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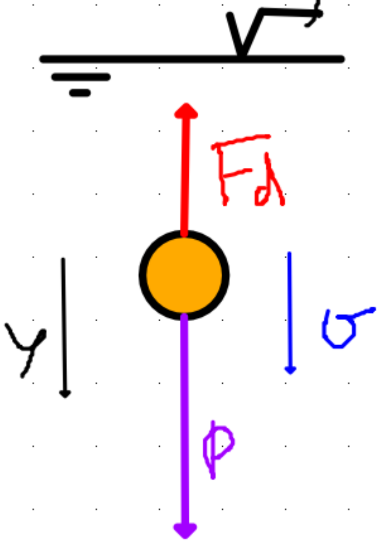
PME 3201 Aula 02 24.08.2023

Turma 21A

Força Viscosa:



$$P = mg - \rho V g = \underbrace{(m - \rho V)}_{m'} g$$

$$m \frac{dv}{dt} = m'g - F_d$$

$m = \rho_{st} V$ $\quad \rho_{st} = 7860 \, kg/m^3$

ρV $\quad \rho = 910 \, kg/m^3$

$Re < 10 \Rightarrow$ força viscosa

$F_d = c\,v$

$Re < 1 \Rightarrow$ Stokes $\quad c = 3\pi \eta D$

$$m\frac{dv}{dt} + cv = m'g$$

Equ. Dif. Ordinária Linear com Coef. Constantes de 1ª Ordem não Homogêneas

v_1 e v_2 são soluções então

$v = \alpha_1 v_1 + \alpha_2 v_2$ também é solução.

equilíbrio: $\frac{dv}{dt} = 0 \Rightarrow c v_{ter} = m'g$

$$v_{ter} = \frac{m'g}{c}$$

$v' = v - v_{ter}$

$\frac{dv}{dt} = \frac{dv'}{dt} \Rightarrow m\frac{dv'}{dt} + c(v' + v_{ter}) = m'g$

$$m\frac{dv'}{dt} + cv' + \cancel{c}\frac{\cancel{m'g}}{\cancel{c}} = \cancel{m'g} = 0$$

$$m\frac{dv}{dt}+cv=0$$

$v=\alpha_1 v_1+\alpha_2 v_2$ and

$$m\frac{dv_1}{dt}+cv_1=0$$

$$m\frac{dv_2}{dt}+cv_2=0$$

$$m\frac{d}{dt}(\alpha_1 v_1+\alpha_2 v_2)+c(\alpha_1 v_1+\alpha_2 v_2)=0$$

$$\alpha_1\left(m\frac{dv_1}{dt}+cv_1\right)+\alpha_2\left(m\frac{dv_2}{dt}+cv_2\right)=0$$

$$\alpha_1 0+\alpha_2 0=0$$

$$m\frac{dv}{dt} + cv + 0 = mg$$

$$m\frac{dv_p}{dt} + cv_p + \left(m\frac{dv_h}{dt} + cv_h\right) = mg$$

onde $m\frac{dv_h}{dt} + cv_h = 0$

é a solução homogênea associada

Portanto:

$$v(t) = v_p(t) + v_h(t)$$

solução geral. solução particular solução homogênea solução homogênea: $v = v_h$

$$m\frac{dv}{dt} + cv = 0$$

$$\frac{dv}{dt} = -\frac{c}{m}v \Rightarrow \frac{dv}{v} = -\frac{c}{m}dt$$

$$\int\frac{dv}{v} = -\frac{c}{m}\int dt \Rightarrow \ln v = -\frac{c}{m}t + K$$

$$v = e^{\left(-\frac{c}{m}t + K\right)} = e^{K} e^{-\frac{c}{m}t} \qquad e^{K} = A$$

$$v(t) = A e^{-c/m\, t} \quad \Big| \quad \forall A \in \mathbb{R}$$

solução homogênea

solução particular: $v = v_p$

$$m\frac{dv}{dt} + cv = m'g$$

$$v(t) = B \Rightarrow \frac{dv}{dt} = 0$$

$$m\,0 + cB = m'g \Rightarrow B = \frac{m'g}{c}$$

$$v(t) = \frac{m'g}{c}$$

solução particular.

Solução geral:

$$v(t) = v_h(t) + v_p(t)$$

$$\boxed{v(t) = Ae^{-t/\tau} + \frac{m'g}{c}}$$

$$\tau = \frac{m}{c}$$

Solução Geral:

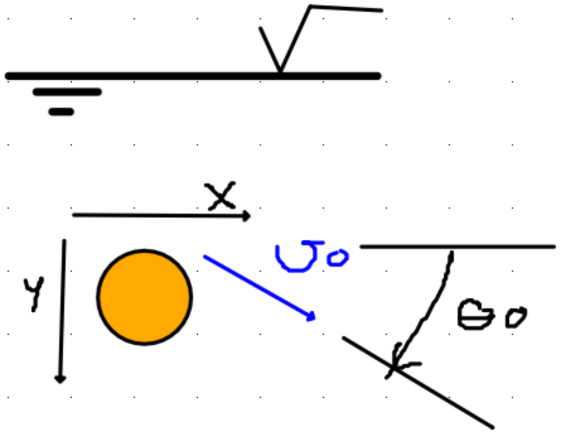
$$v(0) = v_0$$

$$v(0) = Ae^{-0} + \frac{m'g}{c} = v_0 \implies A = v_0 - \frac{m'g}{c}$$

$$= v_0 - v_{ter}$$

$$\boxed{v(t) = v_0 e^{-t/\tau} + v_{ter}\left(1 - e^{-t/\tau}\right)}$$

$$\tau = m/c$$



$$\vec{v}_0 = \underbrace{v_0 \cos\theta_0}_{v_{x0}}\,\hat{i} + \underbrace{v_0 \,\text{sen}\,\theta_0}_{v_{y0}}\,\hat{j}$$

$$\vec{v}_0 = v_{x0}\,\hat{i} + v_{y0}\,\hat{j}$$

$$\vec{F}_d = -c\,\vec{v} \Rightarrow -c v_x\,\hat{i} - c v_y\,\hat{j}$$

$$m\vec{a} = \vec{P} + \vec{F}_d$$

$$\vec{P} = mg\,\hat{j}$$

$$m\vec{a} = m a_x\,\hat{i} + m a_y\,\hat{j} = mg\,\hat{j} - c v_x\,\hat{i} - c v_y\,\hat{j}$$

$$\begin{cases} m a_x = -c v_x \\ m a_y = mg - c v_y \end{cases}$$

equações desacopladas

en x: $$m\frac{dv_x}{dt} + c\,v_x = 0$$

$$\longrightarrow \quad v_x = v_{x0}\left(1 - e^{-t/\tau}\right)$$

en y: $$m\frac{dv_y}{dt} + c\,v_y = mg$$

$$\longrightarrow \quad v_y = v_{y0}\,e^{-t/\tau} + v_{ter}\left(1 - e^{-t/\tau}\right)$$

$$\frac{dx}{dt} = v_x \quad \Longrightarrow \quad x(t) = v_{x0}\,\tau\left(1 - e^{-t/\tau}\right)$$

$$\frac{dy}{dt} = v_y \quad \Longrightarrow \quad y(t) = \left(v_{y0} + v_{ter}\right)\tau\left(1 - e^{-t/\tau}\right) - v_{ter}\,t$$

bdandut

...

$$y = \frac{v_{y0} + v_{ter}}{v_{x0}} \cdot x + v_{ter}\,\tau \ln\left(1 - \frac{x}{v_{x0}\tau}\right)$$

Caminho da Esfera!

Alcance R (Range)

sem resistência: $R_{vac} = 2\frac{v_{x0}\, v_{y0}}{g}$ (Balística)

com resistência viscosa:

$$\frac{v_{y0} + v_{ter}}{v_{x0}} \cdot R + v_{ter}\,\tau \ln\left(1 - \frac{R}{v_{x0}\tau}\right) = 0$$

$R = ?$

$$\ln(1-\epsilon) \qquad \epsilon = \frac{R}{v_{x0}\tau}$$

Série de Taylor! hip. ϵ pequeno

$$\ln(1-\epsilon) = -\left(\epsilon + \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3 + \dots\right)$$

assim!

$$\left[\frac{v_{y0} + \cancel{v_{ter}}}{v_{x0}}\right] R - v_{ter}\tau\left[\cancel{\frac{R}{v_{x0}\tau}} + \frac{1}{2}\left(\frac{R}{v_{x0}\tau}\right)^2 + \frac{1}{3}\left(\frac{R}{v_{x0}\tau}\right)^3\right]$$

$$\frac{v_{y0}}{v_{x0}} R - \left(\frac{R}{v_{x0}\cancel{\tau}}\right)^2 v_{ter}\cancel{\tau}\left[\frac{1}{2} + \frac{1}{3}\frac{R}{v_{x0}\tau}\right] = 0$$

$$\frac{v_{y0}}{v_{x0}} - R\left(\frac{1}{v_{x0}\cancel{\tau}}\right)^2 v_{ter}\,\cancel{\tau}\left[\frac{1}{2} + \frac{1}{3}\,\frac{R}{v_{x0}\,\tau}\right] = 0$$

$$\vdots$$

$$R = \frac{2 v_{x0} v_{y0}}{g} - \frac{2}{3 v_{x0}\tau} R^2$$

$$R = R_{vac} - \frac{2}{3 v_{x0}\tau} \cdot R^2$$

$$R \simeq R_{vac} \Rightarrow R \simeq R_{vac}\left(1 - \frac{4}{3}\,\frac{v_{y0}}{v_{ter}}\right)$$

Com Resistência Hidrodinâmica:

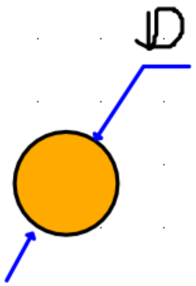
com $10^3 < Re < 10^5 \Rightarrow C_D(v) = C_D = 0{,}47$ (cte)

coeficiente de arrasto para esfera

Força de Arrasto Hidrodinâmica

$$\boxed{F_D = \frac{1}{2}\rho C_D \cdot A v^2}$$

$$A = \frac{\pi D^2}{4}$$



$$f_d = f_0 + f_1 v + f_2 v^2$$

mov. horizontal:

$$m\frac{dv}{dt} = -bv^2 \qquad b = \frac{1}{2}\rho C_D A$$

$$\frac{dv}{v^2} = -\frac{b}{m}dt$$

$$\int_{v_0}^{v}\frac{dv}{v^2} = -\frac{b}{m}\int_0^t dt \Rightarrow \left(\frac{1}{v_0} - \frac{1}{v}\right) = -\frac{b}{m}t$$

$$v(t) = \frac{v_0}{1 + bv_0 t/m} = \frac{v_0}{1 + t/\tau}$$

$$\tau = \frac{m}{b v_0}$$ (é um parâmetro)

$$\frac{dx}{dt} = v = \frac{v_0}{1 - t/\tau}$$

$$\int dx = \int v\,dt \Rightarrow x(t) = x_0 + v_0 \tau \ln(1 + t/\tau)$$

na vertical:

$$m\frac{dv}{dt} = m'g - bv^2$$

equilíbrio: $\frac{dv}{dt} = 0 \Rightarrow m'g - bv_{ter}^2 = 0$

$$v_{ter} = \sqrt{\frac{m'g}{b}}$$

$$\frac{dv}{dt} = \frac{m'}{m}g - \frac{bv^2}{m}$$

supondo sólido:

$m \simeq m'$

$$\frac{dv}{dt} = g - \frac{bv^2}{m} \qquad v_{ter}^2 \simeq \frac{mg}{b} \Rightarrow \frac{b}{m} = \frac{g}{v_{ter}^2}$$

$$\frac{dv}{dt} = g\left(1 - \frac{v^2}{v_{ter}^2}\right)$$

$$\frac{dv}{\left(1 - v^2/v_{ter}^2\right)} = g\,dt$$

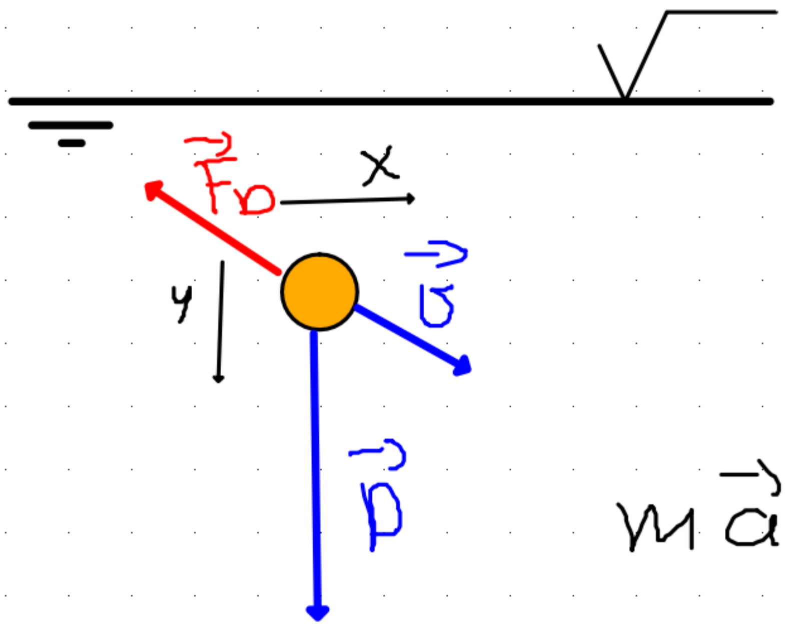
$$\vdots$$

$$\frac{v_{ter}}{g}\,\mathrm{arctanh}\left(\frac{v}{v_{ter}}\right) = t$$

$$v(t) = v_{ter} \tanh\left(\frac{gt}{v_{ter}}\right)$$

$$\tanh(\theta) = \frac{e^{+\theta} - e^{-\theta}}{e^{+\theta} + e^{-\theta}}$$

$$\frac{dy}{dt} = v(t) \Rightarrow y = \frac{v_{ter}^2}{g} \ln\left[\cosh\left(\frac{gt}{v_{ter}}\right)\right]$$



$$\vec{v} = v_x \hat{i} + v_y \hat{j}$$

$$m\vec{a} = \vec{P} + \vec{F}_D$$

$$v^2 = v_x^2 + v_y^2$$

$$\vec{a} = a_x \hat{i} + a_y \hat{j}$$

$$\vec{P} = mg\hat{j}$$

$$\vec{F}_D = -\left(\frac{1}{2}\rho C_D A v^2\right)\frac{v}{|v|}$$

(I) $$m\frac{dv_x}{dt} = -\left(\frac{1}{2}\rho C_D A\left(v_x^2+v_y^2\right)\right)\cdot\frac{v_x}{\sqrt{v_x^2+v_y^2}}$$

$$\frac{\vec{v}}{|\vec{v}|} = \frac{v_x\hat{i}+v_y\hat{j}}{\sqrt{v_x^2+v_y^2}}$$

(II) $$m\frac{dv_y}{dt} = W'g - \left(\frac{1}{2}\rho C_D A\left(v_x^2+v_y^2\right)\right)\cdot\frac{v_y}{\sqrt{v_x^2+v_y^2}}$$

as equações (I) e (II) são acopladas:

Sem solução analítica!

$$\frac{dv_x}{dt} = -\frac{b}{m}\sqrt{v_x^2+v_y^2}\cdot v_x$$

$$\frac{dv_y}{dt} = \frac{m'}{m}g - \frac{b}{m}\sqrt{v_x^2+v_y^2}\cdot v_y$$

equilibrio:

$$\frac{dv_x}{dt} = 0 \Rightarrow \sqrt{v_x^2+v_y^2}\cdot v_x = 0$$

$$\frac{dv_y}{dt} = 0 \Rightarrow \frac{b}{m}\sqrt{v_x^2+v_y^2}\cdot v_x = \frac{m'}{m}g$$

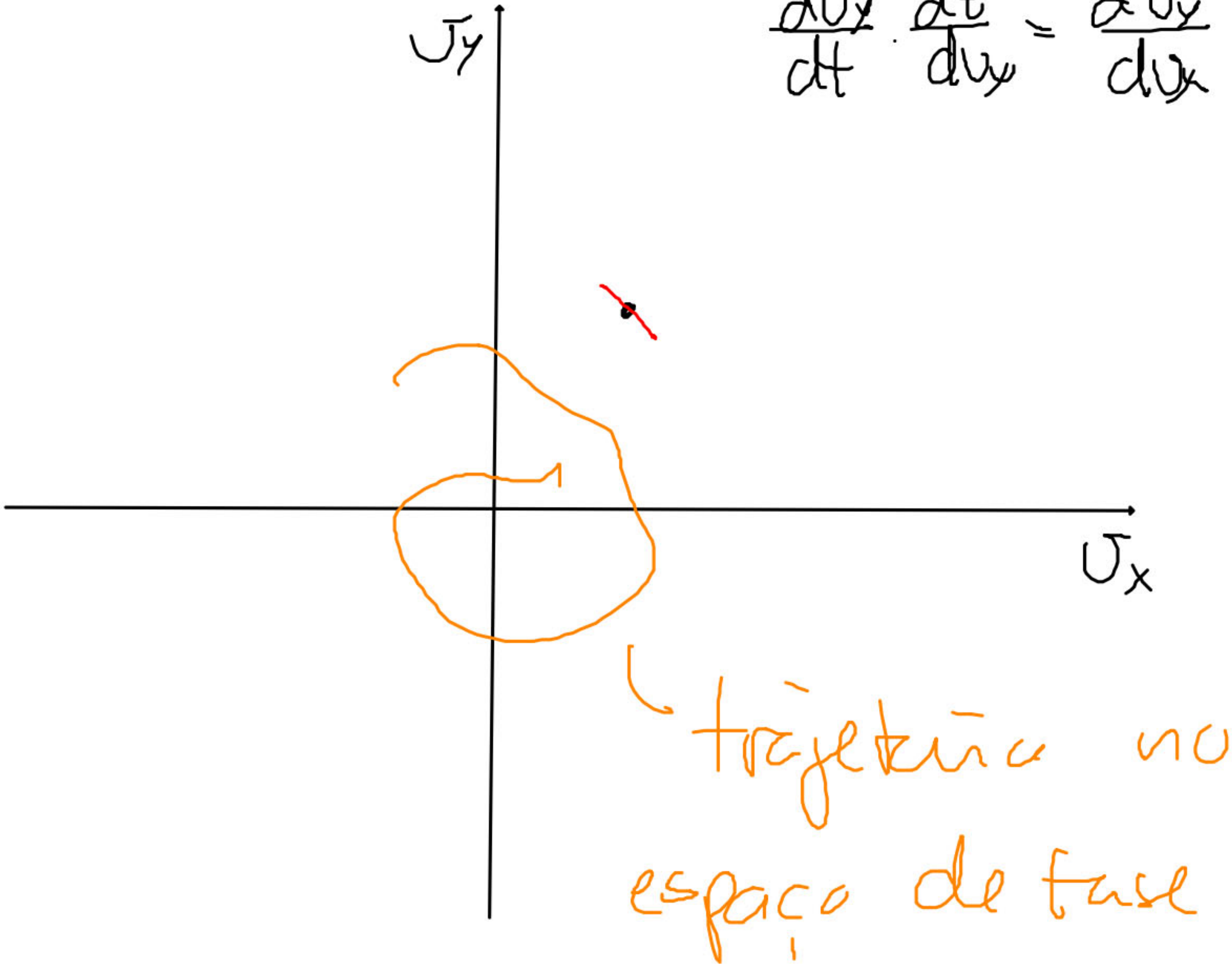
$$\frac{dv_x}{dt} = -\frac{b}{m}\sqrt{v_x^2 + v_y^2} \cdot v_x$$

$$\frac{dv_y}{dt} = \frac{m'}{m}g - \frac{b}{m}\sqrt{v_x^2 + v_y^2} \cdot v_y$$

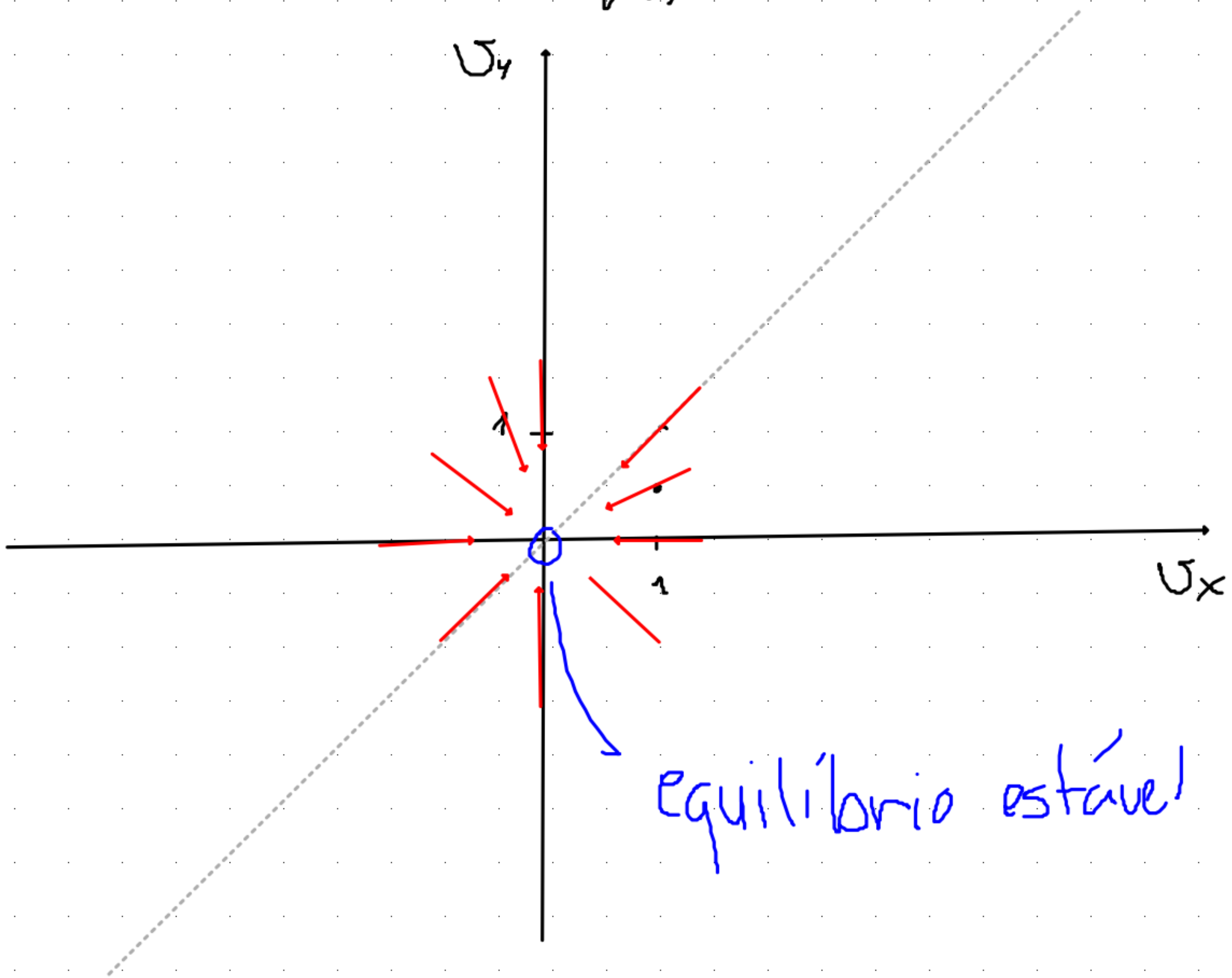
$$\frac{dv_y}{dv_x} = \frac{\frac{dv_y}{dt}}{\frac{dv_x}{dt}} = \frac{\frac{m'}{m}g - \frac{b}{m}\sqrt{v_x^2 + v_y^2} \cdot v_y}{-\frac{b}{m}\sqrt{v_x^2 + v_y^2} \cdot v_x}$$

$$\frac{dv_y}{dv_x} = \frac{v_y}{v_x} - \frac{m'}{b}\frac{g}{\sqrt{v_x^2 + v_y^2} \cdot v_x}$$

$$\frac{dv_y}{dt} \cdot \frac{dt}{dv_x} = \frac{dv_y}{dv_x}$$

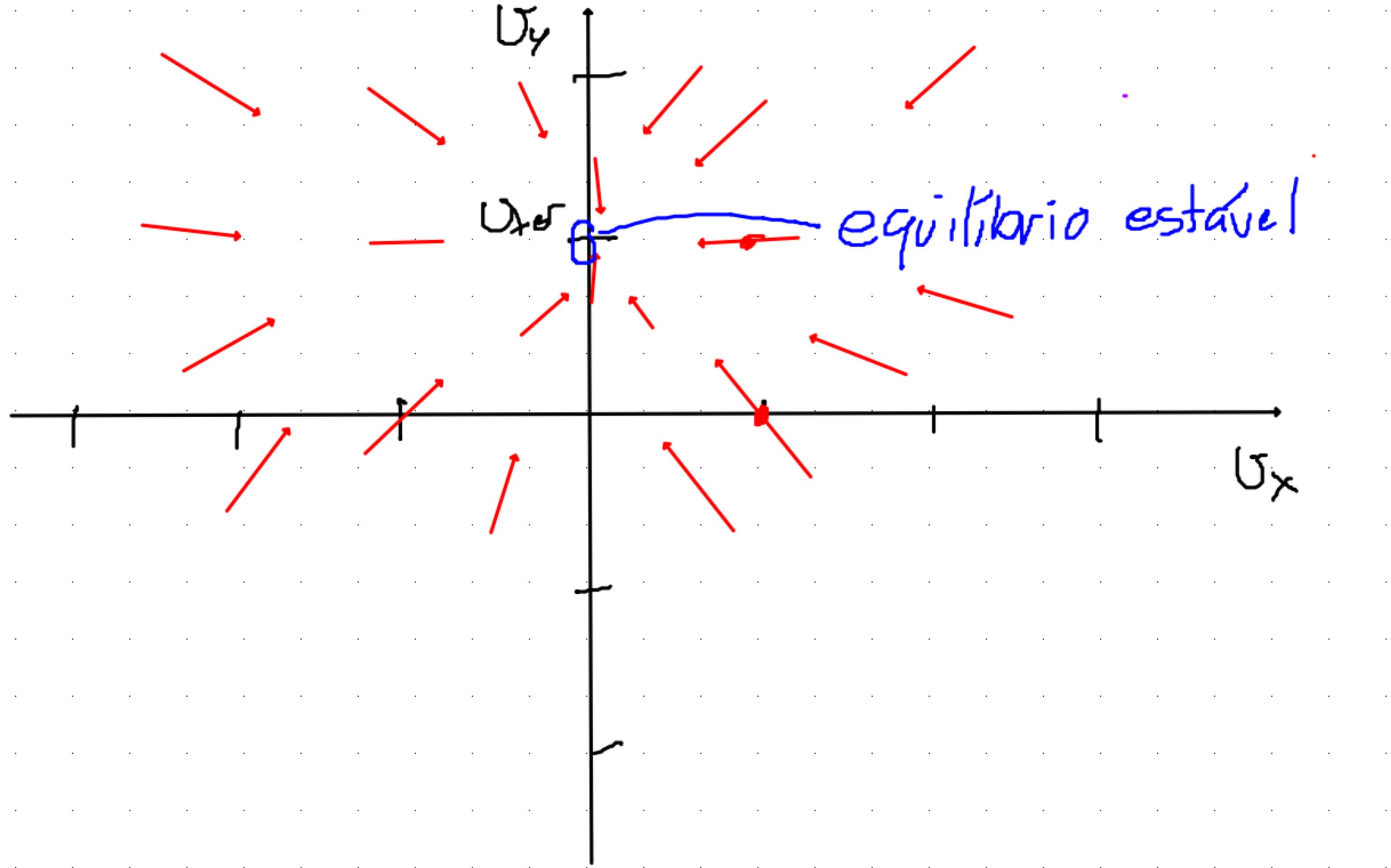


se $g = 0 \Rightarrow \dfrac{dv_y}{dv_x} = \dfrac{v_y}{v_x}$



$$g \neq 0 \Rightarrow \frac{dv_y}{dv_x} = \frac{v_y}{v_x} - \frac{m'}{b} \frac{g}{\sqrt{v_x^2+v_y^2} \cdot v_x}$$

$$= \frac{v_y}{v_x} - \frac{v_{ter}^2}{\sqrt{v_x^2+v_y^2} \cdot v_x}$$

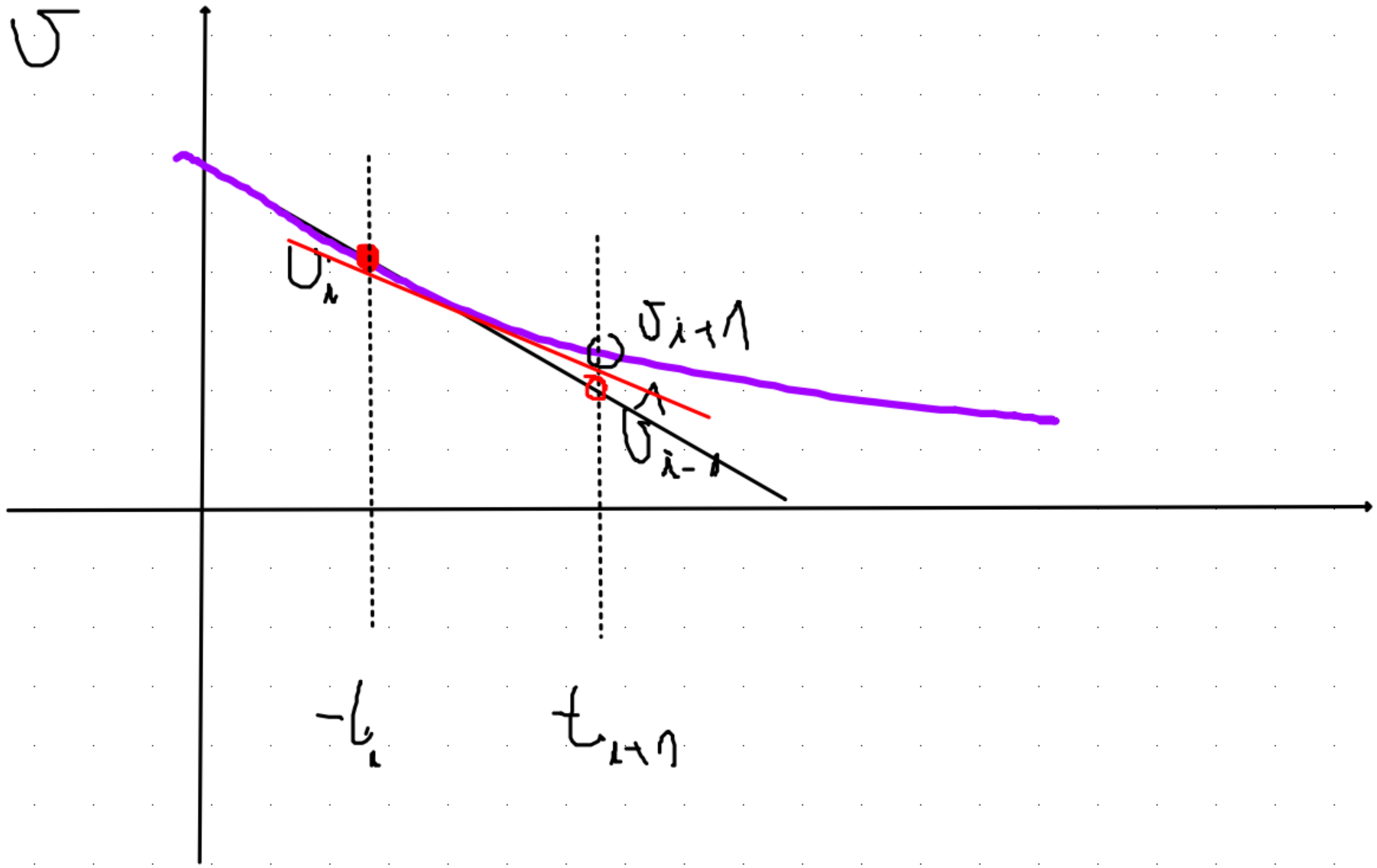


$$\frac{dv}{dt} = f(v, t) \qquad f_i = f(v_i, t_i)$$

$$\downarrow$$

$$v_{i+1} = v_i + f_i h$$ euler explicito

$$v_{i+1} = v_i + \hat{f}_{i+1} h$$ euler implicito



v
v_i
v_{i+1}
$\hat{v}_{i+1}$
t_i
t_{i+1}