsilon^{j w_0 t} \check{I}_{a2}' \end{aligned}\right\} \quad (158)$$

The last two equations give

$$\left.\begin{aligned} I_{u1}' &= -\frac{M\dfrac{d}{d\,t}}{R_u + L_u\dfrac{d}{d\,t}}\,\epsilon^{-jw_0t}\,\check{I}_{a1}' \\ I_{u2}' &= -\frac{M\dfrac{d}{d\,t}}{R_u + L_u\dfrac{d}{d\,t}}\,\epsilon^{jw_0t}\,I_{a2}' \end{aligned}\right\} \quad (\mathbf{159})$$

which on substitution in the first two equations of (**158**) give the equations

$$\left.\begin{aligned} &\left\{R_a + L_a\frac{d}{d\,t} - \frac{M^2\dfrac{d}{d\,t}\left(\dfrac{d}{d\,t} - j\,w_0\right)}{R_u + L_u\left(\dfrac{d}{d\,t} - j\,w_0\right)}\right\}\check{I}_{a1}' \\ &\qquad + Z_{a0}{}^1\,\check{I}_{a1}' + Z_{a2}'\,\check{I}_{a2}' = \check{E}_{a1} \\ &Z_{a1}{}^1\,\check{I}_{a1}' + \left\{R_a + L_a\frac{d}{d\,t}\right. \\ &\left. - \frac{M^2\dfrac{d}{d\,t}\left(\dfrac{d}{d\,t} + j\,w_0\right)}{R_u + L_u\left(\dfrac{d}{d\,t} + j\,w_0\right)}\right\}\check{I}_{a2}' + Z_{a0}{}^1\,\check{I}_{a2}' = 0 \end{aligned}\right\} \quad (\mathbf{160})$$

or if

$$\check{E}_{a1} = E_{a1}\,\epsilon^{jw_0t} \quad (\mathbf{161})$$

the impedances Z_{a0}, Z_{a1}, Z_{a2} become ordinary impedance for an electrical angular velocity w_0 and equations (**160**) become

$$\left.\begin{aligned} &(R_a + j\,w\,L_a + Z_{a0}{}^1)\,\check{I}_{a1}' + Z_{a2}'\,\check{I}_{a}'{}_2 = \check{E}_{a1} \\ &Z_{a1}'\,\check{I}_{a1}' + \{Z_{a0}' + (R_a + K_2{}^2\,R_u) + j\,2\,w_0\,(L_a - K_2{}^2\,L_u) - \\ &\qquad\qquad \tfrac{1}{2}\,K_2{}^2\,R_u\}\,\check{I}_{a2}' = O \end{aligned}\right\} \quad (\mathbf{162})$$

It is apparent that in the generator the impedances

$$R_a + j\,w_0\,L_a = Z_1'$$

and $\{(R_a + K_2{}^2\,R_u) + j\,2\,w_0\,(L_a - K_2{}^2\,L_u) - \tfrac{1}{2}\,K^2 R_u\} = Z_2'$

take the place of Z_1 and Z_2 in the symmetrical induction motor operating on an unsymmetrical circuit, and we may express equation (**162**)

$$\left.\begin{aligned}(Z_{a0}' + Z_1')\, \check{I}_{a1}' + Z_{a2}'\, \check{I}_{a2}' &= \check{E}_{a1}\\ Z_a'\, \check{I}_{a1}' + (Z_{a0}' + Z_2')\, I_{a2}' &= O\end{aligned}\right\} \quad (\mathbf{163})$$

which gives

$$\check{I}_{a2}' = -\frac{Z_{a1}'}{Z_{a0} + Z_2'}\, \check{I}_{a1}'$$

$$\check{I}_{a1}' = \frac{\check{E}_{a1}}{(Z_{a0}' + Z_1') - \dfrac{Z_{a1}'\, Z_{a2}'}{Z_{a0}' + Z_2'}}$$

Or in more symmetrical form

$$\left.\begin{aligned}\check{I}_{a1}' &= \frac{Z_{a0}' + Z_2'}{(Z_{a0}' + Z_1')\,(Z_{a0}' + Z_2') - Z_{a1}'\, Z_{a2}'}\, \check{E}_{a1}\\ \check{I}_{a2}' &= -\frac{Z_{a1}^1}{(Z_{a0}' + Z_1')\,(Z_{a0}' + Z_2') - Z_{a1}'\, Z_{a2}'}\, \check{E}_{a1}\end{aligned}\right\} \quad (\mathbf{164})$$

From (**159**) we have for the damper currents

$$\left.\begin{aligned}\check{I}_{u1}' &= O \text{ if } R_u > O\\ \check{I}_{u2}' &= -\check{K}_2\, I_{a2}\, \epsilon^{j\,2w_0t}\\ &\text{where } \check{K}_2 = j\,\frac{2\,w_0\,M}{R_u + j\,2\,w_0\,L_u}\end{aligned}\right\} \quad (\mathbf{165})$$

A particular case of interest is when the load is a Synchronous Motor or Induction Motor with unsymmetrical line impedances in series—Equation (**163**) becomes

$$\left.\begin{aligned}(Z_{a0}' + Z_1' + Z_1)\, \check{I}_{a1}' + Z_{a2}'\, I_{a2}' &= \check{E}_{a2}\\ Z_{a1}\, \check{I}_{a1}' + (Z_{a0}' + Z_2' + Z_2)\, I_{a2}' &= O\\ I_{a1}' &= \frac{Z_{a0}' + Z_2' + Z_2}{(Z_{a0}'+Z_1'+Z_1)\;(Z_{a0}'+Z_2'+Z_2) - Z_{a1}\,Z_{a2}}\, \check{E}_{a1}\\ I_{a2}' &= \frac{Z_{a1}}{(Z_{a0}'+Z_1'+Z_1)\;(Z_{a0}'+Z_2'\;Z_2) - Z_{a1}\,Z_{a2}}\, \check{E}_{a1}\end{aligned}\right\} \quad (\mathbf{166})$$

An important case is that of a generator feeding into a symmetrical motor and an unsymmetrical load. Let the motor currents be

$\check{I}_a$, $\check{I}_b$, $\check{I}_c$, those of the load I_a', I_b', I_c' and the load impedances Z_a', Z_b', Z_c'. The equations of this system will be

$$\left.\begin{aligned} S^1\,\check{E}_{a1} &= S^1\,\{Z_1'\,(\check{I}_{a1}+\check{I}_{a1}')+Z_{a0}'\,\check{I}_{a1}'+Z_{a2}'\,\check{I}_{a2}'\} \\ S^1\,\check{E}_{a1} &= S^1\,\{Z_1'\,(\check{I}_{a1}+\check{I}_{a1}')+Z_1\,\check{I}_{a1}\} \\ S^2\,O &= S^2\,\{Z_2'\,(\check{I}_{a2}+\check{I}_{a2}')+Z_{a0}'\,\check{I}_{a2}+Z_{a1}'\,\check{I}_{a1}'\} \\ S^2\,O &= S^2\,\{Z_2'\,(\check{I}_{a2}+\check{I}_{a2}')+Z_2\,\check{I}_{a2}\} \end{aligned}\right\} \quad (167)$$

Or, omitting the sequence symbols and re-arranging—

$$\left.\begin{aligned} \check{E}_{a1} &= Z_1'\,\check{I}_{a1}+(Z_1'+Z_{a0}')\,\check{I}_{a1}'+Z_{a2}'\,\check{I}_{a2}' \\ \check{E}_{a1} &= (Z_1'+Z_1)\,\check{I}_{a1}+Z_1'\,\check{I}_{a1}^1 \\ O &= Z_2'\,\check{I}_{a2}+Z_{a1}'\,\check{I}_{a1}^1+(Z_2'+Z_{a0}')\,\check{I}_{a2}' \\ O &= (Z_2'+Z_2)\,\check{I}_{a2}+Z_2'\,\check{I}_{a2}' \end{aligned}\right\} \quad (168)$$

These equations can be further simplified as follows:

$$\left.\begin{aligned} O &= (Z_2'+Z_2)\,\check{I}_{a2}+Z_2'\,\check{I}_{a2}' \\ O &= -\,Z_2\,\check{I}_{a2}+Z_{a1}'\,\check{I}_{a1}'+Z_{a0}'\,\check{I}_{a2}' \\ O &= -\,Z_1\,\check{I}_{a1}+Z_{a0}'\,\check{I}_{a0}'+Z_{a2}'\,I_{a2}' \\ \check{E}_{a1} &= (Z_1'+Z_1)\,\check{I}_{a1}+Z_1'\,\check{I}_{a1}' \end{aligned}\right\} \quad (169)$$

A set of simultaneous equations which may be easily solved.

The Single-Phase Generator is an Important Case of the Three-Phase Generator Operated on an Unbalanced Load

Let the impedance of the single-phase load be Z and let us suppose it to be made up of three star connected impedances

$$Z_a' = 3\,Z_x + \frac{Z}{2}$$

$$Z_b' = \frac{Z}{2}$$

$$Z_c' = \frac{Z}{2}$$

the value of Z_x in the limit being infinity. Then we have

$$\left.\begin{aligned} Z_{a0}' &= Z_x + \frac{Z}{2} \\ Z_{a1}' &= Z_x \\ Z_{a2}' &= Z_x \end{aligned}\right\} \qquad (\mathbf{170})$$

Equation (**164**) in the limit when Z_x becomes infinite reduces to

$$\begin{aligned} \check{I}_{a1}' &= \frac{\check{E}_{a1}}{Z + Z_1' + Z_2'} \\ \check{I}'_{a2} &= -\frac{\check{E}_{a1}}{Z + Z_1' + Z_2'} \end{aligned} \qquad (\mathbf{171})$$

The single-phase load being across the phase $B\ C$, the single-phase current I will therefore be equal to $\check{I}_c$ or

$$\left.\begin{aligned} \check{I} &= \frac{j\sqrt{3}\,\check{E}_{a1}}{Z + Z_1' + Z_2'} \\ &= \frac{\check{E}_{bc}}{Z + Z_1' + Z_2'} \end{aligned}\right\} \qquad (\mathbf{172})$$

$$\left.\begin{aligned} \check{I}_{u1} &= O \text{ if } R_u > O \\ \check{I}_{u2} &= -j\frac{1}{\sqrt{3}}\check{K}_2\,\check{I}\,\epsilon^{ju_0t} \\ \check{I}_{u2} &= -\frac{j\,\check{K}_2}{\sqrt{3}}\,I\,\epsilon^{j2u_0t} \end{aligned}\right\} \qquad (\mathbf{173})$$

$\check{I}_{u2}$ is double normal frequency

$$\left.\begin{aligned} P_I + j\,Q_I &= 3\,I^2\,Z \\ P_L + j\,Q_L &= 3\,I^2\,(Z_1' + Z_2') \\ (P + j\,Q) + (P_H + j\,Q_H) &= 3\,\check{E}_{bc}\,(\hat{I} + \check{I}) \end{aligned}\right\} \qquad (\mathbf{174})$$

In the case of the generally unbalanced three-phase load

$$\left.\begin{aligned} P_1 + j\,Q_1 &= 3\,\{(I_{a1}^2 + I_{a2}^2)\,Z_{a0}' \\ &\qquad + \check{I}_{a1}\,\hat{I}_{a2}\,Z_{a2}' + \hat{I}_{a1}\,\check{I}_{a2}\,Z_{a1}'\} \\ P_L + j\,Q_L &= 3\,\{I_{a1}^2\,Z_1' + I_{a2}^2\,Z_2'\} \\ (P + j\,Q) + (P_H + j\,Q_H) &= 3\,\check{E}_{a1}\,(\check{I}_{a2} + \hat{I}_{a2}) \end{aligned}\right| \qquad (\mathbf{175})$$

When the generator has harmonics in its wave form equations (**162**) must be written

$$\left.\begin{aligned} &(R_a + j\,w\,L_a + Z_{a0}')\,\check{I}_{a1}' + Z_{a2}'\,I_{a2}' = \check{E}_{a1} \\ &\quad Z_{a1}'\,I_{a1}' + \{Z_{a0}' + (R_a + K_2^2\,R_u) \\ &\qquad + j\,2\,w\,(L_a - K_2^2\,L_u) - \tfrac{1}{2}\,a^2\,R_u\}\,I_{a2}' = \check{E}_{a2} \end{aligned}\right\} \quad \textbf{(176)}$$

Where $\check{E}_{a1}$ is finite, $\check{E}_{a2}$ is zero and vice versa, the frequencies being different in each case, we have therefore a solution for each frequency depending on the phase and amplitude and phase sequence of the e. m. f. of this frequency generated. Of course the values of Z_1' and Z_2' change with each frequency on account of the change in the reactance with frequency, and a value must be taken for w conforming with the frequency of the harmonic under consideration.

Symmetrical Synchronous Motor, Synchronous Condenser, Etc.

As in the case of the generator, the synchronous motor has two impedances, one to the positive phase sequence current of a given frequency and the other to the negative phase sequence current of the same frequency. But, since there is no quantity in the positive phase sequence impedance corresponding to the virtual resistance which indicates mechanical work in an induction motor, its equivalent is furnished by the excitation of the field. Let us denote the e. m. f. due to the field excitation by $S^1\,\check{E}_{a1}'$ assuming it to be for the present a simple harmonic three-phase system. Let P_0 be the output of the motor which will include the windage and iron losses assumed to be constant. Then for the synchronous motor on a balanced circuit of e. m. f. $S^1\,\check{E}_{a1}$ we have

$$S^1\,\check{E}_{a1} = S^1\,\{\check{I}_{a1}'(R_a' + j\,w\,L_a') + \check{E}_{a1}'\} \qquad \textbf{(177)}$$

$$S^0\,\check{E}_{a1}\,\hat{I}_{a1} = S^0\left\{I_{a1}^2\,(R_a' + j\,w\,L_a') + \frac{P_0}{3} - j\,\frac{Q_0}{3}\right\} \qquad \textbf{(178)}$$

Where Q_0 is the imaginary part of the product, $\check{E}_{a1}'\,\hat{I}_{a1}$. (**178**) reduces to

$$E_{a1}\,I_{a1}\cos\alpha = I_{a1}^2\,R_a' + \frac{P_0}{3} \qquad \textbf{(179)}$$

Where $\cos\alpha$ is the required operating power factor. Solving for I_{a1}

$$I_{a1} = \frac{E_{a1} \cos \alpha}{2 R_a^1} \left\{ 1 \pm \sqrt{1 - \frac{4 R_a' P_0}{3 E_{a1}^2 \cos^2 \alpha}} \right\} \qquad (180)$$

$$\check{I}_{a1} = \check{E}_{a1} \frac{\cos \alpha}{2 R_{a1}} \left\{ 1 \pm \sqrt{1 - \frac{4 R_{a1}' P_0}{3 E_{a1}^2 \cos^2 \alpha}} \right\} (\cos \alpha - j \sin \alpha) \qquad (181)$$

The apparent impedance of the motor is

$$\frac{2 R_1 \sec \alpha}{1 \pm \sqrt{1 - \frac{4 P_0}{3 E_a^2 \cos^2 \alpha}}} (\cos \alpha + j \sin \alpha) \qquad (182)$$

and

$$\check{E}_{a1}^1 = \check{E}_{a1} \left[1 - \frac{\cos \alpha}{2 R_a^1} \left\{ 1 \pm \sqrt{1 - \frac{4 R_a' P_0}{3 E_{a1}^2 \cos^2 \alpha}} \right\} (\cos \alpha - j \sin \alpha) (R_a' + j w L_a') \right] \qquad (183)$$

The same equations apply to the case of the synchronous condenser with the difference that the mechanical work is that required to overcome the iron and windage losses only.

If we take

$$\left. \begin{aligned} \check{E}_{a1} &= E_{a1} (\cos \alpha + j \sin \alpha) \epsilon^{j w_0 t} = (A_1 + j B_1) \epsilon^{j w_0 t} \\ \check{E}_{a1}' &= (A_1' + j B_1') \epsilon^{j w_0 T} \end{aligned} \right\} \qquad (184)$$

we have

$$\check{I}_{a1} = \frac{A_1}{2 R_a^1} \left(1 \pm \sqrt{1 - \frac{4 R_a' P_0}{3 A_1^2}} \right) \epsilon^{j w_0 t} \qquad (185)$$

$$A_1' = \frac{A_1}{2} \left(1 \pm \sqrt{1 - \frac{4 R_a' P_0}{3 A_1^2}} \right) \epsilon^{j w_0 t} \qquad (186)$$

$$B_1' = \left\{ B_1 - \frac{j w L_a' A_1}{2 R_a'} \left(1 \pm \sqrt{1 - \frac{4 R_a' P_0}{3 A_1^2}} \right) \right\} \epsilon^{j w_0 t} \qquad (187)$$

Since α may be a positive or negative angle, the sine may be positive or negative for a positive cosine, and therefore the power factor will be leading or lagging accordingly as B_1 is negative or positive respectively. The double signs throughout are due to the fact that for any given load and power factor there are always two theoretically possible running conditions. However, since we are concerned only with that one which will give the max. operating efficiency, that is the condition that gives I_{a1} the lesser value, for a given value of P_0 the equations may be written

$$\left.\begin{aligned} \check{I}_{a1} &= \frac{A_1}{2\,R_a^1}\left(1 - \sqrt{1 - \frac{4\,R_a'\,P_0}{3\,A_1^2}}\right)\epsilon^{jw_0t} \\ A_1' &= \frac{A_1}{2}\left(1 + \sqrt{1 - \frac{4\,R_a'\,P_0}{3\,A_1^2}}\right)\epsilon^{jw_0t} \\ B_1' &= \left\{B_1 - \frac{j\,w_0\,L_a'\,A_1}{2\,R_a'}\left(1 - \sqrt{1 - \frac{4\,R_a'\,P_0}{3\,A_1^2}}\right)\right\} \end{aligned}\right\} \quad \textbf{(188)}$$

And corresponding values for (**180**), (**181**), (**182**) and (**183**) may be obtained by omitting the positive sign in these equations.

Another condition of operation is obtained by inspection of (**180**), due to the fact that I_{a1} must be a real quantity

$$\frac{4\;R_a'\;P_0}{3\,E_{a1}^2\cos^2\alpha}\ \text{must be} > 1 \quad \textbf{(189)}$$

this is the condition of stability. In terms of (**184**) it becomes

$$\frac{4\,R_a'\,P_0}{3\,A_1^2}\ \text{must be} > 1 \quad \textbf{(190)}$$

The same conditions apply to the synchronous condenser, the total mechanical load in this case being the iron loss and windage and friction losses.

Proceeding now to operation with unbalanced circuits having sine waves the motor also having a sine wave. In addition to equation (**177**) we shall have

$$S^2\,\check{E}_{a2} = S^2\,Z_2'\,\check{I}_{a2} \quad \textbf{(191)}$$

The mechanical power delivered through the operation of this negative phase sequence e. m. f. is given by P_N where

$$P_N = -\,3\;I_{a2}^2\,\frac{R_a'}{2} \quad \textbf{(192)}$$

this quantity must therefore be subtracted from the value of P_0 in all the equations in which P_0 appears when unbalanced circuits are used in connection with equations (**177**) to (**190**) inclusive. These equations, however, give the conditions for maintaining a given mechanical load and a given power factor in the positive phase sequence component, but in practise what is required is the combined power factor of the whole system, or the conditions to give a certain combined factor while delivering a given mechanical load; this may be obtained as follows:

The negative phase sequence component is a perfectly definite impedance and is independent of the load, and therefore the zero frequency part of the product $E_{a2}\, I_{a2}$ may be set down as

$$\check{E}_{a2}\, \hat{I}_{a2} = \frac{P_2}{3} + j\,\frac{Q_2}{3} \tag{193}$$

we have also for the positive phase sequence power delivered

$$\begin{aligned}(A_1 + j\,B_1)\; I_{a1} = I_{a1}^2\, R_a' + \frac{P_0}{3} - \frac{P_N}{3} \\ + j\,(w\, I_{a1}\, L_a' + B_1^1)\; I_{a1}\end{aligned} \tag{194}$$

And the power factor is given by cos α, where

$$\tan \alpha = \frac{I_{a1}\, B_1 + \dfrac{Q_2}{3}}{I_{a1}\, A_1 + \dfrac{P_2}{3}} \tag{195}$$

From (**194**) we have

$$A_1\, I_{a1} = I_{a1}^2\, R_a^1 + \frac{P_0}{3} - \frac{P_N}{3} \tag{196}$$

$$B_1 = w\, I_{a1}\, L_a' + B_1' \tag{197}$$

$$A_1^2 + B_1^2 = E_{a1}^2 \tag{198}$$

The simplest method of solving these equations is by means of curves. Taking arbitrary values of I_{a1}, B_1 and A_1 are chosen consistent with (**198**) so as to satisfy (**195**), $\frac{P_0}{3}$ A_1' and B_1' are then obtained from (**196**) and (**197**). If there are harmonics in the impressed e. m. f. but there are none in the wave form of the machine, the machine will have a definite impedance to the positive and negative phase sequence components of each harmonic, so that there will be a definite amount of mechanical work contributed by each harmonic which must be subtracted from the total work to be done to give the amount of work contributed by the positive phase sequence fundamental component, the equations will be identical to (**193**), (**194**), (**195**), (**196**),

(197) and **(198)**, if we take P_N to mean the total mechanical work done by the harmonics both positive and negative phase sequence and P_2 and Q_2 to represent the products

$$\Sigma\ ({}_n\check{E}_{a1}\quad {}_n\hat{I}_{a1} + {}_n\check{E}_{a2}\quad {}_n\hat{I}_{a2})$$

the zero frequency part only being taken into account.

When harmonics are present both in the impressed wave and in the generated wave, the problem becomes too complicated to treat generally, but specific cases can be worked out without much difficulty.

Phase Converters and Balancers

The phase converter is a machine to transform energy from single-phase or pulsating form to polyphase or non-pulsating form or vice versa to transform energy from polyphase to single-phase. The transformation may not be complete, that is to say, the polyphase system may not be perfectly balanced when supplied from a single-phase source through the medium of a phase converter. Phase converters may be roughly divided into two classes, namely—shunt type and series type.

Induction Motor or Synchronous Condenser Operating as a Phase Converter of the Shunt Type to Supply a Symmetrical Induction Motor or Synchronous Motor

Let Z_1 and Z_2 be the positive and negative phase sequence impedances of the motor, Z_1', Z_2' those of the phase converter. Let $S^1\ \check{E}_{a1}$ and $S^2\ \check{E}_{a2}$ be the positive and negative phase sequence components of the star e. m. f. impressed on the motor as a result of the operation. The single-phase supply will be one side of the delta e. m. f. $S\ \check{E}_{bc}$ which has positive and negative phase sequence components $S^1\ E_{bc1}$ and $S^2\ E_{bc2}$ the single-phase supply being $\check{E}_{bc} = \check{E}_{bc1} + \check{E}_{bc2}$.

The value of Z_2' may be considered fixed for all practical purposes and since in the induction motor phase converter the speed is practically no-load speed, Z' is practically the no-load impedance plus a real part obtained by increasing the real part of the no-load impedance by the ratio of the normal no-load losses to these same losses plus $\frac{1}{2}$ the secondary losses due to the phase converter currents. The latter may be calculated roughly as even a large error in its value will have an inappreciable effect on the actual results. We have therefore

$$\left.\begin{aligned} S^1\,\check{E}_{a1} &= -\,S^1\,j\,\frac{\check{E}_{bc}}{\sqrt{3}} \\ S^2\,\check{E}_{a2} &= \quad S^2\,j\,\frac{\check{E}_{bc2}}{\sqrt{3}} \end{aligned}\right\} \quad (199)$$

$$\left.\begin{aligned} S_1^1\,\check{I}_{a1}' &= -\,S^1\,j\,\frac{\check{E}_{bc1}}{\sqrt{3}\,Z_1'} \\ S^1\,\check{I}_{a1} &= -\,S^1\,j\,\frac{\check{E}_{bc1}}{\sqrt{3}\,Z_1} \end{aligned}\right\} \quad (200)$$

$$\left.\begin{aligned} S^2\,\check{I}_{a2}' &= \quad S^2\,j\,\frac{\check{E}_{bc2}}{\sqrt{3}\,Z_2'} \\ S^2\,\check{I}_{a2} &= \quad S^2\,j\,\frac{\check{E}_{bc2}}{\sqrt{3}\,Z_2} \end{aligned}\right\} \quad (201)$$

In the common lead of motor and converter we have

$$\check{I}_{a1}' + \check{I}_{a2}' + \check{I}_{a1} + \check{I}_{a2} = O \qquad (202)$$

or, substituting from (**200**) and (**201**)

$$\check{E}_{bc2}\left(\frac{1}{Z_2'} + \frac{1}{Z_2}\right) = \check{E}_{bc1}\left(\frac{1}{Z_1'} + \frac{1}{Z_1}\right) \qquad (203)$$

$$\frac{\check{E}_{bc1}}{\check{E}_{bc2}} = \frac{\dfrac{1}{Z_2'} + \dfrac{1}{Z_2}}{\dfrac{1}{Z_1'} + \dfrac{1}{Z_1}} \qquad (204)$$

$$\check{E}_{bc1} = \frac{\dfrac{1}{Z_2} + \dfrac{1}{Z_2'}}{\left(\dfrac{1}{Z_1} + \dfrac{1}{Z_1'}\right) + \left(\dfrac{1}{Z_2} + \dfrac{1}{Z_2'}\right)}\,\check{E}_{bc} \qquad (205)$$

$$\check{E}_{bc2} = \frac{\dfrac{1}{Z_1} + \dfrac{1}{Z_1'}}{\left(\dfrac{1}{Z_1} + \dfrac{1}{Z_1'}\right) + \left(\dfrac{1}{Z_2} + \dfrac{1}{Z_2'}\right)}\,\check{E}_{bc} \qquad (206)$$

which give the complete solution for all the quantities required with the aid of equations (**200**) and (**201**). For the supply current $\check{I}$

$$\left.\begin{aligned} \check{I} &= \check{I}_{bc1} + \check{I}_{bc2} + I_{bc1}' + I_{bc2}' \\ S\,\check{I}_{bc} &= S^1\,\check{I}_{bc1} + S^2\,\check{I}_{bc2} \\ S\,\check{E}_{bc} &= S^1\,\check{E}_{bc1} + S^2\,E_{bc2} \end{aligned}\right\} \qquad (\mathbf{207})$$

$$P_1 + j\,Q_1 = \check{E}_{bc}\,\hat{I} \qquad (\mathbf{208})$$

In order to obtain a perfect balance we may consider the addition of an e. m. f. $S^2\,j\,\frac{E_{x2}}{\sqrt{3}}$ in series with the phase converter whose value must be a function of the load and the phase converter impedances, and therefore equation (**201**) will be replaced by

$$S^2\,\check{I}_{a2}' = S^2\left(j\,\frac{\check{E}_{bc2}}{\sqrt{3}\,Z_2'} + j\,\frac{\check{E}_{x2}}{\sqrt{3}\,Z_2'}\right) \qquad (\mathbf{209})$$

$$S^2\,\check{I}_{a2} = S^2\,j\,\frac{\check{E}_{bc}}{\sqrt{3}\,Z_2}$$

and since the balance is perfect $\check{E}_{bc2}$ is zero, and therefore

$$S^2\,j\,\frac{\check{E}_{x2}}{\sqrt{3}} = S^2\,Z_2'\,\check{I}_{a2}' \qquad (\mathbf{210})$$

An e. m. f. equal and of opposite phase to the negative phase sequence drop through the phase converter is required to produce a perfect balance.

Carrying out the solution in the same manner as in the imperfect converter, we obtain

$$\check{E}_{bc2} = \frac{\frac{1}{Z_1} + \frac{1}{Z_1'}}{\frac{1}{Z_2} + \frac{1}{Z_2'}}\,E_{bc} - \frac{\frac{1}{Z_2'}}{\frac{1}{Z_2} + \frac{1}{Z_2'}}\,\check{E}_{x2} \qquad (\mathbf{211})$$

and since $\check{E}_{bc2}$ is zero and $\check{E}_{bc1} = \check{E}_{bc}$ the single-phase impressed e. m. f., we obtain

$$\check{E}_{x2} = Z_2'\left(\frac{1}{Z_1} + \frac{1}{Z_1^1}\right)\check{E}_{bc} \qquad (\mathbf{212})$$

and therefore from (**210**)

$$S^2\,\check{I}_{a2}' = S^2\,j\left(\frac{1}{Z_1} + \frac{1}{Z_1'}\right)\frac{\check{E}_{bc}}{\sqrt{3}} \qquad (\mathbf{213})$$

$$S^1 \check{I}_{a1}{}^1 = - S^1 j \frac{\check{E}_{bc}}{\sqrt{3} Z_1{}'} \tag{214}$$

$$S^2 \check{I}_{a2} = O \tag{215}$$

$$S^1 \check{I}_{a1} = - S^1 j \frac{\check{E}_{bc}}{\sqrt{3}\ Z_1} \tag{216}$$

Figs. 9, 10, 11 and 12 are vector diagrams of some of the principal compensated shunt-type phase converters. There will be no difficulty in following out these diagrams if the principles of this paper have been grasped.

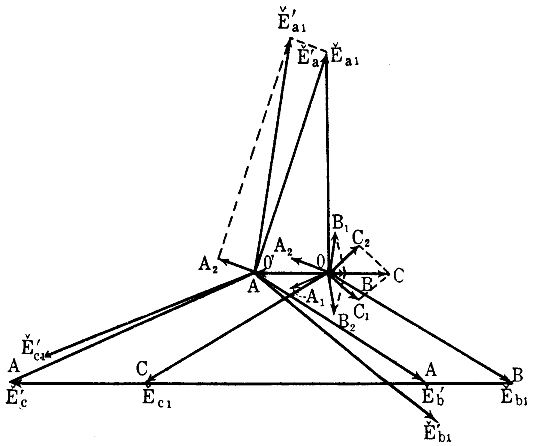

FIG. 9—VECTOR DIAGRAM OF SHUNT-TYPE PHASE CONVERTER OPERATED FROM TRANSFORMER SO AS TO DELIVER BALANCED CURRENTS

Terminal voltages of phase converter $S\check{E}'_a$

Terminal voltages of motor $S^1\check{E}_{a1}$

Negative phase sequence e.m.fs. in phase converter $S^2\ (OA_2)$

The Phase Balancer is a device to maintain symmetry of e. m. fs. at a given point in a polyphase system. It may consist of an induction motor or synchronous condenser with an auxiliary machine connected in series to supply an e. m. f. always proportional to the product of the negative phase sequence current passing through the machine and the negative phase sequence impedance of the balancer. It therefore has the effect of annulling the impedance of the machine to the flow of negative phase sequence current. Thus, in a symmetrical polyphase network, where we have an unbalanced system of currents due to certain conditions

$$S\ \check{I}_a = S^1\ \check{I}_{a1} + S^2\ \check{I}_{a2} \tag{217}$$

If a balancer be placed at the proper point the component $S^2\,\check{I}_{a2}$ will circulate between the loads and the phase balancer, the other component $S^1\,I_{a1}$ being furnished from the power house. On the other hand, if there be a dissymmetry in the impedance of the system up to the phase balancer, the latter will draw a negative phase sequence current sufficient to counteract the unbalance due to any symmetrical load by causing the proper amount of negative phase sequence current to flow to produce a balance.

The balancer may be made inherently self-balancing by inserting in series with it a machine which is self-exciting and is able

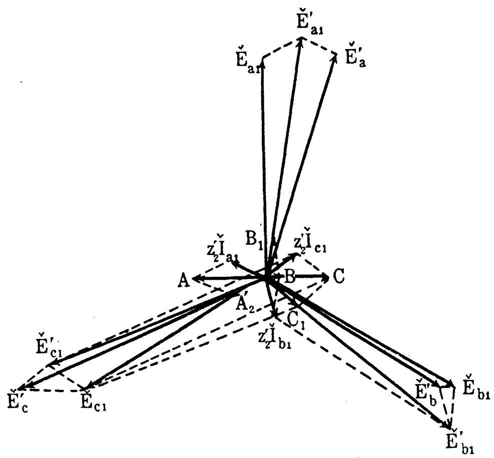

FIG. 10—VECTOR DIAGRAM SHOWING RELATIONS BETWEEN MOTOR TERMINAL E.M.F'S., CONVERTER TERMINAL E.M.FS., AND SYMMETRICAL GENERATED E.M.F'S., SAME CONNECTION AS FOR FIG. 9.

Negative phase sequence drops in phase converter $S^2 Z_2' \check{I}_{a1}$
Conjugate positive phase sequence e.m.fs. $S^1(ABC)$

to furnish an e. m. f. equal to the negative phase sequence impedance drop. The combination thus has zero impedance to negative phase sequence currents. If in the neighborhood of a phase balancer the loads have impedances

$$S\,Z_a = S^0\,Z_{a0} + S^1\,Z_{a1} + S^2\,Z_{a2}$$

The equations of the system are

$$\left.\begin{aligned} S^1\,\check{E}_{a1} &= S^1\,Z_{a0}\,\check{I}_{a1} + S^1\,Z_{a2}\,\check{I}_{a2} \\ S^2\,E_{a2} &= O = S^2\,Z_{a0}\,\check{I}_{a2} + S^2\,Z_{a1}\,\check{I}_{a1} \end{aligned}\right\} \quad \textbf{(218)}$$

The currents in the phase balancer are

$$- S^2\, \check{I}_{a2} \text{ and } S^1\, \frac{\check{E}_{a1}}{Z_1{}'}$$

The solution of (**218**) gives $S^2\, \check{I}_{a2}$ and $S^1\, \check{I}_{a1}$, the former of which are the phase balancer currents. The solution is

$$\left.\begin{aligned} \check{I}_{a1} &= \frac{Z_{a0}}{Z_{a0}^2 - Z_{a1}\, Z_{a2}}\; \check{E}_{a1} \\ \check{I}_{a2} &= - \frac{Z_{a1}}{Z_{a0}^2 - Z_{a1}\, Z_{a2}}\; \check{E}_{a1} \end{aligned}\right\} \quad (\mathbf{219})$$

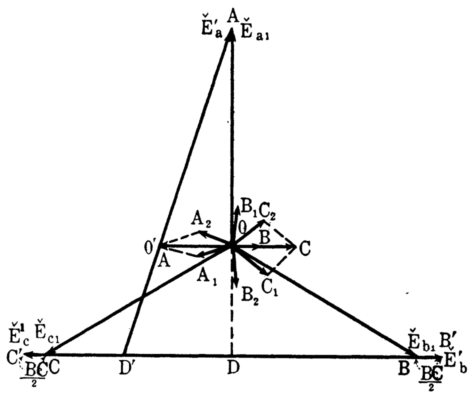

FIG. 11—VECTOR DIAGRAM OF SHUNT-TYPE PHASE CONVERTER SCOTT CONNECTED WITH COMPENSATION BY TRANSFORMER TAPS
Terminal voltages of converter $D'A$ and $B'C^1$
Terminal voltages of motor $S^1 \check{E}_{a1}$

The phase balancer is a voltage balancer and will maintain balanced e. m. f. for any condition of impedance, and if the impedance of the mains is unsymmetrical it will draw a sufficient amount of wattless negative phase sequence current through these mains to produce an e. m. f. balance at its terminals. Hence the complete solution requires consideration of all the connections in the network between the supply point and the balancer. Two equations for each mesh and connection are required, one of the positive phase sequence e. m. fs. and the other of the negative phase sequence e. m. f., and these equations may be solved in the usual way.

Series Phase Converter. In discussing the various reactions in rotating machines we have made use of the terms "positive phase sequence impedance" and "negative phase sequence impedance." These terms are definite enough when dealing with relations between machines whose generated e. m. fs. all have the same phase sequence, but require further definition when we are dealing with relations between machines whose e. m. fs. have different phase sequence. We shall retain the symbols Z_1 and Z_2 for the values of the positive and negative phase sequence impedances, depending upon the sequence symbol S to define whether these impedances apply to a negative or positive phase sequence current. Thus, the phase sequence of the currents and

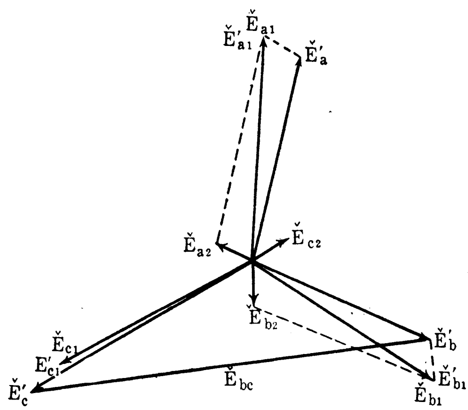

FIG. 12—VECTOR DIAGRAM OF SHUNT-TYPE PHASE CONVERTER WITH AUXILIARY ROTATING COMPENSATOR TO EFFECT A PERFECT BALANCE

Terminal voltages of phase converter $S\,\check{E}_a'$
Terminal voltages of motor $S^1\check{E}_{a1}$
Terminal voltages of compensator $S^2\check{E}_{a2}$

e. m. f. will be defined by the apparatus supplying and receiving power and the impedances of the transmitting apparatus will be defined in relation to these currents. As an example a motor series connected in counter phase sequence relation in a circuit and driven in a positive direction will have impedances

$$\begin{aligned}&\text{positive phase sequence } Z_2\\&\text{negative phase sequence } Z_1\end{aligned} \tag{220}$$

Where an auxiliary machine is defined as being of negative phase sequence relation to other machines, it will have impedances as given above to the positive and negative phase sequence currents passing through the other machines.

A single-phase transformer winding tapped at the middle point may be regarded as an unbalanced three-phase system where

$$\check{E}_a = O, \check{E}_b = + \check{E}_s, \check{E}_c = - \check{E}_s$$

$2\,\check{E}_s$ being the single-phase e. m. f The system may be represented by the equation

$$S\,\check{E}_a = S^1\,\check{E}_{a1} + S^2\,\check{E}_{a2}$$

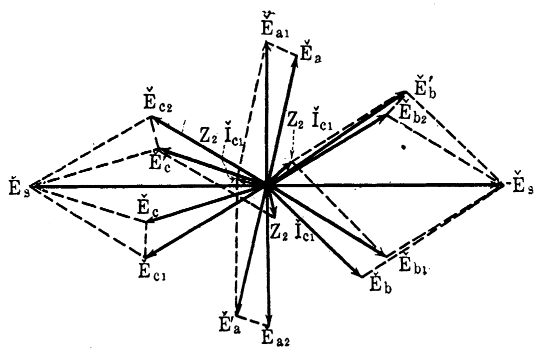

FIG. 13—VECTOR DIAGRAM OF SERIES-TYPE CONVERTER.
NO LOAD E.M.F'S. ACROSS MOTOR TERMINALS $S_1\check{E}_{a1}$
NO LOAD E.M.F'S. ACROSS CONVERTER TERMINALS $S^2\check{E}_{a2}$
SINGLE-PHASE E.M.F'S. $2\check{E}_s$
E.M.F. ACROSS TERMINAL OF MOTOR UNDER LOAD $\check{E}_a\check{E}_b\check{E}_c$
E.M.F. ACROSS TERMINAL OF CONVERTER UNDER LOAD $\check{E}'_a\check{E}'_b\check{E}'_c$

$$\left.\begin{aligned} \text{where } \check{E}_{a1} &= \quad j\,\frac{\check{E}_s}{\sqrt{3}} \\ \check{E}_{a2} &= -\,j\,\frac{\check{E}_s}{\sqrt{3}} \end{aligned}\right\} \qquad \textbf{(221)}$$

If, therefore between the single-phase source of power and the load we interpose a polyphase machine with e. m. f. $-\ S^2\,\check{E}_{a2}$, we shall have at the load terminals the e. m. f. $S^1\,\check{E}_{a1}$. If we use an induction-type phase converter it will have impedances to motor currents as follows

$$\left.\begin{aligned} &\text{To positive phase sequence } Z_2{}' \\ &\text{To negative phase sequence } Z_1{}' \end{aligned}\right\} \qquad \textbf{(222)}$$

we therefore have the relations

$$S^1 \check{E}_{a1} = S^1 \check{I}_{a1} (Z_1 + Z_2') \tag{223}$$

$$S^2 \check{E}_{a2} = S^2 \check{I}_{a2} (Z_2 + Z_1') \tag{224}$$

If the converter is doing no mechanical work, Z_1' is large compared with Z_2' or Z_2, and therefore the component of negative phase sequence is small in the motor. The value of Z_1' depends upon the slip of the phase converter which will depend on the mechanical load it carries as well as on the load carried by the motors. Approximately the load currents due to the motors produce the equivalent at the phase converter of a mechanical load equal to one-half the rotor loss of the phase converter due to these load currents. Substituting the values given in (**221**) for $S^1 \check{E}_{a1}$ and $S^2 \check{E}_{a2}$; we obtain

$$\left.\begin{aligned} S^1 j \frac{\check{E}_s}{\sqrt{3}} &= S^1 \check{I}_{a1} (Z_1 + Z_2') \\ - S^2 j \frac{E_s}{\sqrt{3}} &= S^2 \check{I}_{a2} (Z_2 + Z_1') \end{aligned}\right\} \tag{225}$$

$$\begin{aligned} S^1 \check{I}_{a1} &= \quad S^1 j \frac{\check{E}_s}{\sqrt{3} (Z_1 + Z_2')} \\ S^2 \check{I}_{a2} &= - S^2 j \frac{\check{E}_s}{\sqrt{3} (Z_2 + Z_1')} \end{aligned} \tag{226}$$

If instead of an induction-type phase converter a synchronous phase converter is used an e. m. f. of negative phase sequence $S^2 \check{E}_{a2}$ the generated e. m. f. of the phase converter must be introduced in equations (**224**) *and* (**225**) *and the value and phase of these e. m. fs. will depend upon the load on the phase converter shaft as well as the load carried by the motors.* The equations will be

$$S^1 \check{E}_{a1} = S^1 \check{I}_{a1} (Z_1 + Z_2') \tag{227}$$

$$S^2 \check{E}_{a2} = S^2 \check{I}_{a2} (Z_2 + Z_1') + S^2 \check{E}_{a2}' \tag{228}$$

or

$$\left.\begin{aligned} S^1 j \frac{\check{E}_s}{\sqrt{3}} &= S^1 \check{I}_{a1} (Z_1 + Z_2') \\ - S^2 j \frac{\check{E}_s}{\sqrt{3}} &= S^2 \check{I}_{a2} (Z_2 + Z_1') + S^2 \check{E}_{a2}' \end{aligned}\right\} \tag{229}$$

The last member of equations (**229**) is the equation of a synchronous condenser. Assuming its windage, iron loss and increased losses due to secondary reactions to be P_0, we have by equation (**160**) of the Section on Synchronous Motors

$$\frac{E_s}{\sqrt{3}} I_{a2} \cos \alpha = I_{a2}^2 (R_2 + R_1') + \frac{P_0}{3} \quad \textbf{(230)}$$

Let

$$\check{I}_{a2} = a_2 + j\, b_2 \quad \textbf{(231)}$$

then (**230**) becomes

$$\frac{E_s}{\sqrt{3}} a_2 = (a_2^2 + b_2^2)(R_2 + R_1') + \frac{P_0}{3} \quad \textbf{(232)}$$

Of the two quantities a_2 and b_2, b_2 alone is arbitrary and depends upon the excitation, a_2 will depend upon the value of b_2 and also upon the losses. Solving therefore for a_2 in terms of b_2, we have

$$a_2 = \frac{E_s}{2\sqrt{3}(R_2 + R_1')} \left\{ 1 - \sqrt{1 - \frac{4(R_2 + R_1')\{3\, b_2^2 (R_2 + R_1') + P_0\}}{E_s^2}} \right\} \quad \textbf{(233)}$$

Since b_2 is arbitrary we may now determine $\cos \alpha_2 = \frac{a_2}{\sqrt{a_2^2 + b_2^2}}$ and the value of $\check{I}_{a2}$ in terms of the impressed e. m. f. will be by (**181**) of Section on Synchronous Motors

$$S^2 \check{I}_{a2} = - S^2 \left[j \frac{\check{E}_s}{\sqrt{3}} \frac{\cos \alpha_2}{2(R_2 + R_1')} \left\{ 1 - \sqrt{1 - \frac{4(R_2 + R_1') P_0}{E_s^2 \cos^2 \alpha_2}} \right\} \epsilon^{j\alpha} \right] \quad \textbf{(234)}$$

The effective value of $\check{I}_{a2}$ in terms of the effective value of $\check{E}_s$ will then be

$$I_{a2} = \frac{E_s}{\sqrt{3}} \frac{\cos \alpha_2}{2(R_2 + R_1')} \left\{ 1 - \sqrt{1 - \frac{4(R_2 + R_1') P_0}{E_s^2 \cos^2 \alpha_2}} \right\} \quad \textbf{(235)}$$

and since the component of the e.m.f. generated in phase with the current is determined only by the magnitude of $\check{I}_{a2}$ and the motor losses, if we define its value by A_2' the quadrature component being B_2' we shall have

$$A_2' = \frac{E_s}{\sqrt{3}} \frac{\cos \alpha_2}{2} \left(1 + \sqrt{1 - \frac{4 (R_2 + R_1') P_0}{E_s^2 \cos^2 \alpha_2}}\right) \tag{236}$$

and

$$B_2' = - \frac{E_s}{\sqrt{3}} \sin \alpha_2 - \frac{w (L_2 + L_2')}{A_2^1} \tag{237}$$

$$= - \frac{E_s}{\sqrt{3}} \left\{ \sin \alpha_2 + \frac{3 w (L_2 + L_1')}{P_0} \cdot \frac{\cos \alpha_2}{2 (R_2 + R_1')} \left(1 - \sqrt{1 - \frac{4 (R_2 + R_1') P_0}{E_s^2 \cos^2 \alpha_2}}\right)\right\} \tag{238}$$

and therefore we have

$$\check{E}_2' = -j \frac{E_s}{\sqrt{3}} \left[\frac{\cos \alpha_2}{2} \left(1 + \sqrt{1 - \frac{4 (R_2 + R_1') P_0}{E_s^2 \cos^2 \alpha_2}}\right) - j \left\{ \sin \alpha_2 + \frac{3 w (L_2 + L_1') \cos \alpha_2}{2 P_0 (R_2 + R_1')} \left(1 - \sqrt{1 - \frac{4 (R_2 + R_1') P_0}{E_s^2 \cos^2 \alpha_2}}\right)\right\}\right] (\epsilon^{jwt}) \tag{239}$$

The impedance of the phase converter to the flow of negative phase sequence current is

$$\frac{2 (R_2 + R_1') \sec \alpha}{1 - \sqrt{1 - \frac{4 (R_2 + R_1') P_0}{E_s^2 \cos^2 \alpha}}} \tag{240}$$

The balance will be at its best when $\check{I}_{a2}$ is a minimum with $\cos \alpha_2$ as the independent variable. This will be the case when $\cos \alpha_2$ is unity; that is to say when b_2 is zero.

Recapitulating the results given above, we have for the general case taking the single-phase e. m. f. $\check{E}_s$ as reference

$$S^1 \check{I}_{a1} = S^1 j \frac{\check{E}_s}{\sqrt{3}\,(Z_1 + Z_2')} \tag{241}$$

$$S^2 \check{I}_{a2} = -j\,(a_2 + j\,b_2) \tag{242}$$

where b_2 is arbitrary and

$$a_2 = \frac{E_s}{2\sqrt{3}\,(R_2 + R_1')} \left\{ 1 - \sqrt{1 - \frac{4\,(R_2 + R_1')\,\{3\,b_2^2\,(R_2 + R_1') + P_0\}}{E_s^2}} \right\} \tag{243}$$

Since b_2 is arbitrary $\cos\alpha_2$ is determined by

$$\cos\alpha_2 = \frac{a_2}{\sqrt{a_2^2 + b_2^2}} \tag{244}$$

we may express $\check{I}_{a2}$ in terms of $\check{E}_s$ by

$$S^2 \check{I}_{a2} = -S^2 j \frac{\check{E}_s}{\sqrt{3}} \frac{\cos\alpha_2}{2\,(R_2 + R_1')} \left\{ 1 - \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2 \cos^2\alpha_2}} \right\} \epsilon^{j\alpha_2} \tag{245}$$

The effective value of $\check{I}_{a2}$ will be

$$I_{a2} = \sqrt{a_2^2 + b_2^2} = \frac{E_s}{\sqrt{3}} \frac{\cos\alpha_2}{2\,(R_2 + R_1')} \left\{ 1 - \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2 \cos^2\alpha_2}} \right\} \tag{246}$$

If A_2' and B_2' are components of $\check{E}_{a2}'$ these being the generated e. m. f. in phase and in quadrature with the current $\check{I}_{a2}$ we shall have

$$\check{E}_{a2}' = -j\,(A_2' + j\,B_2') \tag{247}$$

and A_2' and B_2' will have the following values

$$A_2' = \frac{E_s}{\sqrt{3}}\ \frac{\cos\alpha_2}{2}\left(1 + \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2\cos^2\alpha_2}}\right) \qquad \textbf{(248)}$$

$$B_2' = -\frac{E_s}{\sqrt{3}}\ \left\{\sin\alpha_2 + \frac{3\,w\,(L_2 + L_1')}{P_0}\cdot\frac{\cos\alpha_2}{2\,(R_2 + R_1')}\left(1 - \sqrt{1 - \frac{4\,(R_2 + Q_1')\,P_0}{E_s^2\cos\alpha^2}}\right)\right\} \qquad \textbf{(249)}$$

and $\check{E}_{a2}'$ expressed in terms of $\check{E}_s$ becomes

$$\check{E}_2' = -j\,\frac{\check{E}_s}{\sqrt{3}}\left[\frac{\cos\alpha_2}{2}\left(1 + \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2\cos^2\alpha_2}}\right) - j\left\{\sin\alpha_2 + \frac{3\,w\,(L_2 + L')\cos\alpha_2}{2\,P_0\,(R_2 + R_1')}\left(1 - \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2\cos^2\alpha_2}}\right)\right\}\right] \qquad \textbf{(250)}$$

The effective impedance of the phase converter to the flow of negative phase sequence currents is

$$\frac{2\,(R_2 + R_1')\sec\alpha_2}{1 - \sqrt{1 - \dfrac{4\,(R_2 + R_1')\,P_0}{E_s^2\cos^2\alpha}}}\ (\cos\alpha_2 - j\sin\alpha_2) \qquad \textbf{(251)}$$

or

$$\frac{E_s^2}{P_0}\cdot\frac{\cos\alpha_2}{2}\left(1 + \sqrt{1 - \frac{4\,(R_2 + R_1')\,P_0}{E_s^2\cos^2\alpha_2}}\right)\epsilon^{-j\alpha_2} \qquad \textbf{(252)}$$

In the above equations $\cos\alpha_2$ is arbitrary or b_2 may be considered arbitrary and $\cos\alpha_2$ will then be determined.

Minimum Unbalance is obtained when $\cos\alpha_2$ is made unity or when b_2 is made zero in equations (**241**) and (**252**).

Perfect Balance is obtained by driving the phase converter mechanically so as to supply the mechanical power P_0 from a separate or symmetrical source. Under this condition a_2 and b_2 both become zero when $\cos\alpha_2$ is unity. The only equation of the system is then (**241**).

Currents and Power Factor in the Single-Phase Supply Circuit of Series Phase Converter.

The e. m. f. is $2\,\check{E}_s$ and the current supplied is

$$\check{I}_s = \frac{\check{I}_b - \check{I}_c}{2}$$

$$= \frac{\check{I}_{b1} - \check{I}_{c1}}{2} + \frac{\check{I}_{b2} - \check{I}_{c2}}{2} \qquad (253)$$

If we take

$$S^1\,\check{I}_{a1} = S^1 j\,(a_1 - j\,b_1) \qquad (254)$$

$$\frac{\check{I}_{b1} - \check{I}_{c1}}{2} = \frac{\sqrt{3}}{2}\,(a_1 - j\,b_1) \qquad (255)$$

Similarly, since under the same conditions

$$S^2\,\check{I}_{a2} = -\,S^2 j\,(a_2 + j\,b_2) \qquad (256)$$

$$\frac{\check{I}_{b2} - \check{I}_{c2}}{2} = \frac{\sqrt{3}}{2}\,(a_2 + j\,b_2) \qquad (257)$$

and therefore

$$\check{I}_s = \frac{\sqrt{3}}{2}\,\{(a_1 + a_2) - j\,(b_1 - b_2)\} \qquad (258)$$

where a_1, b_1, a_2, b_2 are to be obtained by means of equations (**243**) to (**254**). The single-phase power factor is given by

$$\tan\,\theta = \frac{b_1 - b_2}{a_1 + a_2} \qquad (259)$$

of these quantities a_2 is usually the smallest and its value may be obtained approximately by assigning to b_2 a value which will make the ratio $\frac{b_1 - b_2}{a_1}$ equal to $\tan\,\theta$, and obtaining the corresponding value of a_2 by (**242**), the value of b_2 may then be recalculated from (**259**) by substituting the tentative value obtained for a_2. This procedure may be repeated until sufficient accuracy has been obtained.

Single-Phase Power Factor in Shunt-Type Phase Converter.

The simplest procedure is to obtain a curve of admittances for varying excitation of the converter and plot the power factor obtained by varying the admittance with a fixed load. The true and wattless power is obtained easily by means of (**208**) whether the system is balanced or unbalanced.

Figs. 14, 15, 16 and 17 are vector diagrams of several methods of using phase converters to supply a balanced 3-phase e. m. f. to a symmetrical load such as an induction motor. The diagrams are all based on a main machine having the same negative phase sequence impedance and the system in each case is

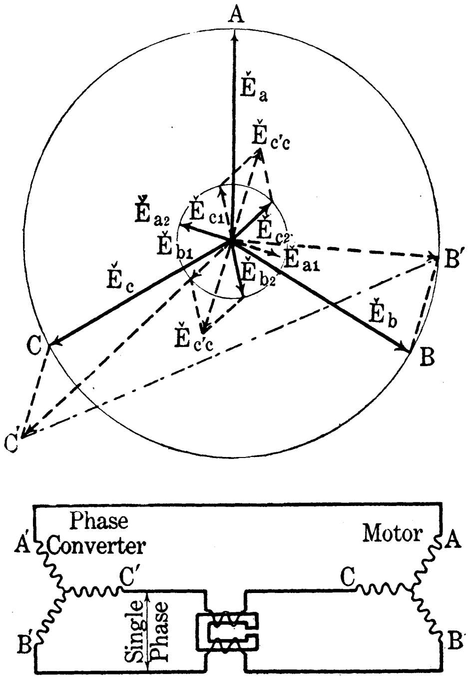

FIG. 14

SINGLE-PHASE IMPRESSED E.M.F. $= B'C'$
MOTOR E.M.F. $= BC$
NEGATIVE PHASE SEQUENCE E.M.FS. $\check{E}_{a2}\check{E}_{b2}\check{E}_{c2}$
CONJUGATE POSITIVE PHASE SEQUENCE E.M.FS. $\check{E}_{a1}\check{E}_{b1}\check{E}_{c1}$
PHASE CONVERTER TERMINAL E.M.F. $AB'C'$

delivering the same amount of power at the same voltage and 3-phase power factor without supplying any wattless power. It will be noted that the scheme Fig. 14 has the lowest single-phase power factor, Fig. 16 the highest and the rest arcing alike. It may be remarked, however, that with the shunt-type schemes adjustments can be made for power factor correction which will result also in better regulation.

APPENDIX I

Cylindrical Fields in Fourier Harmonics

When we have a diametrical coil around a cylinder concentric with another cylinder which forms the return magnetic path, and the length of the gap is uniform and the coil dimensions are very small, the field across the gap takes the form of a square topped

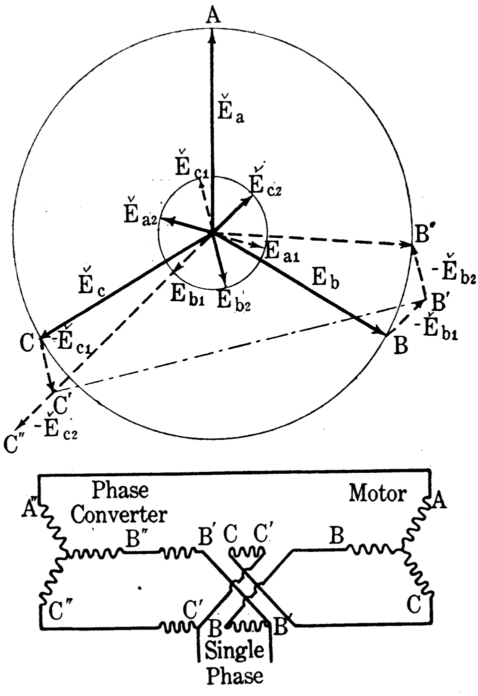

FIG. 15

SINGLE-PHASE IMPRESSED E.M.F. $= B'C'$
MOTOR E.M.F. $= BC$
PHASE CONVERTER E.M.F. $= B''C''$
NEGATIVE PHASE SEQUENCE E.M.F $\check{E}_{a2}\check{E}_{b2}\check{E}_{c2}$
CONJUGATE POSITIVE PHASE SEQUENCE E.M.F. $\check{E}_a{}^1 E_{b1} E_{c1}$
PHASE CONVERTER TERMINAL E.M.F. $AB''C''$

wave, which may be expressed in the form of a Fourier series with the plane of symmetry of the coil as reference plane, and its Fourier expansion is

$$\mathcal{B} = \frac{4\,B}{\pi}\left(\cos\theta - \tfrac{1}{3}\cos 3\,\theta + \frac{1}{S}\cos S\,\theta - \,.\,.\,+\,.\,.\right) \qquad (1)$$

where B is the average induction in the air gap.

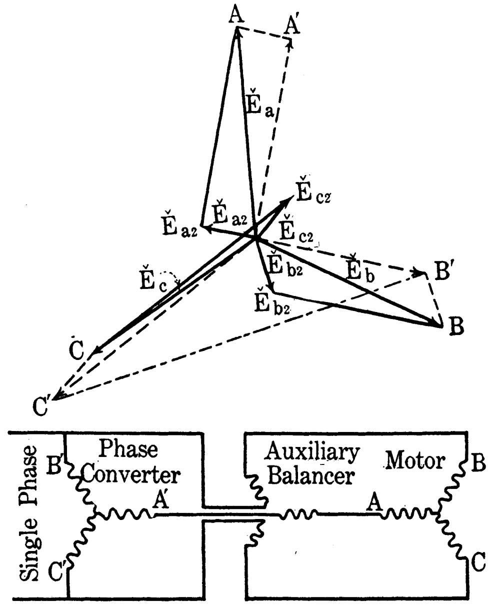

Fig. 16—Phase Converter with Auxiliary Balancer.

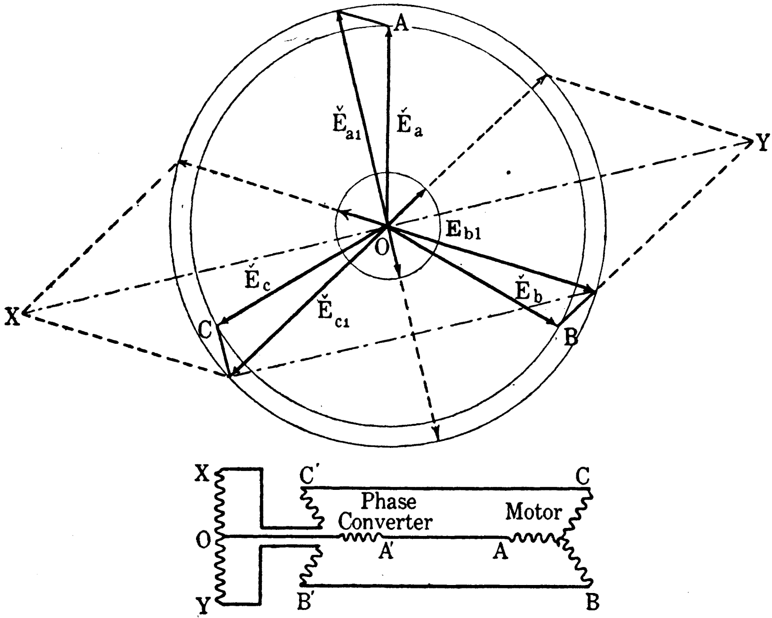

Fig. 17

Single-Phase Impressed e.m.f. $= XY$

Motor e.m.f. $= ABC$

There is a 2 to 1 Transformation of e.m.f. from Single-Phase to Three-Phase in This Connection

With pitch less than π the curve will have a different form, the amplitude being greater on one side of the plane of the coils than on the other, the areas of each wave will remain the same and second harmonic terms will appear. Let $2\, m_0\, \pi$ be the new pitch then the average amplitude of the induction will be the same as before, namely B, and the value on one side of the coil will be $2\,(1 - m_0)B$ and on the other side $2\, m_0\, B$ so that the total flux will be the same on either side. To obtain the values of the coefficients we have

$$2\,(1 - m_0 B \int_0^{m_0 \pi} \cos n\, \theta\, d\, \theta + 2\, m_0\, B \int_{m_0 \pi}^{2\pi} \cos n\, \theta\, d\, \theta = \frac{\pi}{2} A_n$$

$$2\,(1 - m_0)\, B \left[\frac{1}{n} \sin n\, \theta\right]_0^{m_0 \pi} - 2\, m_0\, B \left[\frac{1}{n} \sin n\, \theta\right]_{m_0 \pi}^{2\pi} = \frac{\pi}{2} A_n$$

$$A_n = \frac{4\, B}{\pi} \left\{ \frac{(1 - m_0) + m_0}{n} \sin n\, m_0\, \pi \right\}$$

$$A_n = \frac{4\, B}{\pi} \left(\frac{1}{n} \sin n\, m_0\, \pi_1 \right) \qquad (2)$$

Let $2\, m_0\, \pi = \frac{2}{3}\, \pi$, then $(1 - m_0)\, \pi = \frac{2}{3}\, \pi$ and

$$\mathfrak{B} = \frac{2\sqrt{3}\, B}{\pi} \left(\cos \theta + \frac{1}{2} \cos 2\, \theta - \frac{1}{4} \cos 4\, \theta - \frac{1}{5} \cos 5\, \theta \right.$$

$$\left. + \frac{1}{7} \cos 7\, \theta + \frac{1}{8} \cos 8\, \theta - \frac{1}{10} \cos 10\, \theta\ .\ \ .\ \ . \right) \qquad (3)$$

A general expression for $\mathfrak{B}$ where B is the average of the positive and negative, maximum value for any pitch coil would be

$$\mathfrak{B} = \frac{4\, B}{\pi} \Sigma \left(\frac{1}{n} \sin n\, m_0\, \pi \cos n\, \theta \right) \qquad (4)$$

and includes all possible coil pitches. If the number of teeth in a pole pitch be n_T; in addition to the average induction as indicated by (4), there will also be a tooth ripple of flux, the maximum value of which will depend upon the average value of the induction at each point. The value of m_0 must be a fraction having n_T as denominator and an integral numerator. The value of the integral numerator is therefore always $m_0\, n_T$. The correct value for the max. induction will therefore be

$$\mathfrak{B}_m = \left\{ \frac{4\,B}{\pi} \Sigma \left(\frac{1}{n} \sin n\, m_0\, \pi \cos n\, \theta \right) \right\} (1 - (-1)^{m_0 n_T} K_T \cos n_T\, \theta) \qquad (5)$$

where K_T is the ratio of the average to the min. air gap. "m_0" must always be chosen so that $m_0\, n_T$ is an integer.

If the length of the average effective air gap in centimeters be d the value of B is given by

$$B = \frac{4\,\pi}{10} \; \frac{I\,N}{2\,d} \text{ gauss}$$

where I is the maximum value of the current in the coil and N is the number of turns. If d is given in inches we may write

$$B = \frac{4\,\pi}{10} \; \frac{I\,N}{2\,d} \times 2.54 \text{ maxwells per square inch.}$$

If we integrate (5) between the limits $(\theta - m_0\,\pi)$ and $(\theta + m_0\,\pi)$ we shall have the total flux φ through the coil

$$\varphi = \frac{4\,B\,r\,l}{\pi} \int_{\theta - m_0\,\pi}^{\theta + m_0\,\pi} \Sigma \left(\frac{1}{n} \sin n\, m_0\, \pi \cos n\, \theta \right) d\,\theta$$

$$- \frac{4\,B\,r\,l}{\pi} (-1)^{m_0\, n_T} \int_{\theta - m_0\, u}^{\theta + m_0\,\pi} \Sigma \left(\frac{1}{n} \sin n\, m_0 \cos n\, \theta \right) K_T \cos n_T\, \theta\, d\,0$$

$$= \frac{4\,B\,re}{\pi} \left[\frac{1}{n^2} \sin n\, m_0\, \pi \sin n\, \theta \right]_{\theta - m_0\,\pi}^{\theta + m_0\,\pi}$$

$$- \frac{4\,B\,r\,l}{\pi} (-1)^{m_0\, n_T} K_T\, \Sigma \frac{1}{n} \sin m_0\, n\, \pi \left[\frac{\sin (n - n_T)\, \theta}{2\,(n - n_T)} + \frac{\sin (n + n_T\, \theta)}{2\,(n + n_T)} \right]_{\theta - m_0\,\pi}^{\theta + m_0\,\pi} \qquad (6)$$

The second expression is zero for all values of θ which are integral multiples of the tooth pitch angle, so long as $m_0\, n$ is also an integer and therefore it is zero for all mutual inductive relations of similar coils on a symmetrical toothed core we therefore have:

The induction through a coil displaced an angle θ from the axis of a similar coil carrying a current giving a mean induction B both coils being wound on the same symmetrical toothed core is

$$\varphi = \frac{8\,B\,r\,l}{\pi}\,\Sigma\left(\frac{1}{n^2}\sin^2 n\,m_0\,\pi\cos n\,\theta\right) \qquad (\mathbf{7})$$

The second term in equation (**6**) also becomes zero when n_T becomes infinite independent of the value of θ. We may therefore safely make use of an imaginary uniformly distributed winding when considering self and mutual impedances. It will also be shown later on, that with certain groupings of windings the second term may be reduced to zero for every value of θ.

If N_1 be the total number of complete loops in one complete pole pitch, we may take $\frac{N_1}{2\,\pi}$ as the density of winding per unit angle of the complete pole pitch. The mutual induction per turn in a coil angularly displaced an angle θ from another coil of winding density $\frac{N_1}{2\,\pi}$ with an effective total air gap $2\,d$ and with windings subtending an angle $2\,m_1\,\pi$ is given by

$$M_1 = \frac{8\,N_1\,r\,l}{10^9\,\pi\,d}\int_{-m_1\pi}^{+m_1\pi}\Sigma\left\{\frac{1}{n^2}\sin^2 n\,m_0\,\pi\cos n(\theta+\theta')\right\}d\,\theta' \text{ henrys} \qquad (\mathbf{8})$$

$$= \frac{8\,N_1\,r\,l}{10^9\,\pi\,d}\,\Sigma\,\frac{1}{n^3}\sin^2 n\,m_0\,\pi\,[\sin n\,(\theta+\theta)'\,]_{\theta'=-m_1\pi}^{\theta'=m_1\pi} \text{ henrys}$$

$$M_1 = \frac{16\,N_1\,r\,l}{10^9\,\pi\,d}\Sigma\left(\frac{1}{n^3}\sin^2 n\,m_0\,\pi\sin n\,m_1\,\pi\cos n\,\theta\right)\text{henrys} \qquad (\mathbf{9})$$

Next, if the loop of which M_1 is the mutual inductance is part of a winding having distribution density of winding $\frac{N_2}{2\,\pi}$ and subtending an angle $2\,m_2\,\pi$ its mutual inductance with the other winding will be

$$M_{12} = \frac{8\,N_1\,N_2\,r\,l}{10^9\,\pi^2\,d}\int_{-m_2\pi}^{m_2\pi}\Sigma\frac{1}{n^3}\sin^2 n\,m_0\,\pi\sin n\,m_1\,\pi \cos n\,(\theta+\theta')\,d\,\theta' \text{ henry} \qquad (\mathbf{10})$$

$$= \frac{8\,N_1\,N_2\,r\,l}{10^9\,\pi^2\,d}\Sigma\,\frac{1}{n^4}\sin^2 n\,m_0\,\pi\sin n\,m_1\,\pi\,[\sin n\,(\theta+\theta')]_{\theta'=-m_2\pi}^{\theta'=m_2\pi} \text{ henrys}$$

$$M_{12} = \frac{16\ N_1\ N_2\ r\ l}{10^9\ \pi^2\ d}\ \Sigma \left(\frac{1}{n^4} \sin^2 n\ m_0\ \pi \sin n\ m_1\ \pi \right.$$

$$\left. \sin n\ m_2\ \pi \cos n\ \theta \right) \text{ henrys} \qquad \mathbf{(11)}$$

This is the general expression for the mutual inductance between two groups of connected coils of like form on the same cylindrical core. It should be noted how much the harmonics have been reduced due to grouping.

When the coils are not of like design as in the case of a rotor and stator and the pitch of the coils is different in one from the other, $\sin n\ m_0\ \pi$ will not appear twice in the equation but one of its values must be replaced by $\sin n\ m_x\ \pi$ where $2\ m_x\ \pi$ is the pitch of the new coil. Equation (**11**) then becomes

$$M_{1a} = \frac{16\ N_1\ N_a\ r\ e}{10^9\ \pi^2\ d}\ \Sigma \left(\frac{1}{n^4} \sin n\ m_0\ \pi \sin n\ m_x\ \pi \right.$$

$$\left. \sin n\ m_1\ \pi \sin n\ m_2\ \pi \cos n\ \theta \right) \text{ henrys} \qquad \mathbf{(12)}$$

This formula is strictly correct when m_x is an integer and when θ is an integral multiple of the tooth pitch. It is true for all values of θ if either m_0 or m_x or both are unity.

By considering the axes of two similar groups of coils as coincident we obtain the value of $\Delta_1\ L_1$ which is part of the self inductance of the group, thus

$$\Delta_1\ L_1 = \frac{16\ N_1^2\ r\ e}{10^9\ \pi^2\ d}\ \Sigma \left(\frac{1}{n^4} \sin^2 n\ m_0\ \pi \sin^2 n\ m_1\ \pi \right) \qquad \mathbf{(13)}$$

The other factor that enters into the self inductance is the slot leakage inductance which depends upon the number of turns in a coil, the number of coils in a group and the width and depth of the slot and the length of the air gap. Since with the value of $\Delta_1\ L_1$ all the field which links the secondary winding has been included, only the portion of the slot leakage which does not link all the turns in the opposed secondary coil should be considered. No hard and fast rule can be made for determining this quantity since it depends upon the shape of the slots, there should be little trouble in making the calculation when the data are given. Denoting this quantity by $\Delta_2\ L_1$ we have

$$L_1 = \Delta_1\ L_1 + \Delta_2\ L_1 \qquad \mathbf{(14)}$$

Symmetrically Grouped Windings. The above formulas give the mutual impedance between groups of coils, each group of which may be unsymmetrical. Generally machines are designed so that, although the individual groups of coils due to fractional pitch may be unsymmetrical, the complete winding is symmetrical. When two coils are together in a slot this may be done by connecting one group of coils opposite the north pole in series with the corresponding group opposite the south pole; that is to say, the group displaced electrically by the angle π. If therefore we take equation (**11**) and consider the mutual induction as due to a group having axis at θ = zero and another having its axis at $\theta = \pi$ with a similarly arranged group of coils having its axis at θ, we find that (**11**) becomes

$$M_{12} = \frac{16\, N_1\, N_2\, r\, l}{10^9\, \pi^2\, d}\, \Sigma \left\{ \frac{1}{n^4} \sin^2 n\, m_0\, \pi \sin n\, m_1\, \pi \right.$$

$$\left. \sin n\, m_2\, \pi\, (1 - \cos n\, \pi)^2 \cos n\, \theta \right\} \text{henrys} \qquad (\mathbf{15})$$

Similarly

$$M_{1a} = \frac{16\, N_1\, N_a\, r\, l}{10^9\, \pi^2\, d}\, \Sigma \left\{ \frac{1}{n^4} \sin n\, m_1\, \pi \sin n\, m_x\, \pi \right.$$

$$\left. \sin n\, m_1\, \pi \sin n\, m_a\, \pi\, (1 - \cos n\, \pi)^2 \cos n\, \theta \right\} \text{henrys} \qquad (\mathbf{16})$$

Since $1 - \cos n\pi$ is zero for all even values of n it is evident that (**15**) and (**16**) contain no even harmonics, moreover the above formulas give the mutual induction between two similarly connected groups of windings, but if $(1 - \cos n\,\pi)$ is used only with the first power these formulas give the mutual impedance between one pair of such symmetrically grouped windings and another single group with axis inclined at an angle θ.

The value of self induction is

$$\Delta_1 L_1 = \frac{16\, N_1^2\, r\, l}{10^9\, \pi^2\, d}\, \Sigma \left\{ \frac{1}{n^4} \sin^2 n\, m_0\, \pi \sin^2 n\, m_1\, \pi \right.$$

$$\left. (1 - \cos n\, \pi)^2 \right\} \qquad (\mathbf{17})$$

$\Delta_2 L_1$ is found in the same manner as before

$$L_1 = \Delta_1 L_1 + \Delta_2 L_1 \qquad (\mathbf{18})$$

It is obvious from (**15**) and (**16**) that the effect of dissymmetry is to introduce more or less double frequency into the wave form of generated e. m. f.

It will be seen from an examination of (**15**) and (**17**) that, for example, a winding of pitch $\frac{2\pi}{3}$ and subtending an angle $\frac{\pi}{3}$ when connected in a symmetrical group of two has the same field form and characteristics as a full pitch winding of the same number of turns subtending an angle $\frac{2\pi}{3}$.

There are many symmetrical forms of winding but all will be found to be covered by the formulas (**15**) and (**16**).

Unsymmetrical Windings. These may take many forms which may be classified:

(1) Dissymmetry of flux form due to even harmonics.

(2) Dissymmetry in axial position of polyphase groups.

(3) Dissymmetry in windings due to incorrect grouping of coils.

(4) Dissymmetry due to unsymmetrical magnetic characteristics of the iron.

Of these various forms of dissymmetry the most common is a combination of (**1**), (**2**) and (**3**). These forms of unsymmetrical windings may all be calculated by the formulas (**11**) to (**16**).

It is to be noted that the mutual inductance between a symmetrical and an unsymmetrical winding is harmonically symmetrical. Hence, if the field of a machine is harmonically symmetrical, the e. m. f. generated will be also harmonically symmetrical whatever may be the form of the windings.

The reciprocal nature of M is fully established by its form, for it is immaterial in obtaining (**16**) whether we start out with the winding whose pitch is m_x or with that whose pitch is m_0, the result will be the same. The effect of saturation will be to tend to alter the values of the coefficients of M but the general form will not vary appreciably. We shall now consider some standard windings of generators and motors.

Three-Phase Symmetrical Full Pitch. Here m_0, m_1 and m_2 are 0.5, 0.1666 and 0.1666 respectively. Using formula (**15**) all the even harmonics disappear and $(1 - \cos n\,\pi)^2$ is equal to 4 or zero.

$$M_{12} = \frac{16\,N_1\,N_2\,r\,l}{10^9\,\pi^2\,d}\left(\cos\theta + \frac{4}{81}\cos 3\,\theta + \frac{1}{625}\cos 5\,\theta + \frac{1}{2401}\cos 7\,\theta + \frac{4}{6561}\cos 9\,\theta + \ldots\right) \qquad (\mathbf{19})$$

Theoretical Symmetrical Three-Phase Winding. Here $m_0 = 0.5, m_1 = m_2 = 0.333$. Using formula (**11**)

$$M_{12} = \frac{3}{4}\,\frac{16\,N_1\,N_2\,r\,l}{10^9\,\pi^2\,d}\left(\cos\theta + \frac{1}{625}\cos 5\,\theta + \frac{1}{2401}\cos 7\,\theta + \frac{1}{14641}\cos 11\,\theta + \ldots\right) \qquad (20)$$

Here the third group of harmonics is entirely eliminated.

Three-Phase Symmetrical $\frac{2\,\pi}{3}$ *Pitch Winding.* Here $m_0 = 0.333, m_1 = m_2 = 0.166$. Using formula (**15**)

$$M = \frac{3}{4}\,\frac{16\,N_1\,N_2\,r\,l}{10^9\,\pi^2\,a}\left(\cos\theta + \frac{1}{625}\cos 5\,\theta + \frac{1}{2461}\cos 7\,\theta + \frac{1}{14641}\cos 11\,\theta + \ldots\right) \qquad (21)$$

which gives the same result as (**20**).

Formulas for Salient Pole Machines

The formulas given in the preceding discussion are appropriate for distributed winding and non-salient poles. Where salient poles are used the field form due to the poles with a given winding will be arbitrary so that with the polar axis as reference we shall have

$$\mathcal{B} = \frac{2\,\pi\,N_a\,I_a}{d}\,\Sigma\,(A_n\cos n\,\theta) \qquad (22)$$

Where $\mathcal{B}$ is the induction through the armature or stator. When the poles are symmetrical $A_n \cos n\,\theta$ might be chosen at once for this condition and in this case we do not require coefficients of mutual induction between pole windings, since the value of $\mathcal{B}$ is obtained by considering the mutual reaction between pole windings to be such as will produce symmetry. We may however assume $\mathcal{B}$ to be perfectly general in form in which case the flux through a coil of pitch $2\,m_0\,\pi$ is

$$\varphi = \frac{4\,\pi\,N_a\,I_a\,r\,l}{10\,d}\,\Sigma\left(\frac{A_n}{n}\sin n\,m_0\,\pi\cos n\,\theta\right) \qquad (23)$$

We have therefore for the mutual induction between one pole and a group of coils at an angle θ and subtending an angle $2\,m_1\,\pi$

$$M_{a1} = \frac{4\ N_a\ N_1\ r\ l}{10^9\ d}\ \Sigma\left(\frac{A_n}{n^2}\ \sin n\ m_0\ \pi \sin n\ m_1\ \pi \cos n\ \theta\right) \quad \textbf{(24)}$$

and where there is symmetry due to grouping of windings, we have

$$M_{a1} = \frac{4\ N_a\ N_1\ r\ l}{10^9\ d}\ \Sigma\left\{\frac{A_n}{n^2}\ \sin n\ m_0\ \pi \sin n\ m_1\ \pi\ (1 - \cos n\ \pi)^2 \cos n\ \theta\right\} \quad \textbf{(25)}$$

where N_a is the number of turns for one pole and (**25**) applies to one pair of poles and the corresponding group of coils. When there are more than one pair of poles in series and the corresponding groups of winding are also in series, if it is desired to consider the mutual inductance of the complete winding, the result given above must be multiplied by the number of pairs of poles.

If in equation (**16**) we take

$$\left.\begin{aligned} \frac{N_a}{2\ \pi}\ \frac{1}{n} \sin n\ m_a\ \pi &= N_a \\ \text{and} \qquad \frac{1}{\pi\ n} &= B_n \end{aligned}\right\} \quad \textbf{(26)}$$

it becomes

$$M_{1a} = \frac{32\ N_1\ N_a\ r\ l}{10^9\ d}\ \Sigma\left\{\frac{B_n}{n^2}\ \sin n\ m_x\ \pi \sin n\ m_0\ \pi\ \sin n\ m_1\ \pi\ (1 - \cos n\ \pi)^2 \cos n\ \theta\right\} \quad \textbf{(27)}$$

which is the expression corresponding to (**25**) starting with the winding flux form. (**25**) and (**27**) must therefore be identical and we have

$$\frac{32\ N_1\ N_a\ r\ e}{10^9\ d}\ B_n \sin n\ m_x\ \pi - \frac{4\ N_a\ N_1\ r\ e}{10^9\ d}\ A_n$$

or

$$\mathfrak{B}_n = \frac{A_u}{8 \sin n\ m_x\ \pi} \quad \textbf{(28)}$$

and

$$\mathfrak{B}_1 = \frac{2\ \pi\ I_1}{10\ d}\ \Sigma\ (B_n \sin n\ m_0\ \pi \cos n\ \theta) \quad \textbf{(29)}$$

and is the induction wave form for a single turn of the winding.

The expression for the mutual inductance between windings of the same core for salient poles is obtained in terms of the pole flux wave form by substituting in the formulas $\frac{A_n}{8 \sin n\, m_x\, \pi}$ for $\frac{1}{n\,\pi}$. We have therefore the following formulas for salient poles.

General expression considering only one pole and one group of coils.

$$\mathfrak{B}_a = \frac{2\,\pi\,N_a\,I_a}{10\,d}\,\Sigma\,(A_n \cos n\,\theta) \qquad \textbf{(a)}$$

$$\mathfrak{B}_1 = \frac{\pi\,I_1}{20\,d}\,\Sigma\left(A_n\,\frac{\sin n\,m_0\,\pi}{\sin n\,m_a\,\pi}\cos n\,\theta\right) \qquad \textbf{(b)}$$

$$M_{a1} = \frac{4\,N_a\,N_1\,r\,l}{10^9\,d}\,\Sigma\left(\frac{A_n}{n^2}\sin n\,m_0\,\pi \sin n\,m_1\,\pi\cos n\,\theta\right) \qquad \textbf{(c)}$$

$$M_{12} = \frac{2\,N_1\,N_2\,r\,l}{10^9\,\pi\,d}\,\Sigma\left(\frac{A_n}{n^3}\;\frac{\sin n\,m_0\,\pi}{\sin n\,m_x\,\pi}\sin n\,m_0\,\pi\sin n\,m_1\,\pi\right.$$
$$\left.\sin n\,m_2\,\pi\cos n\,\theta\right) \qquad \textbf{(d)}$$

$$\Delta_1\,L_a = \frac{4\,\pi\,N_a^2\,r\,l}{10^9\,d}\,\Sigma\left(\frac{A_n}{n}\sin n\,m_x\,\pi\right) \qquad \textbf{(e)}$$

$$\Delta_1\,L_1 = \frac{2\,N_1^2\,r\,l}{10^9\,\pi\,d}\,\Sigma\left(\frac{A_n}{n}\;\frac{\sin^2 n\,m_0\,\pi\sin^2 n\,m_1\,\pi}{\sin n\,m_x\,\pi}\right) \qquad \textbf{(f)}$$

General expressions considering only poles to be symmetrical. Considered on the basis of two poles, N_a being turns on one pole.

$$\mathfrak{B}_a = \frac{2\,\pi\,N_a\,I_a}{10\,d}\,\Sigma\,\{A_n\,(1-\cos n\,\pi)\cos n\,\theta\} \qquad \textbf{(a')}$$

$$\mathfrak{B}_1 = \frac{\pi\,I_1}{20\,d}\,\Sigma\left\{A_n\,\frac{\sin n\,m_0\,\pi}{\sin n\,m_x\,\pi}\,(1-\cos n\,\pi)\cos n\,\theta\right\} \qquad \textbf{(b')}$$

$$M_{a1} = \frac{4\,N_a\,N_1\,r\,l}{10^9\,d}\,\Sigma\left\{\frac{A_n}{n^2}\sin n\,m_0\,\pi\sin n\,m_1\,\pi\right.$$
$$\left.(1-\cos n\,\pi)\cos n\,\theta\right\} \qquad \textbf{(c')}$$

$$M_{12} = \frac{2\, N_1\, N_2\, r\, l}{10^9\, \pi\, d}\, \Sigma \left\{ \frac{A_n}{n^3}\; \frac{\sin n\, m_0\, \pi}{\sin n\, m_x\, \pi}\, \sin n\, m_0\, \pi\, \sin n\, m_1\, \pi \right.$$

$$\left. \sin n\, m_2\, \pi\, \cos n\, \theta \right\} \text{henrys} \quad (\mathbf{d}')$$

$$\Delta_1 L_a = \frac{4\, \pi\, N_a^2\, r\, l}{10^9\, d}\, \Sigma \left\{ \frac{A_n}{n} \sin n\, m_x\, \pi\, (1 - \cos n\, \pi) \right\} \quad (\mathbf{e}')$$

$$\Delta_1 L_1 = \frac{2\, N_1^2\, r\, l}{10^9\, \pi\, d}\, \Sigma \left\{ \frac{A_n}{n}\; \frac{\sin^2 n\, m_0\, \pi\, \sin^2 n\, m_1\, \pi}{\sin n\, m_x\, \pi} \right\} \quad (\mathbf{f}')$$

General expression with both polar and winding symmetry.

$$\mathcal{B}_a = \frac{2\, \pi\, N_a\, I_a}{10\, d}\, \Sigma\, \{A_n\, (1 - \cos n\, \pi)\, \cos n\, \theta) \quad (\mathbf{a}'')$$

$$\mathcal{B}_1 = \frac{\pi\, I_1}{20\, d}\, \Sigma \left\{ A_n\, \frac{\sin n\, m_0\, \pi}{\sin n\, m_x\, \pi}\, (1 - \cos n\, \pi)\, \cos n\, \theta \right\} \quad (\mathbf{b}'')$$

$$M_{a1} = \frac{4\, N_a\, N_1\, r\, l}{10^9\, d}\, \Sigma \left\{ \frac{A_n}{n^2} \sin n\, m_0\, \pi\, \sin n\, m_1\, \pi\, (1 - \cos n\, \pi)^2 \right.$$

$$\left. \cos n\, \theta \right\} \quad (\mathbf{c}'')$$

$$M_{12} = \frac{2\, N_1\, N_2\, r\, l}{10^9\, \pi\, d}\, \Sigma \left\{ \frac{A_n}{n^3}\; \frac{\sin n\, m_0\, \pi}{\sin n\, m_x\, \pi}\, \sin n\, m_0\, \pi\, \sin n\, m_1\, \pi \right.$$

$$\left. \sin n\, m_2\, \pi\, (1 - \cos n\, \pi)^2 \cos n\, \theta \right\} \quad (\mathbf{d}'')$$

$$\Delta_1 L_a = \frac{4\, \pi\, N_a^2\, r\, l}{10^9\, d}\, \Sigma \left\{ \frac{A_n}{n} \sin n\, m_x\, \pi\, (1 - \cos n\, \pi)^2 \right\} \quad (\mathbf{e}'')$$

$$\Delta_1 L_1 = \frac{2\, N_1^2\, r\, l}{10^9\, \pi\, d}\, \Sigma \left\{ \frac{A_n}{n}\; \frac{\sin^2 n\, m_0\, \pi\, \sin^2 n\, m_1\, \pi}{\sin n\, m_x\, \pi} \right.$$

$$\left. (1 - \cos n\, \pi)^2 \right\} \quad (\mathbf{f}'')$$

In using any of the formulas given above for machines having more than two poles, it must be divided by the number of pairs of poles and likewise the expression for M or $\Delta_1 L$ must be multiplied by the number of pairs of poles, which leaves the formula for these quantities unchanged.

Let us next consider the actual induction in the air gap with a distributed winding operating with three-phase currents. Let i_{m1} be the magnetizing current of the first phase i_{m2} and i_{m3} those of the other phases. The induction due to one group of coils of phase 1 is

$$\mathcal{B}_1 = \frac{8\,N_1\,i_{m1}}{10\,\pi\,d}\,\Sigma\left\{\frac{1}{n^2}\sin n\,m_0\,\pi\sin n\,m_1\,\pi\cos n\,\theta\right\} \quad (30)$$

and if the phase displacement of 2 and 3 from 1 be φ_{12} and φ_{13}

$$\mathcal{B}_2 = \frac{8\,N_2\,i_{m2}}{10\,\pi\,d}\,\Sigma\left\{\frac{1}{n^2}\sin n\,m_0\,\pi\sin n\,m_2\,\pi\cos(n\theta - \varphi_{12})\right\} \quad (31)$$

$$\mathcal{B}_3 = \frac{8\,N_3\,i_{m3}}{10\,\pi\,d}\,\Sigma\left\{\frac{1}{n^2}\sin n\,m_0\,\pi\sin n\,m_2\,\pi\cos(n\,\theta - \varphi_{13})\right\} \quad (32)$$

For symmetrically grouped coils the formulas become

$$\mathcal{B}_1 = \frac{8\,N_1\,i_{m1}}{10\,\pi\,d}\,\Sigma\left\{\frac{1}{n^2}\sin n\,m_0\,\pi\sin n\,m_1\,\pi\,(1 - \cos n\,\pi)\cos n\,\theta\right\} \quad (33)$$

$$\mathcal{B}_2 = \frac{8\,N_2\,i_{m2}}{10\,\pi\,d}\,\Sigma\left\{\frac{1}{n^2}\sin n\,m_0\,\pi\sin n\,m_2\,\pi(1 - \cos n\,\pi)\cos m\,(\theta - \phi_{12})\right\} \quad (34)$$

$$\mathcal{B}_3 = \frac{8\,N_3\,i_{m3}}{10\,\pi\,d}\,\Sigma\left\{\frac{1}{n^2}\sin n\,m_0\,\pi\sin n\,m_3\,\pi\,(1 - \cos n\,\pi)\cos m\,(\theta - \varphi_{13})\right\} \quad (35)$$

For a symmetrical three-phase motor with full pitch coils $m_0 = 0.5$, $m_1 = m_2 = m_3 = 0.166$ (**33**), (**39**) and (**35**) become of the four

$$\mathcal{B}_1 = \frac{8\,N_1\,i_{m1}}{10\,\pi\,d}\left\{\cos\theta - \frac{2}{9}\cos 3\,\theta + \frac{1}{25}\cos 5\,\theta + \frac{1}{49}\cos 7\,\theta - \frac{2}{81}\cos 9\,\theta + \frac{1}{121}\cos 11\,\theta + \frac{1}{169}\cos 13\,\theta + \right\} \quad (36)$$

which is the field due to one group of coils alone. The wave is flattened by the third group of harmonics but all the other harmonics are peaking values. There is therefore a decided gain in such a wave form of flux since it permits of high fundamental flux density.

The maximum value of flux is approximately

$$B_{max} = 0.823 \cdot \frac{8\, N_1\, i_m}{10\, \pi\, d} \text{ gaus} \tag{37}$$

where d is given in centimeters.

$$B_{max} = \frac{1.67\, N_1\, i_m}{\pi\, d} \quad \text{maxwells per square inch,}$$

with d given in inches.

For the total winding the resultant induction will be the sum of B_1, B_2, and B. If we take the symmetrical winding with angles between planes of symmetry

$\varphi_{12} = \dfrac{2\,\pi}{3}$ and $\varphi_{13} = \dfrac{4\,\pi}{3}$, we have

$$\left.\begin{aligned} \cos n\,\theta &= \frac{\epsilon^{j\,n\,\theta}}{2} + \frac{\epsilon^{-j\,n\,\theta}}{2} \\ \cos n\left(\theta - \frac{2\,\pi}{3}\right) &= a^{-n}\,\frac{\epsilon^{j\,n\,\theta}}{2} + a^{n}\,\frac{\epsilon^{-j\,n\,\theta}}{2} \\ \cos n\left(\theta - \frac{4\,\pi}{3}\right) &= a^{n}\,\frac{\epsilon^{j\,n\,\theta}}{2} + a^{-n}\,\frac{\epsilon^{-j\,n\,\theta}}{2} \end{aligned}\right\} \tag{38}$$

If we multiply these three quantities successively by $\check{I}_{m1}$, $a^2\check{I}_{m1}$, $a\check{I}_{m1}$ and add, we have

$$\check{I}_{m1}\left\{\frac{\epsilon^{j\,n\,\theta}}{2}\;(1 + a^{-(n-2)} + a^{(n+1)})\left(+ \frac{\epsilon^{-j\,n\,\theta}}{2}\right.\right.$$

$$\left.\left. \times\,(1 + a^{n+2} + a^{-(n-1)})\right)\right\} \tag{39}$$

and giving n successive odd values from 1 up, we find for (**39**) the following values

$n = 1$ (**39**) becomes $\dfrac{3}{2}\;\check{I}_{m1}\,\epsilon^{-j\,\theta}$

$n = 3$ " " 0

$n = 5$ " " $\frac{3}{2} \check{I}_{m1} \epsilon^{j 5 \theta}$

$n = 7$ " " $\frac{3}{2} \check{I}_{m1} \epsilon^{-j 7 \theta}$

$n = 7$ " " 0

$n = 11$ " " $\frac{3}{2} \check{I}_{m1} \epsilon^{j 11 \theta}$

$n = n$ " " $2 \check{I}_{m1} \sin^2 \frac{2 n \pi}{3} \epsilon^{-j \frac{2}{\sqrt{3}} \sin \frac{2 n \pi}{3} n \theta}$

We may therefore express $\mathfrak{B}$ by
$\mathfrak{B}$ = real part of

$$\frac{16 N_1 \check{I}_{m1}}{10 \pi d} \Sigma \left\{ \frac{1}{n^2} \sin n m_0 \pi \sin n m_1 \pi (1 - \cos n \pi) \right.$$
$$\left. \times \sin^2 \frac{2 n \pi}{3} \epsilon^{-j \frac{2}{\sqrt{3}} \sin \frac{2 n \pi}{3} n \theta} \right\} \quad \textbf{(40)}$$

It will be obvious that if we proceed around the cylinder in the negative direction of rotation at an angular speed w and $\check{I}_{m1}$ is equal to $I_{m1} \epsilon^{j w t}$, for $n = 1$ the value of B_1 will remain constant and real, hence B_1 must be a constant field rotating at angular velocity w in the negative direction. The value of B may be expressed in harmonic form, but in this form it does not illustrate the rotating field theory so aptly. The harmonic form is given below and is simpler in appearance than **(40)**.

$$\mathfrak{B} = \frac{16 N_1 i_{m1}}{10 \pi d} \Sigma \left(\frac{1}{n^2} \sin n m_0 \pi \sin n m_1 \pi (1 - \cos n \pi) \right.$$
$$\left. \sin^2 \frac{2 n \pi}{3} \cos n \theta \right) \quad \textbf{(41)}$$

For a symmetrical three-phase motor with full pitch coil ($m_0 = 0.5 \; m_1 = 0.166$) $\mathfrak{B}$ becomes

$$\mathfrak{B} = \frac{12 N_1 i_{m1}}{10 \pi d} \Sigma \left\{ \cos \theta + \frac{1}{25} \cos 5 \theta + \frac{1}{49} \cos 7 \theta \right.$$
$$\left. + \frac{1}{121} \cos 11 \theta + \frac{1}{169} \cos 13 \theta + \; . \; . \right\} \quad \textbf{(42)}$$

This gives for the maximum induction approximately

$$\mathcal{B}_{max} = \frac{1.075 \times 12\, N_1\, i_{m1}}{10\, \pi\, d} = \frac{1.29\, N_1\, i_m}{\pi\, d} \text{ gauss} \qquad (43)$$

where d is measured in centimeters.

$$\mathcal{B}_{max} = \frac{3.28 \times N_1\, i_{m1}}{\pi\, d} \text{ maxwell per square inch} \qquad (44)$$

where d is measured in inches and N is the total number of turns per pair of poles.

APPENDIX II

Graphical Construction for Obtaining Symmetrical Components

The graphica method for finding the symmetrical components of $S(\check{E}_a)$ given in the text serves as a geometrical interpretation

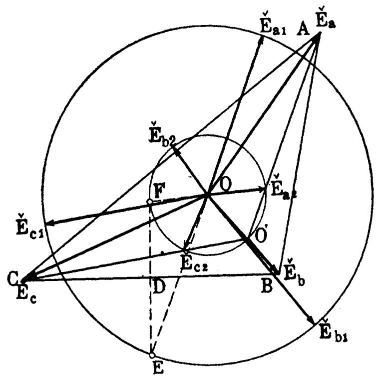

FIG. 18—GRAPHICAL METHOD FOR OBTAINING THE SYMMETRICAL COMPONENTS OF THE THREE-PHASE VECTORS $\check{E}_a$, $\check{E}_b$, $\check{E}_c$

of equation (**5**), but the graphical method shown in Fig. 18 is much simpler and more convenient, the construction is as follows. Find E and F the centroids of the two equilateral triangles with $B\,C$ as base: With 0 the centroid of triangle $A\,B\,C$ as centre describe the two circles passing through E and F then; $E\,O$ extended till it touches the circle through E at the opposite end of the diameter gives $\check{E}_{a1}$; $\check{E}_{b1}$ and $\check{E}_{c1}$ are obtained by laying of points on the circle 60 degrees from E. Similarly if $F\,O$ is extended to meet the circle through F at the opposite end of the diameter we obtain $\check{E}_{a2}$; and $\check{E}_{b2}$ and $\check{E}_{c2}$ are obtained by the same construction as before.

The proof of this construction is as follows: If G and H are the apexes (not shown in figure) of the equilateral triangles having $B\,C$ as base

$$O\,E = \frac{O\,B + O\,C + O\,G}{3} = \frac{-\,O\,A + O\,G}{3} = \frac{-\,\check{E}_a + O\,G}{3}$$

$$O\,F = \frac{O\,B + O\,C + O\,H}{3} = \frac{-\,O\,A + O\,H}{3} = \frac{-\,\check{E}_a + O\,H}{3}$$

$$\begin{aligned}
-\,a^2 \,.\, O\,C &= -\,a^2\,(O\,D + D\,C)\\
-\,a\;.\,O\,B &= -\,a\,(O\,D + D\,B) = -\,a\,(O\,D - D\,C)\\
-\,a^2\,.\,O\,C - a\,.\,O\,B &= -\,(a + a^2)\,O\,D + (a - a^2)\,.\,D\,C\\
&= O\,D + j\,\sqrt{3}\,D\,C\\
&= O\,G
\end{aligned}$$

That is

$$O\,G = -\,(a\,\check{E}_b + a^2\,\check{E}_c)$$

Similarly

$$O\,H = -\,(a^2\,\check{E}_b + a\,\check{E}_c)$$

and therefor

$$O\,E = -\,\frac{\check{E}_a + a\,\check{E}_b + a^2\,\check{E}_c}{3} = -\,\check{E}_{a1}$$

and

$$O\,F = -\,\frac{\check{E}_a + a^2\,\check{E}_b + a\,\check{E}_c}{3} = -\,\check{E}_{a2}$$

The construction when $\check{E}_a + \check{E}_b + \check{E}_c$ is not zero is so obvious that it is not necessary to show it here.

If lines be drawn from E_a to $\check{E}_{a2}$, E_b to $\check{E}_{b2}$, E_c to $\check{E}_{c2}$ these lines will be parallel to $\check{E}_a'$ $\check{E}_b'$ and $\check{E}_c'$ respectively and will meet at a point O'. The vectors $O'\,E_a$ $O'\,E_b$ and $O'\,E_c$ give the values of the e. m. f. across each member of a star delta bank of transformers when operated on the three-phase circuit $S\,(\check{E}_a)$ with ratios changed so as to give a balanced secondary triangle of e. m. f's.

Discussion on "Method of Symmetrical Co-Ordinates Applied to the Solution of Polyphase Networks" (Fortescue), Atlantic City, N. J., June 28, 1918.

J. Slepian (by letter): During the past eighteen months it has been my very good fortune to have been in close contact with Mr. Fortescue and to have had many interesting discussions on the ideas embodied in this paper. Since I have had so long to think over and digest these ideas, I think I may be pardoned for going at great length here into the viewpoint I have reached. I feel all the greater need for a long discussion, because on reading the paper I see that the great wealth of material to be presented in a limited space has crowded out much detailed explanation and appeal to analogy, which in discussions with Mr. Fortescue have contributed so much to my clearer conception of the ideas presented here.

The method given here had its origin in considering the operation of balanced induction machines under unbalanced conditions. Using the "coordinates" proposed here, the theory of these machines may be given with beautiful simplicity. I think I am right in saying that the utility of the method is practically entirely limited to the case of rotating induction machines. Purely static apparatus treated in this way does not show any simplification. When one considers, however, that almost every practical alternating-current circuit contains at least one rotating machine, namely the generator, the broad field of application of the method becomes apparent.

The root of the ideas given here is old and was given early in treatments of single-phase motors. The simple constant rotating nature of the flux in a balanced polyphase motor was well known. It was discovered that the more complicated flux in a single-phase motor could be resolved into the sum of two such constant rotating fluxes of opposite rotations, with the magnitude of these fluxes not necessarily equal. It was observed also, that if the balanced polyphase states, which would give independently each of these rotating fields, were superimposed, the resultant state would give correctly the currents. voltages, torques, etc., of the single-phase motor. For a recent discussion of this, see Mr. Lamme's paper, page 627, Volume I, Transactions 1918.

Mr. Fortescue has generalized this method of resolution of the flux in a polyphase machine under unbalanced conditions into the sum of fluxes, each corresponding to a balanced condition, to a similar resolution of an unbalanced system of any polyphase quantities whatever.

In my discussion here I shall, for simplicity, confine myself to the three-phase case. To illustrate the resolution of a set of unbalanced three-phase currents into symmetrical or balanced components, consider three arbitrary currents I_a I_b I_c flowing into the terminals of a three-phase, star-connected apparatus. (Fig. 1). Mr. Fortescue shows that this set of currents may be had by adding together the following three sets of currents.

Method of Symmetrical Coordinates. First a current I_{ao} in each phase. These three currents then are all equal and all in phase. They combine to give a current in the neutral. Second: a current I_{ao} in phase a, a current $a^2 I_{a1}$, in phase b, and a current $a I_{a1}$ in phase c, where

$$a = -\tfrac{1}{2} + j\,\tfrac{1}{2}\sqrt{3}$$

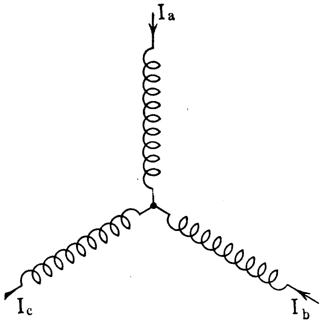

FIG. 1

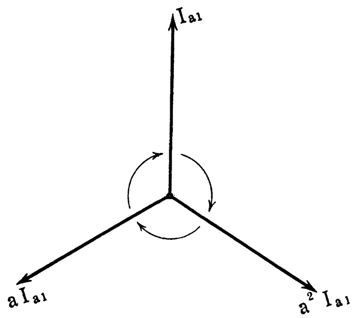

FIG. 2

is a cube root of unity. This set of currents is clearly a balanced polyphase set of what is called positive sequence. (Fig. 2). Third a current I_{a2} in phase a, a current $a\ I_{a2}$ in phase B and a current $a^2\ I_{a2}$ in phase c. (Fig. 3.). This set of currents is clearly a balanced polyphase set of what is called negative phase sequence. If these three sets of currents be made to flow simultaneously in the three-phase apparatus the resultant current in

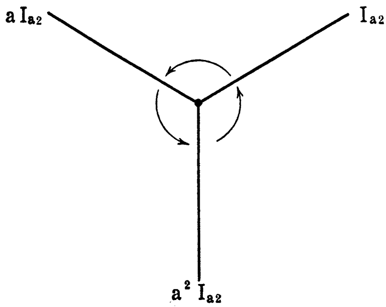

FIG. 3

each phase will be respectively I_a, I_b, I_c. The values of I_{a0}, I_{a1}, I_{a2} are obtained from equations (**5**). They are:

$$I_{a0} = 1/3\ (I_a + I_b + I_c).$$
$$I_{a1} = 1/3\ (I_a + aI_b + a^2\ I_c).$$
$$I_{a2} = 1/3\ (I_a + a^2 I_b + aI_c).$$

In a similar way, any set of three voltages to neutral acting on the three-phase apparatus can be resolved into the sum of three sets of balanced voltages of zero phase sequence, positive phase sequence, and negative phase sequence, respectively.

The sequence operators, S^0, S^1 and S^2, which Mr. Fortescue introduces, have a close analogy, it seems to me, with the j used in the usual treatment of alternating currents. In the elementary theory, it is found useful to resolve an alternating quantity into two components, one in phase with some reference alternating quantity, and the other component in phase quadrature. If i_1 and i_2 are the magnitudes respectively, of the in-phase and the quadrature components, their resultant I, is denoted by $i = i_1 + ji_2$. Here j may be looked upon as a unit alternating quantity in phase quadrature with a reference alternating quantity. Thus, the equation $i = li_1 + j\, i_2$ states that the quantity i may be obtained by adding together i_1 times a unit in phase quantity, and i_2 times a unit quadrature component.

In the same way, the symbols S°, S^1, S^2 may be regarded as unit polyphase vectors* of zero, positive and negative phase sequence. Thus $S^\circ = (1, 1, 1)$, if referring to currents, represents a unit current in each phase of a three-phase apparatus the three currents being in phase with some reference alternating quantity. Similarly, $S^1 = (1, a^2, a)$ represents a unit balanced, three-phase current, of positive phase sequence, in the three-phase apparatus, the current in the first phase being in phase with the reference alternating quantity. Likewise $S^2 = (1, a, a^2)$ represents a unit balanced, three-phase current of negative phase sequence. Thus the equation (**15**) $S\,(I_a) = S^\circ I_o + S^1 I_{a1} + S^2 I_{a2}$ states that a system of three currents $S\,(I_a)$ is equal to the sum of three sets of currents, the first set being of the type S°, that is of zero phase sequence, the second set of the type S^1, that is of positive phase sequence, and the third set of type S^2, that is of negative phase sequence. The three currents of any set are obtained by multiplying the three currents of the corresponding unit polyphase vector by the complex quantity indicated. Thus the currents in the second set are obtained by multiplying the three currents, 1, a_2, a, respectively, by I_{a1}.

The utility of resolving three-phase currents and voltages in this way, when applied to rotating balanced machines, lies in this fact, that a symmetrical set of voltages of any phase sequence applied to the machine will produce a symmetrical set of currents of the *same phase sequence*, and that a symmetrical set of currents of any phase sequence flowing into the machine will produce a symmetrical set of terminal voltages of the same phase sequence. This fact is well known in the theory usually given of balanced polyphase apparatus, although it is seldom explicitly stated. Other methods of dividing a set of unbalanced three-phase quantities acting on a balanced machine into components would

*This very fortunate term is due to Mr. C. T. Allcutt and expresses the idea of the paragraph most succinctly.

not have this simplicity. For example: suppose the three actual currents in the three phases, and the three actual voltages of the terminals relative to neutral were taken as the components or coordinate of the currents and voltages. Then one component of current alone, that is a current in one phase only, would not produce only the corresponding component of voltage, but would produce voltages in all three phases.

Neglecting saturation, for a given rotor speed, the symmetrical currents produced in a balanced machine by symmetrical applied voltage, are proportional and in a definite phase relation to the symmetrical applied voltage. Thus the currents may be obtained by multiplying the individual members of the symmetrical set of applied voltages by some complex number. Similarly, a symmetrical set of currents flowing into a balanced machine produces a symmetrical set of terminal voltages which may be obtained by multiplying the individual members of the symmetrical set of currents by some complex number. The first complex number could be called the symmetrical admittance of the machine, and the second the symmetrical impedance.

This symmetrical impedance and admittance will be different, of course, for symmetrical components of different phase sequence. Thus for a star-connected, ungrounded neutral machine, the admittance for symmetrical voltage of zero phase sequence will be zero; if the neutral is grounded, the impedance to zero phase sequence current will be principally the leakage reactance between phases; if the rotor is running near synchronism in the sense of positive phase sequence, the impedance Z_1 to positive phase sequence current will be large, while the impedance Z_2 to negative phase sequence will be small. The complete expressions for these impedances Z_1, Z_2, are given in equations (**122**), (**123**). The relations between current and voltage components are given in equations (**120**) and (**121**). In these four equations is concentrated the whole theory of symmetrical machines operating under unbalanced conditions. But so simple are these equations and their physical meaning so clear, that once understood, they enable us to predict qualitatively, without computation, the behavior of rotating balanced machines under any unbalanced condition whatever.

Consider, for example, the simple picture of the action of a phase balancer which the above treatment gives. The balancer is merely a balanced machine across the polyphase line running near synchronism. It offers high impedance for the symmetrical voltage of normal or positive phase sequence, but offers a very low impedance for any negative phase sequence component of voltage. Thus the negative phase sequence voltage is "short-circuited out," and balance on the line is preserved.

The harmful effects of slight unbalance in the terminal voltage of a polyphase machine upon the machine's rating is also clearly shown. Since the negative sequence impedance is very low, a small negative phase sequence voltage will produce large negative phase sequence currents with their attendant heating. This suggests a better quantity to denote the degree of unbalance of a line than the one commonly used. It is the ratio of the negative phase sequence component of the line voltage to the positive phase sequence component.

Balanced stationary apparatus also enjoys the simplicity pointed out in balanced rotating machines, namely that symmetrical voltages of any phase sequence produce symmetrical currents of the same phase sequence, but here the impedances for positive and negative phase sequence are always the same. The zero phase sequence impedance is generally different.

When we pass to unbalanced apparatus, things become more complicated. Symmetrical currents of one phase sequence no longer produce e. m. fs. of that phase sequence alone, but the total e. m. fs. are unsymmetrical and contain components of other phase sequences. Let us consider in detail an unbalanced star impedance, with grounded neutral. Let the phase impedances be Z_a, Z_b, Z_c.

Let us first study the e. m. fs. to neutral produced by a zero phase sequence current. If I_{ao}, I_{ao}, I_{ao}, are the three currents, the three e. m. fs. are $Z_a I_{ao}$, $Z_b I_{ao}$, $Z_a I_{ao}$. Resolving these three e. m. fs. into symmetrical components, we find by equations (**5**) that the zeroth order component is:

$$\tfrac{1}{3}\,(Z_a I_{ao} + Z_b I_{ao} + Z_c I_{ao}) = \tfrac{1}{3}\,(Z_a + Z_b + Z_c)\, I_{ao}$$

The positive phase sequence components will be:

$$\tfrac{1}{3}\,(Z_a I_{ao} + aZ_b I_{ao} + a^2 Z_c I_{ao}) = \tfrac{1}{3}\,(Z_a + aZ_b + a^2 Z_c)\, I_{ao}$$

The negative phase sequence component will be:

$$\tfrac{1}{3}\,(Z_a I_{ao} + a^2 Z_b I_{ao} + aZ_c I_{ao}) = \tfrac{1}{3}\,(Z_a + a^2 Z_b + aZ_c)\, I_{ao}$$

Thus the total e. m. f. using the symbols $S^\circ S'\ S^2$ would be written:

$$S^\circ \left[\tfrac{1}{3}\,(Z_a + Z_b + Z_c)\, I_{ao}\right] + S^1 \left[\tfrac{1}{3}(Z_a + aZ_b + a^2 Z_c) I_{ao}\right] + S^2 \left[\tfrac{1}{3}(Z_a + a^2 Z_b + aZ_c\right] I_{ao} = S^\circ\,(Z_{ao} I_{ao}) + S'\,(Z_{a1} I_{ao}) + S^2\,(Z_{a2} I_{ao}).$$

Where
$$Z_{ao} = \tfrac{1}{3}\,(Z_a + Z_b + Z_c)$$
$$Z_{a1} = \tfrac{1}{3}\,(Z_a + aZ_b + a^2 Z_c)$$
$$Z_{a2} = \tfrac{1}{3}\,(Z_a + a^2 Z_b + aZ_c)$$

Now consider the e. m. f's. given by the positive phase sequence currents, I_a, $a^2 I_{a1}$, aI_{a1}. They will be $Z_a I_{a1}$, $Z_b a^2 I_{a1}$ $Z_c a I_{a1}$.

Resolve the system of em. f's. into symmetrical components. We find for the zero phase sequence component:

$$\tfrac{1}{3}\,(Z_a I_{a1} + Z_b a^2 I_{a1} + Z_c a I_{a1}) = \tfrac{1}{3}\,(Z_a + a^2 Z_b + aZ_c)\, I_{a1} = Z_{a2} I_{a1}$$

For the positive phase sequence component we find

$$\tfrac{1}{3}\,(Z_a I_{a1} + a Z_b a^2 I_{a1} + a^2 Z_c a I_{a1}) = \tfrac{1}{3}\,(Z_a + Z_b + Z_c)\, I_{a1} = Z_{a0} I_{a1}$$

For the negative phase sequence component we find

$$\tfrac{1}{3}\,(Z_a I_{a1} + a^2 Z_b a^2 I_{a1} + a Z_c a I_{a1}) = \tfrac{1}{3}\,(Z_a + a Z_b + a^2 Z_c)\, I_{a1} = Z_{a1} I_{a1}$$

Lastly, consider the e. m. f.'s produced by negative phase sequence currents, I_{a2}, $a\ I_{a2}$, $a^2 I_{a2}$. The zero phase sequence component will be

$$\tfrac{1}{3}\,(Z_a I_{a2} + Z_b a I_{a2} + Z_c a^2 I_{a2}) = \tfrac{1}{3}\,(Z_a + a Z_b + a^2 Z_c)\, I_{a2} = Z_{a1} I_{a2}.$$

The positive phase sequence e. m. f. component will be

$$\tfrac{1}{3}\,(Za I_{a2} + a Z_b a I_{a2} + a^2 Z_c a^2 I_{a2}) = \tfrac{1}{3}\,(Z_a + a^2 Z_b + a Z_c)\, I_{a2} = Z_{a2} I_{a2}.$$

The negative phase sequence e. m. f. component will be

$$\tfrac{1}{3}\,(Z_a I_{a2} + a^2 Z_t a I_{a2} + a Z_c a^2 I_{a2}) = \tfrac{1}{3}\,(Z_a + Z_t + Z_c)\, I_{a2} = Z_{a0} I_{a2}$$

The components of voltage are then expressed easily in terms of the quantities Z_{a0}, Z_{a1}, Z_{a2}, defined in equations (**8**).

Let us bring these results together in tabular form where the relations between them can be observed.

Currents	Voltage Components		
	S° or zero phase sequence	S^1 or positive phase sequence	S^2 or negative phase sequence
Zero phase sequence, $S^{\circ}\ (I_{ao}) = (I_{ao}\ I_{ao}\ I_{ao})$	$Z_{ao} I_{ao}$	$Z_{a1} I_{ao}$	$Z_{a2} I_{ao}$
Positive phase sequence, $S^1\ (I_{a1}) = (I_{a1},\ a^2 I_{a1},\ a I_{a1})$	$Z_{a2} I_{a1}$	$Z_{ao} I_{a1}$	$Z_{a1} I_{a1}$
Negative phase sequence $S^2\ (I_{a2}) = (I_{a2},\ a I_{a2},\ a^2 I_{a2})$	$Z_{a1} I_{a2}$	$Z_{a2} I_{a2}$	$Z_{ao} I_{ao}$

Studying the table above, we see that each symmetrical component of current gives rise to symmetrical voltages of all three phase sequences. We notice first the e. m. f. component which is of the same phase sequence as the current by which it is produced, is obtained by multiplying the producing currents by Z_{ao}. That is the unbalanced apparatus may be said to have one component of impedance, Z_{ao}, which gives e. m. f. 's of the same phase sequence as the currents. This component of impedance does not change the exponent of the S symbol defining the phase sequence of the currents. Thus Z_{ao} alone corresponding to the currents $S^{\circ}\ (I_{ao})$ gives the e. m. fs. $S^{\circ}\ (Z_{a1} I_{ao})$; corresponding to the

Currents S^1 (I_{a1}) gives the e. m. fs. S^1 $(Z_{ao}I_{a1})$; and corresponding to the

Currents S^2 (I_{a2}) gives the e. m. fs. S^2 $(Z_2\ I_{a2})$.

Now let us examine how Z_{a1} enters in the above table. Corresponding to the zero phase sequence.

Currents S^0 (I_{ao}) we find the e. m. fs. S^1 $(Z_{a1}I_{ao})$ of positive phase sequence. Corresponding to the positive phase sequence.

Currents S^1 (I_{a1}) we find the e. m. fs. S^2 $(Z_{a1}I_{a1})$ of negative phase sequence. Corresponding to the negative phase sequence.

Currents S^2 (I_{a2}) we find the e. m. fs. S^0 $(Z_{a1}I_{a2})$ of zero phase sequence.

If we agree now to the following definitions of the symbols S when affected with higher exponents:

$$S^3 = S^\circ;\ S^4 = S^1;\ S^5 = S^2;\ S^6 = S^\circ;\ S^7 = S^1;\ \text{etc.}$$

We see that the way in which Z_{a1} terms enter can be summarized in this way:

The unbalanced apparatus has a component of impedance, Z_{a1}, which increases the exponent of the phase sequence symbol by unity.

Lastly, let us see how Z_{a2} enters the above table. We find that:

Currents S° (I_{ao}) give e. m. fs. S^2 $(Z_{a2}I_{ao})$.
Currents S^1 (I_{a1}) give e. m. fs. S° $(Z_{a2}I_{a1})$.
Currents S^2 (I_{a2}) give e. m. fs. S^1 $(Z_{a2}I_{a2})$.

This may be summarized by saying that the unbalanced apparatus has a component of impedance Z_{a2} which increases the exponent of the sequence symbol by two.

All these results will be obtained automatically if we suppose that symbols S°, S^1, S^2, are attached respectively to Z_{ao}, Z_{a1}, Z_{a2}, and when multiplying the current components, S° (I_{ao}), S^1 (I_{a1}), S^2 (I_{a2}), the exponential laws hold. Thus:

$$\begin{aligned}
S^\circ (Z_{ao})\, S^\circ (I_{ao}) &= S^\circ (Z_{ao}\, I_{ao});\ S^1 (Z_{a1})\, S^\circ\cdot (I_{ao}) = S^1 (Z_{a1}I_{ao}).\\
S^\circ (Z_{ao})\, S^1 (I_{a1}) &= S^1 (Z_{ao}\, I_{a1});\ S^1 (Z_{a1})\, S^\circ (I_{a1}) = S^2 (Z_{a1}\, I_{a1}).\\
S^\circ (Z_{ao})\, S^2 (I_{a2}) &= S^2 (Z_{ao}\, I_{a2});\ S^1 (Z_{a1})\, S^2 (I_{ao}) = S^3 (Z_{a1}\, I_{a2}).\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = S^\circ (Z_{a1}\, I_{a2}).
\end{aligned}$$

$$\begin{aligned}
S^2 (Z_{a2})\, S^\circ(I_{ao}) &= S^2 (Z_{a2}I_{ao})\\
S^2 (Z_{a2})\, S^1(I_{a1}) &= S^3 (Z_{a2}I_{a1}) = S^0(Z_{a2}I_{a1})\\
S^2 (Z_{a2})\, S^2 (I_{a2}) &= S^4 (Z_{a2}I_{a2}) = S^1 (Z_{a2}I_{a2})
\end{aligned}$$

The whole result can be expressed by writing the total impedance, $S(Z_a) = S^\circ Z_{ao} + S^1 Z_{a1} + S^2 Z_{a2}$.

To get the e. m. fs. corresponding to any currents, we multiply together the total impedance by the total current, following merely the rules of algebra, and interpreting higher powers of S as described above. Thus we get:

$$\begin{aligned}
S(E_a) &= S(Z_a)\, S(I_a)\\
&= (S^\circ Z_{ao} + S^1 Z_{a1} + S^2 Z_{a2})\, (S^0 I_{ao} + S^1 I_{a1} + S^2 I_{a2})\\
&= S^\circ(Z_{ao}I_{ao} + Z_{a1}I_{a2} + Z_{a2}I_{a1}) + S^1(Z_{ao}\, I_{a1}\\
&\qquad + Z_{a1}I_{ao} + Z_{a2}I_{a2}) + S^2 (Z_{ao}I_{a2} + Z_{a1}I_{a1} + Z_{a2}I_{ao})
\end{aligned}$$

We get here the foundation of a complex three-phase algebra, which performs the same functions for three-phase systems as the usual complex quantity does for simple alternating current.

This leads to another interpretation or mode of viewing the sequence symbols, S°, S^1, S^2, and here again we may profit by analogy with the j in the complex algebra of simple alternating quantities. Suppose i is a current in phase with some reference alternating quantity. If this current flows through a resistance r. the resulting e. m. f., ri is also in phase with reference alternating quantity. If the resistance is of unit value the resulting e. m. f. is i. Suppose, however, that the current flows through a reactance whose impedance is x ohms. The resulting e. m. f. will be $x\,I$ in magnitude, but it will be in phase quadrature with the reference alternating quantity. This is taken care of in the usual theory by affecting the expression for the reactance with a j, thus xj, and making j have the property that when multiplying a vector or alternating quantity it does not change the value of the vector but merely advances its phase by ninety degrees This is a different meaning from what was given before. j here no longer represents a unit vector or alternating quantity. It now represents an operator, or a symbol for advancing the phase of any vector which it multiplies. Where j by itself is referred to as a vector, it should be understood that ji is meant; where i is a unit vector in phase with the reference vector.

It is clear that operating twice successively by j upon a vector merely reverses the phase of that vector. Thus $j\,(j.i) = -1$, which leads to the rule of multiplication $J^2 = -1$. Lastly, we see that the e. m. f. induced by a current flowing through an impedance may be obtained correctly both in phase and magnitude by multiplying the current by a number of the form $r + jx$, which represents the complete impedance. Thus the complex algebra of simple alternating quantities is born.

We may attach significance as operators in a similar way to the symbols S°, S^1, S^2. Now they shall no longer represent polyphase vectors, but shall merely be operators which when written next to a polyphase vector, change it into a polyphase vector of another phase sequence. The quantities which the S's affect shall be considered as polyphase vectors. Thus the symbol I shall stand for three currents, each equal to I. Operating on I with S° means multiplying each of these currents by unity. Thus S° is a unity operator and does not change the polyphase vector upon which it operates. A separate symbol for it might have been omitted.

$S^1\,I$ or S^1 operating on I, means that the three currents I, I, I, are to be multiplied by 1, a^2, a, respectively; or that the first current is to be unchanged, the second to be advanced in phase by 120 deg. and the third by 240 deg. S' by itself shall not mean anything unless it is understood to be followed by 1 in which case it stands for the unit polyphase vector of positive, rotation obtained by operating with S^1 on the three currents (1, 1, 1).

Similarly, S^2I or S^2 operating on I, means that the three currents I, I, I, are to be multiplied by I, a, a^2, respectively; or that the first current is to be unchanged, the second advanced in phase by 240 deg. and the third by 120 deg. S^2 by itself shall not mean anything unless it is understood to be I, in which case it stands for the unit polyphase vector of negative phase rotation obtained by operating upon the polyphase quantity (I, I, I).

With the meaning now given to S°, S^1, S^2, it at once follows that they satisfy the law of exponents. For example, S^1 (S^1I) means leave the first current I unchanged; advance the second current I, in phase by 120 deg., and then again by 120 deg; advance the third current, I, by 240 deg. and than again by 240 deg. It is clear that the final results are exactly the same as the results of operating with S^2 on I. Also it is clear that multiplication by an operators and a constant Z is commutative. Thus $ZS^1I = S^1Z_{a1}I$. This serves as the foundation of the complex three-phase algebra.

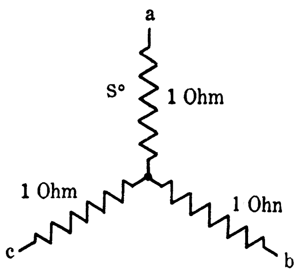

FIG. 4

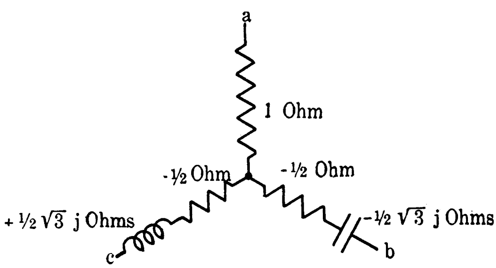

FIG. 5

In the complex algebra of simple alternating quantities, it is easy to give an illustration of the operator J as in impedance. In fact, a reactive impedance of one ohm gives for any current an e. m. f. which may be obtained from the current by multiplying by J. Can we similarly illustrate the three operators S°, S^1, S^2.

S°, of course, is easy. Take a grounded star of resistances, each of one ohm. Then we find for the currents $S^\circ\ (I) = (I, I, I)$ the voltages, $(I, I, I) = S^\circ I$; for the currents $S^2\ (I) = (I, aI, a^2I)$ the voltages $(I, aI, a^2I) = S^2I$.

Now for S^1. Take a grounded star in which the impedance of phase a is 1 ohm, of phase b, $(-\frac{1}{2} - \frac{1}{2}\sqrt{3J})$ ohms $= a^2$ ohms, and of phase c, $(-\frac{1}{2} + \frac{1}{2}\sqrt{3J})$ ohms $= a$ ohms. Now we find for the currents $S^\circ\ (I) = (I, I, I)$, the e. m. fs. $(I, a^2I, aI) = S^1I = S^1\ (S^\circ I)$ for the currents $S^1\ (I) = (I, a^2I, aI)$, the e. m. fs. $(I, aI, a^2I) = S^2I = S'\ (S'I)$; for the currents $S^2\ (I) = (I, aI, a^2I)$, the e. m. fs. $(I, I, I) = S^\circ I = S^1\ (S^2I)$. Fig.5.

Lastly, S^2. Take a grounded star in which the impedance of phase a is 1 ohm, of phase b, $(-\frac{1}{2} + \frac{1}{2}\sqrt{3J})$ ohms, and of phase c, $(-\frac{1}{2} - \frac{1}{2}\sqrt{3}j)$ ohms. Then we find for the currents $S^{\circ}(I) = (I, I, I)$, the e. m. fs. $(I, aI, a^2I) = S^2I = S^2S^{\circ}I$; for the currents $S'(I) = (I, a^2I, aI)$ the e. m. fs. $(I, I, I) = S^{\circ}I = S^2S'I$; for the currents $S^2(I) = (I, aI, a^2I)$ the e. m. fs. $(I, a^2I, aI) = S^1I = S^2S^2I$. Fig. 6.

In these examples, negative resistances appear. This is not surprising, as it is clear that unbalanced apparatus, when current of one phase sequence flows, may feed energy into impressed

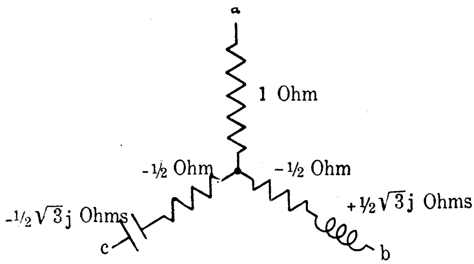

FIG. 6

e. m. fs. of another phase sequence. The negative resistance may be represented physically by a series commutator a-c. motor, driven at constant speed in the opposite of its motoring direction of rotation.

The resolution of an arbitrary three-phase star impedance into its three-phase components is merely the problem of finding three three-phase impedances of the S°, S^1 and S^2 type, respectively, which put in series as in Fig. 7 will reproduce the given three-phase star.

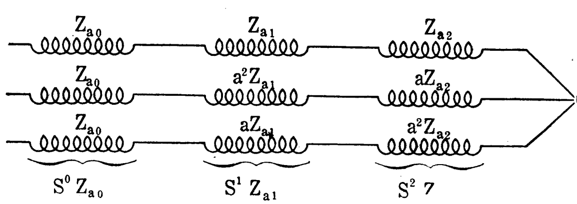

FIG. 7

It is clear that admittances may be treated in the same way as impedances above. Thus the general three-phase admittance may be written in terms of its components, thus:

$$S(Y_a) = S^{\circ}Y_{ao} + S^1Y_{a1} + S^2Y_{a2}$$

The currents produced by a general three-phase voltage $S(E_a) = S^{\circ}E_{ao} + S^1E_{a1} + S^2E_{a2}$ will be:

$$S\,(I_a) = (S^\circ Y_{ao} + S^1 Y_{a1} + S^2 Y_{a2})\quad (S^\circ E_{ao} + S^1 E_{a1} + S^2 E_{a2}).$$
$$= S^\circ\,(Y\ E_{ao} + Y_{a1}E_{a2} + Y_{a2}E_{a1})$$
$$+ S^1\,(Y_{ao}\,E_{a1} + Y_{a1}E_{ao} + Y_{a2}\,E_{a2})$$
$$+ S^2(Y_{ao}\,E_{a2} + Y_{a1}E_{a1} + Y_{a2}E_{ao}).$$

In the complex algebra of simple alternating currents we pass formally from an impedance $R + jx$ to the corresponding admittance by taking the reciprocal, $\frac{l}{r + jx}$ and multiplying numerator and denominator by $r - jx$ to reduce it to the standard form.

$$\text{Thus } \frac{l}{r + Jx} \times \frac{r-jx}{r-jx} = \frac{r}{r^2 + x^2} - j\,\frac{x}{r^2 + x^2}$$

Similarly in this three-phase complex algebra we obtain the admittance corresponding to an impedance $S^\circ Z_{ao} + S^1 Z_{a1} + S\,Z_{a2}$ by taking the reciprocal $\frac{1}{S^\circ Z_{ao} + S^1 Z_{a1} + SZ_{a2}}$. But now to reduce to standard form we must multiply numerator and denominator by

$$S^\circ\,(Z_{ao} + aS^1Z_{a1} + a^2S^2Z_{a2})\,(S^\circ Z_{ao} + a^2S^1Z_{a1} + aS^2Z_{a2}).$$

Thus:

$$\frac{1}{S^\circ Z_{ao} + S^1Z_{a1} + S^2Z_{a2}} \cdot \frac{S^\circ Z_{ao} + aS^1Z_{a1} + a^2S^2Z_{a2}}{S^\circ Z_{ao} + aS^1Z_{a1} + a^2S^2Z_{a2}} \cdot \frac{S^\circ\,Z_{ao} + a^2S^1Z_{a1} + aS^2Z_{a2}}{S^\circ\,Z_{ao} + a^2S^1Z_{a1} + aS^2Z_{a2}}$$

$$= \frac{S^\circ\,(Z_{ao}^2 + a^2Z_{a1}Z_{a2} + aZ_{a1}Z_{a2}) + S^1(a^2Z_{ao}Z_{a1} + aZ_{ao}Z_{a1} + Z^2_{a2}) + S^2(a^2Z_{ao}Z_{a2} + Z^2_{a1} + aZ_{ao}Z_{a2})}{D}$$

$$= \frac{S^\circ\,(Z^2_{ao} - Z_{a1}\,Z_{a2}) + S^1\,(Z^2_{a2} - Z_{ao}\,Z_{a1}) + S^2\,(Z^2_{a1} - Z_{ao}\,Z_{a2})}{S^\circ(Z^3_{ao} + Z^3_{a1} + Z^3_{a2} - 3\,Z_{ao}\,Z_{a1}\,Z_{a2})}$$

Remembering that S° is an operator which does not change the quantity upon which it acts, and therefore that a symbol for it might have been omitted, we may leave off the S° in the denominator and get for the final admittance:

$$S\,(Y_a) = S^\circ\,\frac{Z^2_{ao} - Z_{a1}\,Z_{a2}}{Z^3_{ao} + Z^3_{a1} + Z^3_{a2} - 3\,Z_{ao}\,Z_{a1}\,Z_{a2}}$$
$$+ S^1\,\frac{Z^2_{a2} - Z_{ao}\,Z_{a1}}{Z^3_{ao} + Z^3_{a1} + Z^3_{a2} - 3\,Z_{ao}\,Z_{a1}\,Z_{a2}}$$
$$+ S^2\,\frac{Z^2_{a1} - Z_{ao}\,Z_{a2}}{Z^3_{ao} + Z^3_{a1} + Z^3_{a2} - 3\,Z_{ao}\,Z_{a1}\,Z_{a2}}$$

The principles just enunciated enable us to put together three-phase apparatus and calculate the resulting three-phase components just as is done in single-phase apparatus. For apparatus in parallel, we add admittances; for apparatus in series we add impedances.

So far, in the static apparatus considered, the mutual inductance between "legs" of the three-phase apparatus has been assumed zero. When these mutual inductances are not zero, things become more complicated. A three-phase apparatus with grounded neutral is, of course, a four terminal network, and as is well known, in the most general case would require $\frac{4 \times 3}{2} = 6$ complex constants to characterize it under steady state conditions. It is clear that a complex three-phase expression $S^o Z_{ao} + S^1 Z_{a1} + S^2 Z_{a2}$ depends on only three complex constants, and hence cannot represent the most general three-phase apparatus.

The relations between the symmetrical components of voltage and current on the general static three-phase network are given in equations (**25**). One way of summarizing these equations would be to say that the three-phase apparatus has three different impedances for the three phase-sequence currents. Thus the impedance to zero phase sequence current as given by equations (**25**) is:

$$S^\circ (Z_{aao} + 2 Z_{bco}) + S^1 (Z_{aa1} - Z_{bc1}) + S^2 (Z_{aa2} - Z_{bc2});$$

the impedance to positive phase sequence currents is

$$S^\circ (Z_{aao} - Z_{bco}) + S^1 (Z_{aa1} + 2 Z_{bc1}) + S^2 (Z_{aa2} - Z_{bc2});$$

and the impedance to negative phase sequence currents is

$$S^\circ (Z_{aao} - Z_{bco}) + S^1 (Z_{aa1} - Z_{bc1}) + S^2 (Z_{aa2} + Z_{bc2}),$$

Another way of summarizing the equations (**25**) would be to say that the actual three-phase apparatus is equivalent to a three-phase network having the three-phase impedance

$$S^\circ (Z_{aao} - Z_{bco}) + S^1 (Z_{aa1} - Z_{bc1}) + S^2 (Z_{aa2} - Z_{bc2}),$$

in series with a network whose impedance to zero, positive and negative phase sequence current are respectively, $S^\circ 3\, Z_{bco}$; $S^1\, 3\, Z_{bc1}$; $S^2\, 3\, Z_{bc2}$.

With the ideas developed in this paper, solutions of problems on symmetrical rotating machines with unbalanced static apparatus may be worked with comparative ease. As an example I shall consider the case of a generator, with ungrounded neutral, acting on a given star load, of leg impedances, Z_a, Z_b, Z_c. Suppose the machine is symmetrical, *i. e.*, non-salient poles, and a damping squirrel cage on the field. In this case the generator has an impedance Z_1 to positive phase sequence current, and a small impedance Z_2 to negative phase sequence currents. The positive phase sequence impedance Z_1 is what has been called the synchronous impedance of the machine such that if E is the no load voltage, $E - Z_1 I$ is the voltage on a positive sequence balanced load I. The problem is to find the currents and voltages on the load, and the voltage between load neutral and generator neutral.

We have given that $I_{ao} = O$.

We calculate the impedance drops of positive phase sequence and put their sum equal to the generated positive sequence voltage. Thus:

$$Z_1 I_{a1} + Z_{ao} I_{a1} + Z_{a2} I_{a2} = E.$$

Similarly, since the generated negative phase sequence voltage is zero,

$$Z_2 I_{a2} + Z_{a1} I_{a1} + Z_{ao} I_{a2} = 0.$$

Solving for I_{a1} and I_{a2}

$$I_{a1} = E \frac{Z_2 + Z_{ao}}{Z_1 Z_2 + (Z_1 + Z_2) Z_{ao} + Z^2_{ao} - Z_{a1} Z_{a2}}$$

$$I_{a2} = E \frac{- Z_{a1}}{Z_1 Z_2 + (Z_1 + Z_2) Z_{ao} + Z^2_{ao} - Z_{a1} Z_{a2}}$$

The terminal voltages will be:

$$E_{a1} = E - Z_1 I_{a1} = Z_{ao} I_{a1} + Z_{a2} I_{a2} = E \,.\, \frac{Z^2_{ao} - Z_{a1} Z_{a2} + Z_2 Z_{ao}}{0}$$

$$E_{a2} = - Z_2 I_{a2} = Z_{a1} I_{a1} + Z_{ao} I_{a2} = E \frac{Z_2 Z_{a1}}{0}$$

The voltage between load neutral and generator neutral will be:

$$E_{ao} = Z_{a1} I_{a2} + Z_{a2} I_{a1} = E \,.\, \frac{- Z^2_{a1} + Z_{ao} Z_{a2} + Z_2 Z_{a2}}{D}$$

A special case of interest is where $Z_b - Z_c$ are zero, that is, a single-phase short circuit. We then have:

$$Z_{ao} = \tfrac{1}{3} Z_a \qquad Z_{a1} = \tfrac{1}{3} Z_{a1} \qquad Z_{a2} = \tfrac{1}{3} Z_a$$

$$I_{a1} = E \frac{(Z_a + 3 Z_2)}{3 Z_1 Z_2 + (Z_1 + Z_2) Z_a}$$

$$I_{a2} = E \frac{-Za}{3 Z_1 Z_2 + (Z_1 + Z_2) Z_a}$$

$$E_{a1} = E \cdot \frac{Z_2 Z_a}{3 Z_1 Z_2 + (Z_1 + Z_2) Z_a}$$

$$E_{a2} = E \cdot \frac{Z_2 Z_a}{3 Z_1 Z_2 + (Z_1 + Z_2) Z_a}$$

$$E_{ao} = E \frac{Z_2 Z_a}{3 Z_1 Z_2 + (Z_1 + Z_2) Z_a}$$

If just previous to the single-phase short circuit the generator was unloaded, Z_a becomes infinite and our final results are:

$$I_{a1} = - I_{a2} = \frac{E}{Z_1 + Z_2}$$

$$E_{a1} = E_{a2} = E_{ao} = E \frac{Z_2}{Z_1 + Z_2}$$

If the generator above did not have a damper winding on the field, it would be an unsymmetrical machine having a single-phase rotor. As we know, the theory of unsymmetrical rotating machines is of considerable complication. Mr. Fortescue has promised a treatment by these methods of the general unsymmetrical rotating machine in a future paper, which I look forward to with great interest.

C. P. Steinmetz: In dealing with calculations or investigations of polyphase systems, or, as usually the case, three-phase systems, the difficulties which we meet are not so much mathematical difficulties, but are what I may call mechanical difficulties. The equations, while mathematically not complicated, lead to expressions which are so complicated and extensive in form as to make any such calculations extremely difficult.

In dealing with the balanced polyphase system, this difficulty has been overcome by the introduction of the equivalent single-phase system, by considering the polyphase system as resolved into a number of single-phase systems, each comprising the circuit from one of the phase wires to the neutral point of the system. This method however, fails in the unbalanced polyphase system—and naturally practically all existing commercial polyphase systems are more or less unbalanced—and the theory of the vector method given to us today in a more extensive description by Mr. Fortescue, gives the solution by showing us in the case of the general three-phase system that it can be resolved into two balanced three-phase systems of opposite phase rotation. We can apply the same plan to other polyphase systems.

V. Karapetoff: Mr. Fortescue deserves the gratitude of the profession for bringing out a new method for numerical computations in unsymmetrical polyphase systems, and also for applying the method to a number of practical cases.

Expressions (**4**) represent the result of solution of certain equations, which equations unfortunately are not given in the paper. The expressions quoted are unnecessarily involved, and it is difficult to see their method of derivation. Moreover, it may be shown that they do not represent the most general case of resolution of an unsymmetrical system of vectors. The following procedure would seem to be preferable.

Definition. A multiple-angle symmetrical polyphase system of vectors is defined as one in which the vectors are numbered not consecutively, but skipping a certain number of vectors. Thus, Fig. 8 represents a triple-angle seven-phase clockwise system, because three angles are comprised between phase 1 and phase 2, and also between phase 2 and 3, etc. Such a generalization of the concept of polyphase systems is useful when the number of phases and the number of angles skipped are prime numbers, so that all the phases may be numbered consecutively, without omitting or repeating any. If the total number of phases n is a prime number, then the total number of possible combinations or multiple systems is $(n-1)$, the angles between the consecutive phases being $\frac{2\pi}{n}$, $2 \times \frac{2\pi}{n}$, . . . $(n-1)\ \frac{2\pi}{n}$. It is assumed that the numbering in *all* the systems is either clockwise or counter-clockwise, so that there is no confusion between a positive and a negative phase sequence.

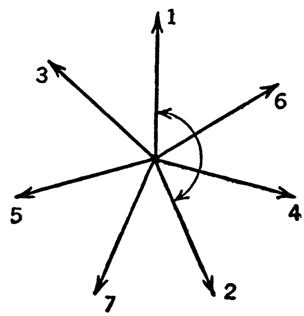

FIG. 8—A TRIPLE-ANGLE SEVEN-PHASE SYMMETRICAL SYSTEM

Theorem 1. An arbitrary system of n unequal and unsymmetrical vectors, *without residue*, may be represented by $(n-1)$ symmetrical multiple n-phase systems. A system without residue is defined as one in which the sum of the vectors is equal to zero. Let the given vectors be $E_1, E_2, \ldots \ldots E_n$, and let the $(n-1)$ unknown systems be denoted by $A, B, \ldots M$, where A is a single-angle system, B is a double-angle system, etc. It is required to prove that the following equations are consistent and may be solved for $A, B, \ldots \ldots M$:

$$\left.\begin{aligned} E_1 &= A_1 + B_1 + \ldots + M_1 \\ E_2 &= A_2 + B_2 + \ldots + M_2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ E_n &= A_n + B_n + \ldots + M_n \end{aligned}\right\} \qquad (\mathbf{1})$$

These equations correspond to eqs. (**4**) in Mr. Fortescue's paper when $\Sigma E = 0$. In order to preserve the same conventions as in the original paper, the component systems are assumed to be numbered clockwise (though the vectors are rotating counter-clockwise). Then if a is an operator which rotates a vector by an angle $2\pi/n$ counter-clockwise, the preceding equations become

$$\left.\begin{array}{l} E_1 = A_1 + B_1 + \ldots + M_1 \\ E_2 = a^{-1} A_1 + a^{-2} B_1 + \ldots + a^{-(n-1)} M_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ E_n = a^{-(n-1)} A_1 + a^{-2(n-1)} B_1 + \ldots a^{-(n-1)(n-1)} M_1 \end{array}\right\} \quad (2)$$

These n equations contain $(n-1)$ unknown vectors $A_1, B_1, \ldots M_1$, but only $(n-1)$ equations are independent of each other, because of the condition $\Sigma E = 0$. To solve for A_1 multiply the second equation by a, the third by a^2, etc. and add them together. The result is

$$n A_1 = E_1 + a E_2 + \ldots + a^{n-1} E_n \quad (3)$$

This checks with the second line of Mr. Fortescue's equations (**4**) By analogy we obtain

$$n B_1 = E_1 + a^2 E_2 + \ldots\ldots + a^{2(n-1)} E_n \ldots\ldots \quad (4)$$

and

$$n M_1 = E_1 + a^{n-1} E_2 + \ldots\ldots a^{(n-1)(n-1)} E_n \quad (5)$$

The theorem is thus proved because equation (**2**) can be solved for the $(n-1)$ unknown vectors in terms of the given E's.

Theorem 2. Referring to theorem 1, if the given system of vectors has a *residue*, say, equal to a vector R, this residue may be split into n arbitrary vectors, $\rho_1, \rho_2, \ldots \rho_n$, in phase or out of phase with one another, and equations (**1**) become

$$\left.\begin{array}{l} E_1 = \rho_1 + A_1 + B_1 + \ldots\ldots + M_1 \\ E_2 = \rho_2 + A_2 + B_2 + \ldots\ldots + M_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ E_n = \rho_n + A_n + B_n + \ldots\ldots + M^2 \end{array}\right\} \quad (6)$$

This proposition follows at once from the fact that the vectors $(E_1 - \rho_1)$, $(E_2 - \rho_2)$, etc. form a system without residue, so that theorem 1 applies to them. Mr. Fortescue considers only a particular case when

$$\begin{array}{l} \rho_1 = \rho_2 = \ldots\ldots = \rho_n = R/n \\ \qquad\qquad\qquad\quad = (E_1 + E_2 + \ldots E_n)\ /n \end{array} \quad (7)$$

and writes this expression as the first term on the right-hand side in all the formulas of his equations (**4**). This is perfectly justifiable in so far as he is after a symmetrical solution, but there may be practical cases in which a more general representation according to equations (**6**) is preferable. In this case the expressions $(E_1 - \rho_1)$, $(E_2 - \rho_2)$, etc. must be used in equations (**3**) to (**5**), instead of E_1, E_2, etc.

The net result is that only a system without residue may be resolved in a perfectly definite manner, while a system with a residue first must be converted into one without residue by subtracting arbitrary parts of the whole residue from each vector.

With a three-phase system the double-angle component simply becomes a three-phase system of an opposite phase rotation, so that it is not necessary to introduce the question of multiple-angle systems. From a practical point of view the paper would gain much if the author would begin directly with the three-phase system on the basis of his equations (**6**), and leave the general consideration of n-phase systems for the end of the paper. For reference purposes one would not have to study the general theory and perhaps get discouraged before obtaining the required bit of specific information.

The introduction of the sequence operator S does not seem to be necessary, at least for the three-phase system. A proper cyclic notation should accomplish the same purpose with much less theory and much less writing. As a matter of fact, the index of S may be obtained correctly from the subscripts in the expression to which it refers. For example, in Mr. Fortescue's formula (**22**) on the first line the sum of the subscripts of $Z\,I$ is $1 + 2 = 3$, and the order of S is 3 or, which is the same, 0. On the second line the sum of the subscripts of $Z\,I$ is $1 + 0 = 1$, and the order of S is 1. The same is true for all the fundamental formulas on pp. 1038 to 1042 as well as in applications. Therefore, it would seem that the operator S ought to be dropped, and the formulas so rewritten that they would apply to any phase of the system in cyclic rotation.

The fundamental formula (**21**) will then simply become

$$E = Z\,I + W_+\,I_+ + W_-\,I_- \qquad (8)$$

Here E, Z, and I apply to any of the three phases, I_+ is the current in the next consecutive phase, and I_- is the current in the preceding phase. For example, if E, Z, and I refer to phase b, I_+ refers to phase c and I_- refers to phase a. At the same time the equation is so written that any of the three phases may be taken as the phase under consideration. W_+ and W_- are the corresponding mutual reactances.

When the mutual inductance between a primary and a secondary polyphase circuit must be considered, Fortescue's equation (**28**) applies, and in the simplified cyclic notation it becomes

$$E = X\,J + X_+ J_+ + X_- J_- \qquad (9)$$

where J, J_+ and J_- are the secondary currents in the three phases. The letter X is used instead of W to indicate that the mutual inductance is between the phases of the primary and the secondary circuit. X without subscript covers the combinations of the "corresponding" phases $a - u$, $b - v$, and $c - w$, while X_+ refers to the combinations $a - v$, $b - w$, $c - u$ of the phases "tied one forward", and X_- refers to the remaining combination of the phases "tied two forward", or "tied one backward".

Each of the quantities in equations (**8**) and (**9**) may be represented by means of symmetrical coordinates in a general manner, without specifying phases *a*, *b* and *c*. Thus,

$$\left.\begin{aligned} I &= I_0 + I_p + I_n \\ I_+ &= I_0 + a\, I_p + a^{-1}\, I_n \\ I_- &= I_0 + a^{-1}\, I_p + a\, I_n \end{aligned}\right\} \qquad (\mathbf{10})$$

Similarly,

$$W_+ = W_0 + a\, W_p + a^{-1}\, W_n \qquad (\mathbf{11})$$

The first equation (**10**) refers to any phase, and states that the current in that phase is a sum of the residue current I_0, plus the vectors I_p and I_n of the symmetrical coordinates *of the same phase*. The subscripts p and n refer to the systems of positive and negative phase rotation respectively. In an n-phase system subscripts 1, 2, 3, etc. should be used in place of $_p$ and $_n$, to indicate single-angle, double-angle, triple-angle, etc. coordinates.

Substituting expressions such as (**10**) and (**11**) into equations (**8**) and (**9**) one obtains comparatively simple polynomials with three groups of terms: those without a, those multiplied by a, and those multiplied by a^{-1}. These final expressions give the coordinates of E, and are the fundamental equations upon which all applications should be based, without any need for the sequence operator S.

Incidentally, I believe that the expression "symmetrical components" is a more correct and descriptive term of the method than the term "symmetrical coordinates", and I should like to see it so changed while it is not too late.

I also suggest that the term "mutual reactance" be used throughout the paper in place of the "mutual impedance". The latter term implies a combined resistance and inductance action, while the formulas in the paper seem to refer to the magnetic inductive action exclusively. If the resistance component is intended to be taken into account, a mathematical definition of the mutual impedance becomes necessary, because two coils are presupposed to have different resistances, and it is not clear how these resistances enter into the expression for mutual impedance.

It may also be mentioned here that there are cases of unsymmetrical polyphase connections which can be solved more readily without resolving the given electrical quantities into their symmetrical components. Several such cases are considered in the present writer's work entitled "Ueber Mehrphasige Stromsysteme bei Ungleichmaessiger Belastung (Enke, Stuttgart, 1900). For example, in the solution of any arbitrary n-phase star-connected system with given voltages E and phase admittances Y, it is convenient to take the voltage e between the two neutral points as the independent variable. Then, for each phase the current $I = Y\,(E - e)$, and according to the first Kirchhoff law

$$\Sigma\, I = \Sigma\, Y\,(E - e) = 0,$$

so that

$$e = (\Sigma\, Y E)\, /\, \Sigma\, Y.$$

Knowing e, the currents in the individual phases are computed from the first equation. In the most general case of a complicated network with a-c. voltages inserted at different places, the Kirchhoff equations of two kinds, written for the vectors of currents and voltages, furnish a complete solution, and one has to consider in individual applications whether or not the resolution into symmetrical components is desirable or not.

A. M. Dudley: Attention has been called to the practical applications of this solution, and one or two may be mentioned here. It has been long known that in the ordinary induction motor in addition to the main rotating working field there may exist several other fields of different amplitude and frequency and phase rotation and that the existence of such fields is probably responsible for the operating freaks sometimes noticed in motors. For example, a motor is sometimes found which will not reverse its mechanical direction of rotation when running light even though the leads be reversed so as to give the opposite direction of phase rotation; or, a two-phase motor will be found which when running light takes power from the line on one phase and returns power to the line on the other, or a, squirrel-cage motor will start from rest with considerably more torque than it is capable of accelerating up to full speed so that it apparently has a so-called sub-synchronous speed. These are actual practical operating conditions often mentioned but never determined quantitatively. Mr. Fortescue's analysis offers the means of such quantitative study.

As a further practical instance may be mentioned the two cases of single-phase connections referred to in the analysis of equations (**141**), (**142**) and (**150**), one of which is equivalent to two coupled polyphase motors connected in parallel and the other to two coupled polyphase motors connected in series. This was fully discussed from another view point in Mr. Lamme's April paper before the Institute and the identity of the two conclusions is a check on Mr. Fortescue's method.

To those who have expressed the idea that the paper is hard to read I would commend Dr. Slepian's discussion also printed herewith as illuminating both to the method and the results. He makes the suggestion that the method as outlined may constitute the basis of a new complex polyphase algebra which will greatly simplify the study of polyphase network problems of all kinds.

Dr. Steinmetz and Prof. Karapetoff have told us that as a mathematical achievement and as a demonstration of theory this paper is a masterpiece. I should like to venture the prediction that as a practical working tool it will eventually come into the greatest usefulness.

Let us then pay our respects to Mr. Fortescue both as a great mathematician and as a scientist and also as the developer of a practical tool which shall make easier the daily task of the engineer in the ranks.

Charles F. Scott: Mathematics and engineering have their varying conditions, and they have their supporters and their critics. To a certain class of engineers the mathematician is an abstract man who starts somewhere and finds something that can lead to consecutive conclusions. On the other hand, a practical man, using that term in the extreme, is apt to rely wholly on experience and to have no use for the mathematics he does not understand. We all know that the really valuable position is the mean between these two, in which the theoretical and the practical do go together.

The reference to papers of a mathematical and explanatory character, and their relations to engineering reminded me somehow of the first Institute paper I came in contact with. It was through the pages of the *Electrical World*. I had been experimenting with alternating currents and was associated with men who knew only slightly more about it than I did. Many things seemed mysterious. Then the paper by Mr. William Stanley on "Alternating-Current Phenomena," appeared, and that mysterious thing which was the stumbling block of about everybody who came up against alternating-current, the simple single-phase phenomenon of current and voltage out of phase, was cleared up. Mr. Stanley drew some triangles and explained a few things. That put me on a new basis. I could see how the mathematical explanation, the physical phenomena, were related, and it gave me a real understanding of the alternating current. A little later came the polyphase, and it was a step from that to the work on the unbalanced polyphase system.

As we advance then, getting into more and more complicated problems, these papers of solution and explanation come to guide the engineer in his work.

What is this paper? When you look at a paper of this sort it looks as if a mathematician had produced it, leading on and on to interesting formulas, one after another. Now, to bring out my point, I want to emphasize and say I do not think this is a product of the mathematician, but of the engineer. Mr. Fortescue did not work these things out from the theoretical, mathematical standpoint, but as an engineer who was confronted with problems in which unbalanced polyphase systems were concerned. It may have been transmission or some kind of particular machinery which involved problems which he did not have the proper tools to solve.

Therefore, in studying the problems and seeing the inadequacy of the mathematical tools at hand, he went to work and constructed tools to perform that work. Mr. Fortescue has done just as some other engineers do, who work first with experiments, then go to mathematics, and then back to the experiments. That combination of first theory and then practise, back and forth, constitutes the work of the engineer who can be constructive and do pioneer work. I think I am right in saying that this is not a paper of a theoretical mathematician, but a paper of a practical engineer, who is developing a new tool for himself and offering it to others.

C. O. Mailloux: This is the second great paper which Mr. Fortescue has presented before this Institute. Much that I had to say in the way of friendly criticism has already been anticipated by Prof. Karapetoff. I sympathize with Prof. Karapetoff's views, and I share them, in regard to the possibilities of simplification of this paper. I thank Mr. Fortescue and many other members of the Institute will also thank him, for having given an introduction to the paper, because that introduction is really the mathematical specifications in accordance with which the paper has been prepared; and to me it was more valuable, as a criterion of the value of the paper, than the paper itself, because the paper itself cannot be read offhand. The bewildering exhibit of subscripts to be found in it is something that will, well, make one pause; but the introduction, to any one who is at all familiar with mathematics, tells exactly the basis on which the paper is written, what it aims to do, and substantially the method by which it accomplished its aim. I am very thankful, indeed, personally, that that was done.

Part I is also very useful, but if it had followed the method of presentation suggested by Prof. Karapetoff, it would have been much simplified. The paper is so valuable, that it would be well worth while to rewrite it for the purpose of making it more easily digestible and more easily accessible to the great body of practical men. It appears to have been written too much from the point of view of the professed mathematician, who seeks first to make a generalization, and then proceeds to experimentation to find practical and special cases. It would have been a better arrangement if that generalization had been put in an appendix, and if the statement had been made that from the general case is derived the particular case which interests us, namely, the case of the three-phase current. I think that to discuss *n-phases* is of academic interest only at present, even though it is possible that there may be some day when it will be of practical importance. Until that seemingly remote day comes, a generalization which includes n-phases might as well lie in an appendix, out of harm's way, and especially where it would not encumber the rest of the discussion. If Mr. Fortescue had written the paper by putting into the appendix, the generalization, the thing which makes it complete and comprehensive, he would have simplified it and made it more useful.

It is because I think so highly of the paper, I would like to see it made clearer. It is not a difficult paper, the mathematics are simple, but they look complex, because one is bewildered by a maze of subscripts and by many "operators." The fundamental idea of the operator is a splendid one, but it is just as well to put that explanation in an appendix, in all its generality, and then use it in the simple form that Prof. Karapetoff suggests. Those who deal with the matter practically, would not then have so many equations to analyze and not so many symbols to deal with.

The analysis of the actions and reactions of the various forces present in electrical circuits, and the interpretation of the equations devised for expressing these actions and reactions, under different conditions, whether they occur in balanced or in unbalanced systems, may be facilitated by remembering that all physical forces and all forms of energy are subject to the same general principles of conservation and equilibrium, and, especially, that Newton's principle of dynamic equilibrium,—namely, that under all conditions and at all times, action and reaction are opposed and equal to each other, exactly—, applies to electrical, as well as to mechanical, forces, powers, and energies.

Analogy requires that, in any electrical circuit, at any instant of time whatever, the forces of action and those of reaction should always be opposing and balancing each other exactly, just as they do in mechanical systems; and it is known that they always do balance exactly. Analogy also requires that, for electrical circuits, when dealing with instantaneous values, the equations of electrodynamic equilibrium should have the same general characteristics as the general equation of dynamic equilibrium or balance for mechanical forces. The latter, it is well known, reduces to an algebraical sum of four terms, representing four distinct kinds or amounts of force, or power, or energy, according to the case, whose resultant, when a state of equilibrium occurs, is equal to zero.

The four kinds or amounts of force, power, or energy, in terms of which every dynamic reaction whatever, in mechanics, can be completely expressed, find exact parallels in electrodynamics. Here, also, we find four kinds of force, power, or energy; we have reversible forces of two kinds, corresponding to those which, in mechanics, produce or result from changes in kinetic energy, or changes in potential energy; we have the dissipated force, which is expended in overcoming ohmic resistance, corresponding to the force lost in overcoming friction in mechanics, and we have, for the fourth kind, the force, power, or energy representing the balancing actions or reactions which are necessary to maintain the equilibrium that must exist at every instant of time between action and reaction. The first two kinds represent energy which is not immediately dissipated but is displaced and stored in the system; the third kind represents energy which is immediately consumed in the system; and the fourth kind represents energy which is put into the system from an outside source, or else which is taken out of the system by overflow into another system or circuit, as may be required to maintain equilibrium.

These considerations show that even in an unbalanced electrical system, so-called, there must still be dynamic balance or equilibrium. The balancing force, power, or energy, according to the case, appears in the fourth term, where it represents the compensating force, power, or energy by which equilibrium is maintained.

Charles L. Fortescue: I maintain that the paper is quite simple. The mathematical portion is not difficult to understand, anybody can follow it if he takes the pains. I admit that the appearance of the equations is cumbersome, but that is almost impossible to overcome. The nature of the subject makes the equations cumbersome.

Prof. Karapetoff in his discussion used the word "stimulation". I wish to say that necessarily in a paper of this kind there are many sources of stimulation, as I pointed out in my introduction, many of the ideas are not new. The idea of symmetrical component three-phase systems is being used more or less by others, but the theory has never been presented systematically, and I think the idea of symmetrical operators is new.

Dr. Steinmetz's and Prof. Karapetoff's discussions are what I may term conjugate discussions. Dr. Steinmetz discusses the paper purely from the practical point of view. He thinks that the system is capable of practical application, and will be of great use for that purpose. Dr. Karapetoff points out the possibility, from a theoretical point of view, of the n-phase system. I wish to say nothing would have given me greater pleasure than to go into that purely theoretical matter much more extensively. It is very fascinating and has great promise, but I felt that there was needed, in this paper, some practical justification for presenting it. The theoretical part was so long drawn out, that I felt it was necessary to show by practical illustrations why I presented it.

I felt that the presentation of mathematical solutions did not alone afford a justification for the paper, but that there must also be a good practical reason for it, and I felt if I went into all the theoretical ramifications of this very interesting subject people would ask: "What is he about? What does he mean by driving us through all this painful stuff without giving us a good reason?" So I thought that the presentation of the theory should be as concise and short as possible, and for that reason I left out a good deal of explanation that some people think ought to be there.

Prof. Karapetoff has apparently lost sight of the fact that unless his arbitrary vectors are all equal to $\dfrac{\check{E}_1 + \check{E}_2 + \ldots \check{E}_n}{n}$ they in turn may be resolved into $(n - 1)$ symmetrical systems and the "residue" will eventually reduce to the same value as that given by me, namely n equal vectors of value

$$\frac{\check{E}_1 + \check{E}_2 + \ldots \check{E}_n}{n}.$$

The form in which the theorem is presented does not preclude the adoption of any artifice such as the subtraction of n arbitrary vectors to make the vector sum equal to zero, when any material advantage is gained thereby. Such an artifice, in fact, is generally adopted in the case of Y-connected circuits where the vector sum is rarely zero on account of the presence of the third harmonic. Furthermore, in any symmetrical system there is a definite impedance to each component symmetrical current including the "residue" as Prof. Karapetoff has termed the constant vector, this is not true if an arbitrary "residue" is adopted.

Prof. Karapetoff brings up another point, the use of sequence symbols. In three-phase systems, once a person has become familiar with the use of this method, it becomes quite a natural system. He thinks quite naturally in terms of symmetrical components, and he usually omits the use of the symbols. In other words, the symbols are not absolutely necessary, but they simply afford a sure method of keeping track of the mechanical part of the mathematical work. If one makes a mistake in carrying out the mathematical work, one can always go back to the sequence symbols and check it up. If a quantity looks unsymmetrical, and a mechanical error is suspected, one can always follow it out with the sequence symbols and be sure of getting the correct result. Furthermore, in dealing with a system of more than three phases, the sequence operator becomes important, because there are so many cross mutual inductances that the formulas become very much involved, and it is necessary to have some reliable mechanical device that will keep one in the right path; the sequence operators are such guides. They are almost indispensable when dealing with power both in three-phase and *n*-phase systems. Those are the principal reasons for their use.

Prof. Karapetoff brings up the question—What is the "mutual impedance?" I have not used the term perhaps quite correctly as I have applied it to cases in which the dissipative forces are not strictly proportional to the velocities or currents. Mutual impedance may be defined in this manner: If we have two terminals of a circuit carrying a given current, and two other terminals of another circuit, the electromotive force produced across the second pair of terminals, given a sine-wave current of a given frequency is, the product of the mutual impedance between the two circuits and the current in the first pair of terminals. In some cases the current follows paths common to both circuits, so that the mutual impedance may have a real component; in other cases the energy component may be due to eddy currents or losses in subsidiary circuits common to both.

Mr. Dudley has pointed out that the practical engineer must always keep track of the theoretical side of his work. The object of all mathematical investigation and theory should be to carry our knowledge of operating conditions further, and to investigate such obscure phenomena as arise from time to time. Very often we find that a new tool has to be devised in order to enable us to carry out our theoretical investigations without becoming involved in too great complication.

I feel that a paper presented before a scientific body that cannot be read by every member is hardly justified. I believe that all can follow this paper if they sincerely try to do so, it may take a little time, but the mathematics is not difficultt.

V. Karapetoff (by letter): In my Theorem 2, ρ_2, ρ_2 etc. are arbitrary vectors such that $\Sigma\ \rho = R$, where R is the total residue of the given system of vectors. Each ρ maybe resolved geometrically into a vector R/n, in phase with total R, and another vector, say ρ, so that $\rho = \rho + (R/n)$. Hence $\Sigma\ \rho + (R/n) = R$, or $\Sigma\ \rho' = 0$. Thus, the system of vectors is without residue and consequently may be resolved into $(n - 1)$ symmetrical systems. The system of n vectors each equal to R/n, in phase with total R, is an *irreducible residue.*

When it is desired to sift out all the symmetrical components from the given vectors, the form R/n for the residue should be used. However, there may be conceivable cases in which there might be some advantage in splitting the residue into certain unequal or unsymmetrical parts, without attempting to extract all of its symmetrical components. For this reason, Theorem 2 is stated in the most general form. I am glad to have my attention called to the fact that only in the case where each ρ is equal to R/n, we have the *true* or the *irreducible* residue.