MODEL THEORETIC STABILITY II

28/4/87 Unfinished business of last term:
One needs to repair the proofs of theorem 35 (forking symmetry), and of lemma 28 (almost over implies d.n.f.). In this version, the proof of lemma 28 depends on thm. 35.

One had proved thm 34, the version over models in terms of heirs. (See pp. 24-25.) T is assumed stable.

THEOREM 35: $Tp(\bar{a}/C\cup\bar{b})$ forks over $Tp(\bar{a}/C)$ iff $Tp(\bar{b}/C\cup\bar{a})$ forks over $Tp(\bar{b}/C)$.

PROOF:
One writes this in terms of d.n.f. and tries to reduce to thm. 34.

So, assume (i): $tp(\bar{b}/\bar{a}\cup C)$ dnf. over $tp(\bar{b}/C)$. One wants $tp(\bar{a}/\bar{b}\cup C)$ dnf over $tp(\bar{a}/C)$.

Claim: wlog. there are $\mathcal{M}_1, \mathcal{M}_2$, such that: (ii) $\mathcal{M}_1 \supseteq \bar{a}\cup C$, and $tp(\bar{b}/\mathcal{M}_1)$ d.n.f. over $tp(\bar{b}/C)$;
(iii) $\mathcal{M}_2 \supseteq C$ such that $tp(\bar{a}/\mathcal{M}_2)$ dnf over $tp(\bar{a}/C)$;
(iv) $tp(\bar{b}/\mathcal{M}_1\cup\mathcal{M}_2)$ is the heir of $tp(\bar{b}/\mathcal{M}_1)$.

Assume these.
Claim (1): $tp(\bar{b}/\mathcal{M}_1\cup\mathcal{M}_2)$ dnf over $tp(\bar{b}/C)$. (Directly from (ii) & (iv).)
Claim (2): $tp(\bar{b}/\mathcal{M}_2\cup\bar{a})$ dnf. over $tp(\bar{b}/\mathcal{M}_2)$. (Since $C\subseteq\mathcal{M}_2$, $\beta(tp(\bar{b}/C))\subseteq cl(tp(\bar{b}/\mathcal{M}_2))$, and $cl(tp(\bar{b}/\mathcal{M}_2))\subseteq cl(tp(\bar{b}/\mathcal{M}_1\cup\mathcal{M}_2))=\beta(tp(\bar{b}/C))$, by claim (1).)

So $tp(\bar{b}/\mathcal{M}_2\cup\bar{a})$ is the heir of $tp(\bar{b}/\mathcal{M}_2)$. By thm. 34, $tp(\bar{a}/\mathcal{M}_2\cup\bar{b})$ is the heir of $tp(\bar{a}/\mathcal{M}_2)$. So $tp(\bar{a}/\mathcal{M}_2\cup\bar{b})$ dnf. over $tp(\bar{a}/\mathcal{M}_2)$, which dnf. over $tp(\bar{a}/C)$, by (iii). So $tp(\bar{a}/C\cup\bar{b})$ dnf. over $tp(\bar{a}/C)$. □

Justifying (ii), (iii) & (iv):
(ii) Chose any model $\mathcal{M}_1 \supseteq \bar{a}\cup C$, and q over $\mathcal{M}_1$, a non-forking extension of $tp(\bar{b}/\bar{a}\cup C)$. There may be a hidden

connection between $\bar{b}$ & $\mathcal{M}_1$. So, one can't assume (ii) for $\mathcal{M}_1, \bar{b}$. Choose $\bar{\beta}$ in an ambient model, realizing q. So $\bar{\beta}$ realizes $tp(\bar{b}/\bar{a}\cup C)$. Wlog (in a bigger ambient model) $\bar{b}$ & $\bar{\beta}$ conjugate over $\bar{a}\cup C$. So, it suffices to prove the result for $\bar{\beta}$. So (ii) holds.

(iii) & (iv): Use a compactness argument in an extended language. One has constants for elements of $\mathcal{M}_1$, and a 1-ary predicate M_2 (whose eventual interpretation is $\mathcal{M}_2$). Take axioms: T + "$M_2 \prec$ universe" + $C \subseteq M_2$ + + $\forall \vec{x}(\vec{x} \in M_2 \to \neg\varphi(\vec{a},\vec{x}))$, if $\varphi(\vec{v},\vec{y}) \in \beta(\bar{a}/C)$ + + $\forall \vec{x}(\vec{x} \in M_2 \to \neg\psi(\vec{b},\vec{x},\vec{\mu}))$, if $\vec{\mu} \in \mathcal{M}_1$, and $\psi(\vec{v},\vec{w}) \notin cl(\bar{b}/\mathcal{M}_1)$. If this is consistent, the interpretation of M_2 gives solution to (iii) & (iv). But any finite subset is consistent, if one interprets M_2 by $\mathcal{M}_1$. □

Now one is going to repair the:

<u>LEMMA 28</u>: (T stable). Let $p \in S_n(A)$, $p \subseteq q$, $q \in S_n(\mathcal{M})$, q almost over A. Then q d.n.f. over A.

<u>DEF</u>: Let $A, B, C \subseteq$ ambient $\mathcal{M}$. Then B & C are <u>independent over A</u> if for every $\bar{b}$ in B, $tp(\bar{b}/C\cup A)$ dnf over $tp(\bar{b}/A)$.

<u>LEMMA 28^{--}</u>: If $A \subseteq B$, there is an $\mathcal{M} \supseteq A$ such that $\mathcal{M}$ & B are independent over A.

<u>PROOF</u>:

The same style of argument as just done.

One wants the consistency of $T_B \cup T_B(\mathcal{M}) \cup (\forall \vec{x} \in \mathcal{M})(\neg\varphi(\vec{b},\vec{x},\vec{\mu}))$, if $\vec{\mu}$ is in A, and $\varphi(\vec{v},\vec{w}) \notin \beta(tp(\bar{b}/A))$. If it is not consistent, one gets $\varphi_1, \ldots, \varphi_r$ not in $\beta(tp(\bar{b}/A))$ but represented in every model. □

(Slight sleight of hand here, since

one has many $\bar{b}$'s.)

<u>NOTE</u>: This is symmetric <u>because</u> <u>of</u> the symmetry of forking.

<u>4/5/87</u> <u>LEMMA 28'</u>: Let A, B & $\mathcal{M}$ be as above, $p \in S_n(A)$, $q \supseteq p$, $q \in S_n(B)$. Then q dnf over p iff $\exists\, q_1 \supseteq p$, $q_1 \in S_n(\mathcal{M})$ s.t. q_2, the heir of q_1 over $B \cup \mathcal{M}$, is an extension of q.

<u>PROOF</u>:
($\Rightarrow$) Assume q dnf. over p. Take q_2 a non-forking extension of q to $B \cup \mathcal{M}$. So q_2 dnf over $q_1 = q_2 \restriction_{\mathcal{M}} \supseteq p$.
($\Leftarrow$) Suppose q_1, q_2 given as in the diagram:

$$\begin{tikzcd} & B \cup \mathcal{M} \quad q_2 & \\ q \quad B \arrow[ur, dash] & & \mathcal{M} \quad q_1 \arrow[ul, dash] \\ & A \quad p \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

Let $\bar{c}$ realize q_2. Then, for any $\bar{b}$ in B, $tp(\bar{c}/\mathcal{M} \cup \bar{b})$ dnf over $\mathcal{M}$, so, by symmetry, $tp(\bar{b}/\mathcal{M} \cup \bar{c})$ dnf over $\mathcal{M}$. Also $tp(\bar{b}/\mathcal{M})$ dnf over A (by independence of B & $\mathcal{M}$), so $tp(\bar{b}/\mathcal{M} \cup \bar{c})$ dnf. over A. So $tp(\bar{b}/A \cup \bar{c})$ dnf over A. Thus $tp(\bar{c}/A \cup \bar{b})$ dnf over A, for any $\bar{b}$ in B, so $q = tp(\bar{c}/B) = tp(\bar{c}/A \cup B)$ dnf over A. $\square$

<u>PROOF OF LEMMA 28</u>:
Take $\mathcal{N}$ independent of $\mathcal{M}$ over A. One has that q is almost over A, by hypothesis. Take a model $\mathcal{M}_1 \supseteq \mathcal{M} \cup \mathcal{N}$. Let $q'(x) = \{\varphi(\bar{x}, \bar{m}_1) : \mathcal{M}_1 \models (d_q\varphi)(\bar{m}_1)\}$. Then $q'(x)$ is a type over $\mathcal{M}_1$, and the heir of q. For each φ, the equivalence classes used to give $(d_q\varphi)$ in $\mathcal{M}$ are represented in $\mathcal{M}_1$, & so in $\mathcal{N}$, giving alternative definition $d_q^*\varphi$ of q'. It follows that $q_2 = q' \restriction_{\mathcal{M} \cup \mathcal{N}}$ is the heir of $q_2 \restriction_{\mathcal{N}}$. This gives the result, by lemma 28'. $\square$

6. More Consequences of the Above

(6.1) THEOREM 36: Suppose T has no order. Then T is stable.

PROOF:

If T has no order, then, by thm. 32, every heir of a type is also a coheir. Suppose T not stable. Then, for any κ, there is (by thm 19) a model $\mathcal{M}$, $|\mathcal{M}|=\kappa$, and a type p over $\mathcal{M}$, s.t. for some $\mathcal{M} \precneqq \mathcal{N}$, p has at least $(2^{2^\kappa})^+$ heirs over $\mathcal{N}$. These must be coheirs, contradicting the following lemma. □

LEMMA 37: For any $\mathcal{M} \subseteq A$, and any p over $\mathcal{M}$, p has $\leq 2^{2^{|\mathcal{M}|}}$ coheirs.

PROOF:

Coheirs are in the closure (in the usual Stone space topology on types) in $S(A)$ of the set of types realized in $\mathcal{M}$. There are $\leq 2^{|\mathcal{M}|}$ such types. □

(6.2) Further topological considerations: (T stable):

One considers now $A \subseteq B$, $DNF(B/A) = \{q \in S_n(B) : q \text{ dnf. over } q\restriction_A\}$.

THEOREM 38: DNF (B/A) is closed.

PROOF:

CASE 1: A is a model.

Then $DNF(B/A) = \{q : q \text{ is the heir of } q\restriction_A\} = \{q : q \text{ is the coheir of } q\restriction_A\}$ (by lemma 30 & thm 32), which is closed.

CASE 2: General case.

Take $\mathcal{M}$ independent of B over A. Then $DNF(\mathcal{M}\cup B/\mathcal{M})$ is closed, by case 1. The image (by restriction) in B is closed, and this is DNF (B/A), by lemma 28⁻. □

LEMMA 40: Let $p \in S_n(A)$. Then, for each $\varphi(\bar{x},\bar{y})$, there is a $(d_1\varphi)(\bar{y})$ s.t. for all $A \to B$, and $\bar{b}$ in B, $(B \subseteq \mathfrak{M})$, $\mathfrak{M} \models (d_1\varphi)(\bar{b})$ iff p has a non-forking extension q with $\varphi(\bar{x},\bar{b}) \in$ $\in q$.

PROOF:

$$(d_1\varphi) = \neg(d^*\neg\varphi). \square$$

THEOREM 41: $DNF(B/A) \twoheadrightarrow S(A)$ is open.

PROOF:

Let $\psi(\bar{x},\bar{b})$ be an open nbd. of $DNF(B/A)$. Let $r = tp(\bar{b}/A)$. Let $\varphi(\bar{x},\bar{y})$ by $\psi(\bar{y},\bar{x})$, and choose $d_1\varphi$ for type r, as in lemma 40. ($\varphi(\bar{b},\bar{x})$ holds for any $\bar{x}$ s.t. $\psi(\bar{x},\bar{b})$.) So, for all $\vec{c}$, $\mathfrak{M} \models (d_1\varphi)(\vec{c})$ iff r has a non-forking extension q to $A \cup \vec{c}$ with $\varphi(\bar{y},\vec{c}) \in q$, where $\mathfrak{M}$ is a model containing $A, B, \& \vec{c}$.

Claim: Let $p \in S(A)$. Then p has a non-forking extension in the nbd determined by $\psi(\bar{x},\bar{b})$ iff $(d_1\varphi)(\bar{x}) \in$ $\in p$. (This obviously proves the thm.)

Suppose p has a non-forking extension q over B, with $\psi(\bar{x},\bar{b}) \in q$. Let $\vec{c}$ realize q. One has $tp(\vec{c}/B)$ dnf over A. So $tp(\vec{c}/A \cup \vec{b})$ dnf over A. So, by simmetry, $tp(\bar{b}/A \cup \vec{c})$ dnf over A. Also, $\varphi(\bar{y},\vec{c})$ is in this type, since $\psi(\vec{c},\vec{b})$. So $\mathfrak{M} \models$ $\models (d_1\varphi)(\vec{c})$. Since $\vec{c}$ was an arbitrary realization of q, $(d_1\varphi)(\vec{x}) \in q$, and so $(d_1\varphi)(\vec{x}) \in p$.

11/5/87 Conversely, suppose $(d_1\varphi)(\vec{x}) \in p$. Let $\vec{c}$ realize p, so $\models (d_1\varphi)(\vec{c})$ holds. So $r(x)$ has a non-forking extension r' to $A \cup \vec{c}$, with $\varphi(\vec{x},\vec{c}) \in r'$. Let $\vec{b}'$ realize r'. In particular, $\models \varphi(\vec{b}',\vec{c})$. Also $\bar{b}$, and $\vec{b}'$ have the same type over A, so $\vec{b}'$ & $\vec{b}$ are A-automorphic, with $\vec{c} \mapsto \vec{c}'$. Then $\models \varphi(\vec{b},\vec{c}')$, $tp(\bar{b}/A \cup \vec{c})$ dnf over A, and $tp(\bar{c}'/A) = p$. By symmetry, $tp(\vec{c}'/A \cup \vec{b})$ dnf over A. Let q^* be a nonforking extension of this type to B. So q^* dnf over A, $q^* \supseteq p$, and $\psi(\bar{x},\vec{b}) \in q^*$. $\square$

(6.3') Strong Types:

(See thm. 29, p. 23)

DEF: Let $\ell(\vec{x}) = \ell(\vec{y}) = n$. One says that $\vec{x}$ and $\vec{y}$ have the <u>same strong type over A</u>, $st_A(\vec{x}) = st_A(\vec{y})$, iff $\vec{x}$ and $\vec{y}$ agree on every element $E \in FE^n(A)$. $STP_n(A)$ denotes the <u>space</u> of strong n-types over A.

EXERC: Construe $STP_n(A)$ as $\varprojlim_{E \in FE^n(A)} \mathcal{M}^n/E$.

($\mathcal{M}^n/E$ = set of equivalence classes in $\mathcal{M}$ – any ambient model – it is finite, & so compact in the discrete topology. Put $\mathcal{M}^n/E_1 \twoheadrightarrow \mathcal{M}^n/E_2$, if $E_1 \subseteq E_2$, via class of $\vec{m}$ (mod E_1) $\to$ class of $\vec{m}$ (mod E_2). Give projective limit topology to $STP_n(A)$ – this is compact – the basic neighbourhoods given as equivalence classes of an A-definable E. One has a natural continuous map $STP_n(A) \twoheadrightarrow S_n(A)$: every $\varphi(\vec{x}, \vec{a})$ has associated the equivalence relation $E_{\varphi(\vec{x},\vec{a})}$, with 2 classes, one $\{\vec{x} : \models \varphi(\vec{x}, \vec{a})\}$, and other $\{\vec{x} : \models \neg\varphi(\vec{x}, \vec{a})\}$. One has, for $A \to B$, a natural map $STP(B) \twoheadrightarrow STP(A)$, such that

$$\begin{tikzcd} STP(B) \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "C"] & STP(A) \arrow[d] \\ S(B) \arrow[r, two heads] & S(A). \end{tikzcd}$$

If $A = \mathcal{M}$, $STP(\mathcal{M}) \twoheadrightarrow S(\mathcal{M})$ is an isomorphism, since the classes are "named" in $\mathcal{M}$.)

One defines non-forking for strong types by descending to types & coming back up, using the above commuting diagram. From this, symmetry is obvious. What about the existence of strong types? (This is not trivially guaranteed by the commuting diagram.)

The problem reduces to the following (via compactness): let $E \in FE^n(A)$; any E-class consistent with p is consistent with a n.f. extension p' over B. Let $\varphi(\vec{x}, \vec{y}, \vec{a})$ define E. Consider $(d_1\varphi)$ as in lemma 40.

Suppose first $\mathcal{B} = \mathcal{M}$. Suppose $\vec{b}$ is in $\mathcal{B}$, and there is no n.f. extension of $p(\vec{x})$ to $\mathcal{B}$ with "$\vec{x}$ is in the Eclass of $\vec{b}$". Then $\mathcal{M} \models \neg(d_1\varphi)(\vec{b},\vec{a})$. More generally, $\{\vec{z} : \mathcal{M} \models \neg(d_1\varphi)(\vec{z},\vec{a})\}$ defines the class of $\vec{z}$ in $\mathcal{M}$ s.t. p has no n.f. extension with $\vec{x} \equiv \vec{z} \bmod (E)$.

12/5/87 Begin with $\bar{p} \in STP(A)$, p the underlying type. One has to get, for any class $\mathcal{C}$ of an A-definable $E \in FE^n(A)$, $\mathcal{C}$ consistent with p, a n.f. extension q over $\mathcal{B}$, q consistent with $\mathcal{C}$. Suppose E is defined by $\varphi(\bar{x},\bar{y},\bar{a})$, $\bar{a}$ in A. Get $d_1\varphi(\bar{y},\bar{z})$ as before. For any $\bar{c}$, $\models d_1\varphi(\bar{c},\bar{a})$ iff there is a non-forking extension p' of p to $A \cup \bar{c}$, such that $\varphi(\bar{x},\bar{c},\bar{a}) \in p'$. [Choose $\mathcal{M} \supseteq A$ realizing p, p' a n.f. extension of p to $\mathcal{M}$ such that $\varphi(\bar{x},\bar{c},\bar{a}) \in p'$; consider $p'' = p' \restriction_{A\cup\bar{c}}$. This is non-forking, so $\mathcal{M} \models (d_1\varphi)(\bar{c},\bar{a})$.] So, now consider $A \to \mathcal{B} \to \mathcal{M}$, for some model $\mathcal{M}$, and $\bar{m}$ in $\mathcal{M}$ s.t. $\varphi(\bar{x},\bar{m},\bar{a})$ is consistent with p. [Class of $\bar{m}$ is a class $\mathcal{C}$ consistent with p.] So p has a n.f. extension to $\mathcal{M}$ including $\varphi(\bar{x},\bar{m},\bar{a})$. $\square$

(6.4) <u>Stationarity of strong types</u>: $\bar{p}$ is called stationary if $\bar{p}$ has a <u>unique</u> n.f. extension to all extensions.

<u>THEOREM 42</u>: Strong types are stationary.

<u>PROOF</u>:

$$\begin{array}{ccccccc}
\bar{p} & STP(A) & \twoheadleftarrow & STP(\mathcal{B})^{q_1,q_2} & \twoheadleftarrow & STP(\mathcal{M}) & r_1, r_2 \\
\downarrow & \downarrow & & \downarrow & & \downarrow & \downarrow \\
p & S(A) & \twoheadleftarrow & S(\mathcal{B}) & \twoheadleftarrow & S(\mathcal{M}) & r_1, r_2
\end{array}$$

Suppose q_1 & q_2 are distinct non-forking extensions of $\bar{p}$. Go to $\mathcal{B} \to \mathcal{M}$. Pull q_1, q_2 back to distinct r_1 & r_2 over $\mathcal{M}$. But for models $\mathcal{M}$, $STP(\mathcal{M}) \to S(\mathcal{M})$ is a homeomorphism, since the names for the equivalence classes are available. Construe r_1, r_2 as types. These are distinct n.f. extensions of p. By the Finite Equiv. Relation Thm (see thm 29, p 23), the realizations of these have necessarily distinct strong types over A. But both have $\bar{p}$ as strong types. □

(6.5) Strong types as types:

LEMMA 43: Suppose $C \subseteq D$, and $tp(\bar{a}^\frown \bar{b}/D)$ dnf. over C. Then $tp(\bar{b}/D \cup \bar{a})$ dnf. over $C \cup \bar{a}$.

PROOF:

By symmetry, $tp(D/C \cup \bar{a} \cup \bar{b})$ dnf. over C. (Really symmetry applied to any finite $\bar{d}$ in D.) So dnf. over $C \cup \bar{a}$. So (by symmetry) $tp(\bar{b}/D \cup \bar{a})$ dnf over $C \cup \bar{a}$. □

LEMMA 44: Same hypotheses. Then $tp(\bar{b}/D)$ dnf over C, and $tp(\bar{a}/D)$ dnf over C.

PROOF:

One has $tp(D/C \cup \bar{a} \cup \bar{b})$ dnf over C. So $tp(D/C \cup \bar{a})$ dnf over C. So $tp(\bar{a}, D)$ dnf over C. Similarly for $\bar{b}$. □

THEOREM 45: The following are equivalent:

(i) $stp(\bar{a}/C) = stp(\bar{b}/C)$;

(ii) for some $\mathcal{M} \supseteq C$, $tp(\bar{a}/\mathcal{M}) = tp(\bar{b}/\mathcal{M})$.

PROOF:

(i)$\Rightarrow$(ii): Choose $\mathcal{M} \supseteq C$ st. $tp(\bar{a}^\frown \bar{b}/\mathcal{M})$ dnf over C. So $tp(\bar{a}/\mathcal{M})$ & $tp(\bar{b}/\mathcal{M})$ dnf over C. These give n.f. extensions of $stp(\bar{a}/C)$, so give the same type (by thm 42).

(ii)$\Rightarrow$(i): (ii) gives $stp(\bar{a}/\mathcal{M}) = stp(\bar{b}/\mathcal{M})$, and these restrict to $stp(\bar{a}/C) = stp(\bar{b}/C)$. □

18/5/87 (6.6) Product of types (T stable):

DEF: Let $p \in S_n(\mathcal{M})$, $q \in S_m(\mathcal{M})$. Pick $\bar{a}$ realizing p. Let q' be the heir of q over $\mathcal{M} \cup \bar{a}$. Pick $\bar{b}$ realizing q'. Let $p \cdot q = tp(\bar{a}^\frown \bar{b}/\mathcal{M}) \in S_{n+m}(\mathcal{M})$.

LEMMA 46: The definition of $p \cdot q$ does not depend on choice of $\bar{a}, \bar{b}$.

PROOF:

Exercise. (Use definability.) $\square$

LEMMA 47: $p \cdot q = tp(\bar{a} \frown \bar{b})$ for any realizations $\bar{a}, \bar{b}$ independent over $\mathcal{M}$.

PROOF:

Clear, by definition of independent (see p. 28). $\square$

"$\cdot$" is in a certain sense commutative. Let $\sigma \in Sym(1, \ldots, n+m)$ sending $x_1 \ldots x_n y_1 \ldots y_m$ to $y_1 \ldots y_m x_1 \ldots x_n$. Then σ acts on S_{n+m}.

LEMMA 48: $p \cdot q = (q \cdot p)^{\sigma}$

PROOF:

Symmetry of independence. $\square$

What about continuity of "$\cdot$"?

It is not in general continuous.

Ex: $n = m = 1$: the inverse image of the nbd. determined by $x_1 = x_2$ is the diagonal $\{(p, q) : p = q\}$ which is closed, but not open.

However:

Exerc: decide what happens for fixed q, i.e., is $p \mapsto p \cdot q$ continuous?

(6.7) Multiplicity & strong types:

Let p be over A; define multiplicity of p, $mult(p)$, to be max # of non-forking extensions of p to $A \to B$, as B varies.

One calculate $mult(p)$ via map $STP(A) \xrightarrow{\pi} S(A)$ (restriction).

THEOREM 49: p has finite multiplicity iff $\pi^{-1}(p)$ is discrete (equivalently, finite, since $\pi^{-1}(p)$ is compact).

PROOF:

Suppose $\pi^{-1}(p)$ is infinite. Choose distinct $\bar{q}_i$ $(i < \omega)$ in it, and choose realizations $\bar{c}_i$. Find $\mathcal{M} \supseteq A$ independent from the $\bar{c}_i$'s. So $tp(\bar{c}_i / \mathcal{M})$ are n.f. extensions of p, $\forall i < \omega$. The $tp(\bar{c}_i / \mathcal{M})$ are distinct, just because $stp(c_i / \mathcal{M})$ are & $\mathcal{M}$ is a model. So mult $(p) \geq \aleph_0$.

Conversely, suppose mult$(p) \geq \aleph_0$. Say, get n.f. $q_i \supseteq p$, q_i over B. Lift q_i to strong types $\bar{q}_i$, & let $\bar{p}_i = \bar{q}_i \restriction_A$. By the Finite Equiv. Relation Thm (see thm 29, p. 23), the $\bar{p}_i$ are distinct. $\Box$

Exerc.: mult$(p) = $ card$(\pi^{-1}(p))$.

Return to (6.3'). (See pp 34-35). Best way to proceed:

LEMMA 50: For any $p \in S(A)$, the space $\pi^{-1}(p)$ ($\subseteq STP(A)$) is homogeneous (i.e., the topol. automorphism group is transitive).

PROOF:

Let $\bar{q}_1, \bar{q}_2 \in \pi^{-1}(p)$. Let $\bar{q}_i = stp(\bar{x}_i / A)$. So $tp(\bar{x}_1/A) = tp(\bar{x}_2/A)$. So $\bar{x}_1$ is A-conjugated to $\bar{x}_2$ in a bigger model via σ, say. So σ induces $\bar{\sigma} \in Aut(\pi^{-1}(p))$, with $\bar{\sigma}(\bar{q}_1) = \bar{q}_2$. $\Box$

THEOREM 51: ($\mathcal{L}$ countable) If mult(p) is infinite, then mult$(p) = 2^{\aleph_0}$.

PROOF:

$\pi^{-1}(p)$ is a perfect space (since it has a limit point & is homogeneous). So card$(\pi^{-1}(p)) \geq 2^{\aleph_0}$. Then look back at proof of thm 49. $\Box$

THEOREM 52: If T is ω-stable, then mult(p) is finite.

PROOF:

If not, get p over A, with mult(p) infinite. Get $A \to B$, $p \subseteq p_i$, p_i over B non-forking. Also get countable $A_0 \to A$, p dnf over $p \restriction_{A_0}$. (Recall that multiplicity $\leq 2^{|\mathcal{L}|}$.) So wlog, A is countable. Now $|\pi^{-1}(p)| = 2^{\aleph_0}$. Go to countable $A \to \mathfrak{M}$, & get $|S(\mathfrak{M})| = 2^{\aleph_0}$. $\Box$

19/5/87 (6.8) Superstability ($\mathcal{L}$ countable):

DEF: T is superstable iff T is λ-stable for $\forall \lambda \geq 2^{\aleph_0}$.

LEMMA 53: (See Pillay, 5.4) Let $p \in S_n(\mathcal{M})$, $|\mathcal{M}| \leq \lambda$, $cl(p) \subsetneq \beta \in O_n(T)$. Then $\exists \mathcal{N} \succ \mathcal{M}$, $|\mathcal{N}| = \lambda$, and λ distinct extensions q_i $(i < \lambda)$ of p with $cl(q_i) = \beta$.

PROOF:

Exercise. $\square$

THEOREM 55: If $O_n(T)$ has an increasing ω-chain, then $\exists \mathcal{M}$, $|\mathcal{M}| = \lambda$, $|S_n(\mathcal{M})| \geq \lambda^{\aleph_0}$, λ arbitrary.

PROOF:

See Pillay, 5.6. $\square$

(This will give non λ-stability, if $\lambda^{\aleph_0} > \lambda$.)

LEMMA 56: Suppose T superstable. Then $O_n(T)$ has no increasing ω-chain.

PROOF:

Use $\lambda > 2^{\aleph_0}$, $\lambda^{\aleph_0} > \lambda$. (Use singular card. of cofinality $\aleph_0$.) $\square$

LEMMA 57: If $O_n(T)$ has no increasing ω-chain, then for every p, there is a finite A_0 st. p dnf over A_0.

PROOF:

If not, for all finite A_0, p forks over A_0. Forking is revealed by finite information (something represented.) So, one can construct an ω-chain in $O_n(T)$:

$$A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots$$

forking revealed. $\square$

THEOREM 58: T is superstable iff for all p, there is a finite A_0 st. p dnf over A_0.

PROOF:

($\Rightarrow$) done.

($\Leftarrow$) suppose $|\mathcal{M}| = \lambda \geq 2^{\aleph_0}$. $\mathcal{M}$ has λ finite subsets. Consider any finite A_0. $STP_k(A_0)$ has card. $\leq 2^{\aleph_0}$. Distinct non-forking extensions of a type over A_0 correspond to distinct elements of $STP_k(A_0)$. So one has $|S_k(\mathcal{M})| \leq \lambda \cdot 2^{\aleph_0} = \lambda$. $\square$

THEOREM 59: (Shelah – stability spectrum)
Let T be stable. Either T is superstable, or for any $\lambda > \aleph_0$, T is λ-stable iff $\lambda = \lambda^{\aleph_0}$.

PROOF:
Suppose T stable, but not superstable. Let $\lambda = \lambda^{\aleph_0}$, and $\mathcal{M} \models T$, $|\mathcal{M}| = \lambda$. Since all types are definable, & one has $\leq \lambda^{\aleph_0}$ definitions, one gets λ-stability.
Suppose T not superstable. Get p which ~~divid~~ forks over any finite subset. So get ω-chain in $On(T)$, so by thm (55) get $\mathcal{M}$, $|\mathcal{M}| = \lambda$, $|S_n(\mathcal{M})| \geq \lambda^{\aleph_0} > \lambda$, so no λ-stability. □

THEOREM 60: T is ω-stable iff:
(i) T is superstable; &
(ii) $S_n(T)$ is countable, $\forall n \geq 1$; &
(iii) for all p, mult(p) is finite.

PROOF:
($\Rightarrow$): (ii) is trivial; (iii) done; & (i) exercise (in fact, ω-stable $\Rightarrow$ λ-stable, for all λ).
($\Leftarrow$) Let $\mathcal{M}$ be a countable model of T. One will show that $S_n(\mathcal{M})$ is countable, for all n.
Let $p \in S_n(\mathcal{M})$. Choose A st. p dnf over A. By finite multiplicity, $p\restriction_A$ determines p up to finitely many possibilities.
<u>Claim</u>: $S_n(A)$ is countable:
Suppose $|A| = k$. One has $S_{n+k}(T)$ is countable, so $S_n(A)$ is countable.
Hence $S_n(\mathcal{M})$ is countable. □

THEOREM 61: Let T be stable. T is ω-stable iff:
(i) $S_n(T)$ is countable, each n; &
(ii) for all countable $\mathcal{M} \models T$, and all $p \in S_n(\mathcal{M})$, p is definable over some finite $A \subseteq \mathcal{M}$.

PROOF:
($\Rightarrow$) (i) is trivial.
Let $p \in S_k(\mathcal{M})$, $\mathcal{M}$ countable. By ω-stability, p dnf over some finite A_0.

So p is definable almost over A_0 (by last term lectures). By thm (60), p has finite multiplicity. Go up to a finite $A_1 \supseteq A_0$ st. $p{\restriction}_{A_1}$ is stationary. So p is definable over A_1.

($\Leftarrow$) (ii) clearly implies stability – use D.L.S.

Let $\mathcal{M}$ be a countable model of T. As in last ~~[illegible]~~ thm, $|S_n(A)| \leq \omega$, for all finite A. If p is definable over A, p dnf over A, and $p{\restriction}_A$ is stationary. So $p{\restriction}_A$ determines p, & so $S_n(\mathcal{M})$ is countable. □

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(6.9) EXAMPLES:

(1) R-modules, for a fixed ring $R(\ni 1)$. Language for abelian groups $+$, $-$, 0, and for each $r \in R$, have $\bar{r}$ 1-ary (interpretation: each $\bar{r}$ gives the action $x \mapsto r.x$ of the ring on the abelian group). Every complete theory of an R-module is stable. (Recall special cases from last term – eg. vector spaces, $\mathbb{Z}$-modules.)

Result to be quoted: for a fixed complete T, every formula is equiv. to a Boolean combination of +ve primitive formulas.

Positive primitive $\varphi(\vec{x})$: $\exists \vec{y}\,(\bigwedge_i \Lambda_i(\vec{x},\vec{y})=0)$, Λ_i a linear term with coeffs from R. (Defines an additive subgroup.)

Look at specific term $\sum_i r_i x_i + \sum_j s_j y_j$.

Suppose $p(x)$ is an 1-type over $A \subseteq \mathcal{M}$. Consider $\varphi(\vec{x},\vec{a})$ in p. This puts x into an A-definable coset of a definable group. There are $\leq$ ~~[illegible]~~ card.(language) definable subgroups. What $\varphi(\vec{x},\vec{a})$ or $\neg\varphi(\vec{x},\vec{a})$ says about x is that it is in a certain A-definable coset, or can say that x is not in it. For a fixed φ, this gives ~~[illegible]~~ $\leq \max(|A|, |R|)$ possibilities.

Assume R countable. Have $\leq |A|.\omega$ possibilities. Get $\leq |A|^{\aleph_0}$ possibilities for p.

(42)

Note: For certain R (even $R=\mathbb{Z}$), one can get each of ω-stable, superstable & not ω-stable, stable & not superstable.

(2) T = theory of an equivalence relation E with just two classes, both infinite. T has Q.E.

T is ω-stable: let $p(x) \in S_1(A)$. Split A as $A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$, in the two classes. Possibilities for x:
(1) $x \in A$; (2) $x \notin A$, but $xE\alpha$, for some $\alpha \in A$; (3) $x \notin A$, & $\neg xE\alpha$, for any $\alpha \in A$ (& so either A_1 or A_2 is empty); (4) $A = \emptyset$. This gives complete type.

Note: $\exists!\, p \in S_1(\emptyset)$; & $mult(p) = 2$.

(3) T = ~~theory~~ theory of an equiv. relation E, with infinitely many classes, all infinite. As in previous example, T is ω-stable. $\exists!\, p \in S_1(\emptyset)$, $mult(p) < \infty$, & clearly not > 1:

Let $M \models T$, & q a non-forking extension of p to M. Consider the formula $\varphi(x,y) = xEy$. Take any N. There is an extension $\bar{p}$ of p to N, st. $\neg xE\bar{n}$ is in $\bar{p}$, $\forall n \in N$. So $\varphi(x,y)$ is not needed by p, so $\varphi \notin \beta(p)$. So $q(x)$ puts x out of all the classes occupied by elements of M. So q is unique. Hence $mult(p) = 1$.

(4) T = theory of infinitely many equiv. relations $E_0, E_1, \ldots$. E_0 has 2 classes, each infinite; E_{n+1} is got ~~by~~ from E_n by chopping each of E_n's classes into 2 infinite bits. It is <u>not ω-stable</u>:

Take a $M \models T$. Pick elements from each E_0-class, E_1-class, ... This gives $\aleph_0$ elements. Get $2^{\aleph_0}$ many types over this set. (Use the binary tree.) (Note: $\exists!\, p \in S_1(\emptyset)$, but $mult(p) = 2^{\aleph_0}$.) Now show T is <u>superstable</u>:

26/5/87 Let $M \models T$, $|M| \geq \aleph_0$. Calculate upper bound for $|S_1(\mathfrak{M})|$. $\mathfrak{M}$ occupies each class of E_n, $\forall n$, but not necessarily every element of $\varprojlim \mathfrak{M}/E_n$. ($= 2^\omega$ as topol. space.) To element $f \in$ $\in \varprojlim \mathfrak{M}/E_n$ (constructed as an element of $\prod \mathfrak{M}/E_n$), let correspond the set Σ_f of 1-types p over $\mathfrak{M}$ st. each of the following is in $p(v)$: v is in the E_n-class $f(n)$. (Can say this by using elements $m_n \in f(n)$ — one doesn't say anything about whether $v = m$, $m \in \mathfrak{M}$.)

Number of 1-types over $\mathfrak{M}$:

T has QE. Type $p(v)$ determined by:

(i) all $v = m$ in it, $m \in \mathfrak{M}$;

(ii) class of v mod E_n, $\forall n$ (i.e., which elements of $\mathfrak{M}$ occupy same E_n-class as v).

In (i) have $|\mathfrak{M}|$ possibilities; in (ii) have $2^{\aleph_0}$ possibilities. So, if $|\mathfrak{M}| \geq 2^{\aleph_0}$, there are only $|\mathfrak{M}|$ possibilities. So T is superstable. □

<u>Note</u>: it follows that any $p \in S_1(A)$ dnf over a finite subset:

* <u>CASE 1</u>: $v = a$ is in p, some $a \in A$. Then p dnf over $\{a\}$.

* <u>CASE 2</u>: $v \neq a$ is in p, all $a \in A$. Then p dnf over $\emptyset$.

<u>Note</u>: For each n, $|S_n(T)| \leq \aleph_0$.

(4-a) Have $E_0, E_1, \ldots$, as before. Now insist that each E_n has infinitely many classes, all infinite. E_{n+1} splits every E_n-class into infinitely many infinite bits. (Replace 2^ω by ω^ω.) T has QE, as before. T is stable (use modified counting argument as in 4), but not superstable.

(5) (Morley): Work with topological space 2^ω (2 with discrete topology). Basis of open sets: $\{f \in 2^\omega : f(n_1) = \ldots = f(n_k) = 0,\ f(m_1) = \ldots = f(m_\ell) = 1\}$. This is a compact (complete, metrizable) space. Morley assigned to each <u>closed</u> $X \subseteq 2^\omega$ a theory T_X.

Language: unary P_i, $i\in\omega$. Motivation: a condition $\exists v P_n(v)$ identifies with $f(n)=1$, for certain $f\in 2^\omega$, and $\neg\exists v P_n(v)$ with $f(n)=0$. The type of X will correspond to f, where $f(x)=1$ iff $P_n(x)$ (and 0 otherwise.) Arrange T_X so that $S_1(T_X)\cong X$.

Fixed X, give axioms for T_X:

<u>Trivial axioms</u>: $(\exists v)(\bigwedge_i^k P_i^{f(i)}(v)) \to (\exists^{\geq m} v)(\text{same})$, $\forall m\in\omega$, where $P_j^{f(j)}=P_j$ if $f(j)=1$, & $P_j^{f(j)}=\neg P_j$, if $f(j)=0$. One axiom for each finite f, and each m.

<u>Axioms specific to X</u>: For every clopen set $\{f: f(n_1)=\dots=f(n_j)=0,\ f(m_1)=\dots=f(m_\ell)=1\}$ in $2^\omega\setminus X$ have axiom $\neg(\exists x)(\bigwedge_1^\ell P_{m_i}(x)\wedge\bigwedge_1^j\neg P_{n_i}(x))$.

For every clopen (as above) which has nonempty intersection with X, have axiom $(\exists^{\geq j}x)(\bigwedge_1^\ell P_{m_i}(x)\wedge\bigwedge_1^j\neg P_{n_i}(x))$, each j.

<u>CLAIMS</u>: (i) T_X is complete & has Q.E.

(ii) The map $S_1(T_X)\to 2^\omega$ given by $p(v)\mapsto f$, where $f(n)=1$ if $P_n(v)\in p$, and $f(n)=0$ if $\neg P_n(v)\in p$, is a <u>homeomorphism</u> with range X.

It is obvious, by first (specific) axioms, that range $\subseteq X$. Proof that range $=X$ is via compactness argument.

1/6/87 Now, for $S_1(A)$: $q\in S_1(A)$ is got from $q\restriction\emptyset$ by specifying the true equalities $x=a$, $a\in A$. (In all this one appeals to known QE for monadic predicate logic - Goldfarb.) S_2, etc, are just $S_1\times S_1$ (essentially).

One has $|S_1(A)|\leq|S_1(\emptyset)|\cdot|A|$. This gives superstability.

Non-forking: let $p\in S_1(A)$.

<u>Case 1</u>: $x=a\in p$, some $a\in A$. Then p dnf over $\{a\}$.

<u>Case 2</u>: $x\neq a\in p$, $\forall a\in A$. Then p dnf over $\emptyset$.

Multiplicities: $m(p)=1$, $\forall p$.

Despite this, the theory is not allways ω-stable. (E.g., if $|X|>\aleph_0$, since then $|S_1(\emptyset)|>\aleph_0$.) So, if $|X|>\aleph_0$ then T_X is superstable but not ω-stable. ($\Rightarrow |X|=2^{\aleph_0}$, by Cantor-Bendixon thm.)

<u>Exercise</u>: X countable $\Rightarrow T_X$ ω-stable.

(There is an exact connection between Cantor-Bendixon rank of X and Morley rank of T_X.)

⑥ Algebraically Closed Fields: $p \in S_1(A)$ is determined as either; (i) x is transc. over A; or (ii) x is algebraic over A, with minimal polynomial (over field generated by A) of degree n. So T is ω-stable.

Multiplicity:

(i) p transc. over A. So p is transc. over $\tilde{A}$ (= alg. closure of A) in any $\mathcal{M}$ ($\supseteq A$). Let p_1, p_2 be non-forking extensions of p to B. (W.l.o.g. B is a field.) One has that $p_i(x) \Rightarrow x$ is transc. over $\tilde{A}$. Let x_i be a realization in $\mathcal{M}$ of p_i. Then $x_1 \sim_{\tilde{A}} x_2$. So x_1, x_2 have the same strong type over A. Therefore $p_1 = p_2$, & so $m(p) = 1$.

(ii) p is alg. over A, minimal polyn. of degree n. Suppose $p(x) \Rightarrow x$ is separable over A. Then all the extensions of p to a model are conjugate over A, & so $m(p) = n$.

Exercise: Calculate $m(p)$ in inseparable case.

Exercise: (i) Connect stability-theoretic independence & algebraic independence.

(ii) Connect stationarity with absolute irreducibility.

⑦ Separably Closed Fields (of fixed char. $p > 0$.) (K is separably closed iff K has no separable alg. extensions.)

Let K be sep. closed of char. p. Let $K^p = \{x^p : x \in K\}$; this is a field.

Eršov invariant of K: (i) n, if $\dim_{K^p} K = p^n$, $n \in \omega$; (ii) ∞, o.w.

Fact: $K \equiv L$ iff K & L have the same Eršov invar. (if they are ACF, invariant is 0.)

Fact: The theory of sep. closed fields of char p has QE in terms of $+, -, \cdot, 0, 1,$ & the predicates $D_n(x_1, \ldots, x_n)$ saying $x_1, \ldots, x_n$ are linear dependent over K^p. (Need reference also to Eršov invariant.) - See C. Wood, J.S.L. (1979).

Fact: (i) Each sep. closed field has stable theories; (ii) the non-alg. closed ones do not have superstable theory.

Stability table for fields:

ω-stable	"only" superstable	only stable
Finite fields & A.C.F. (A.J. Macintyre, 1971)	No theory ⟵ (Cherlin-Shelah)	Separably closed (Shelah, 1972)

2/6/87 ## (7) Chain Conditions for Stable Groups:

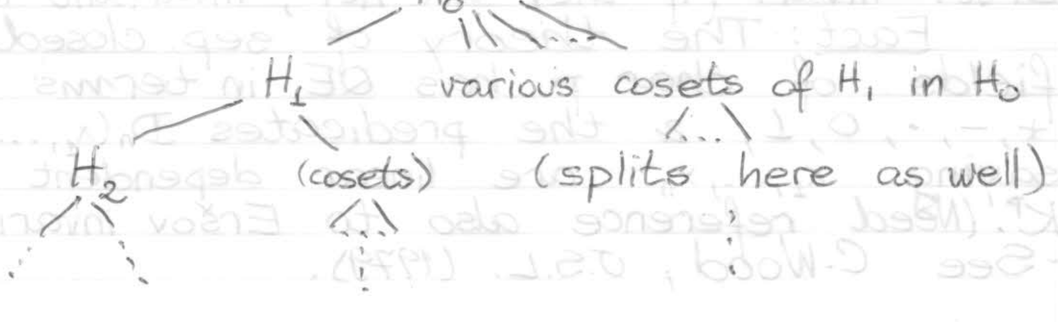
(DEF: $\mathfrak{M}$ is (κ-)stable if $Th(\mathfrak{M})$ is (κ-)stable.)

LEMMA 62: Suppose G is an ω-stable gp. Then there is no infinite descending chain of definable subgroups of G^k, any $k \in \omega$. [Note: definable subgroup of G^k means a subgroup $H < G^k$ such that $H = \{\vec{g} = (g_1, \dots, g_k) : G \models \varphi(\vec{g})\}$, where φ is a formula over G.]

PROOF:

Suppose that $H_0 \supsetneqq H_1 \supsetneqq H_2 \supsetneqq \dots$ is a descending chain of definable subgroups of G^k, H_i defined by φ_i. Each H_i is def. from finitely many elements of G. G with constants for those elements is still ω-stable. So, wlog., the H_i's are defined from $\emptyset$.

Picture:



Choose a binary tree:

$$\begin{tikzcd}[column sep=small]
 & & & H_0 \arrow[dll, dash] \arrow[drr, dash] & & & \\
 & H_1 \arrow[dl, dash] \arrow[dr, dash] & & & & \text{another coset (of } H_1) \arrow[dl, dash] \arrow[dr, dash] & \\
H_2 & & \text{coset} & & \text{coset} & & \text{coset (of } H_2)
\end{tikzcd}$$

This gives $2^{\aleph_0}$ k-types over any countable set A which intersects all the cosets in this tree. Hence G is not ω-stable, contradiction. $\square$

<u>LEMMA 63</u>: Suppose G is superstable. Then there is no descending chain of definable $H_i \subseteq G^k$ with $[H_i : H_{i+1}] \geq \aleph_0$.

<u>PROOF</u>:
Use Löwenheim-Skolem to get a model of power λ, with each $[H_i : H_{i+1}] = \lambda$. By same argument as above, get $\lambda^{\aleph_0}$ k-types, contradicting superstability, for suitable λ. $\square$

<u>NON-EXAMPLE</u>: Let $G = \mathbb{Z}^\omega$. G is stable. Write additively. Put $H_0 = G$, $H_i = 2G, \ldots, H_n = (n+1)!\, G$, etc. So $[H_0 : H_1] = 2^{\aleph_0}, \ldots, [H_n : H_{n+1}] = (n+1)^{\aleph_0}$, etc.

<u>REMARK</u>: $\mathbb{Z}$ is superstable. Use known Q.E. Let $\mathcal{M} \models Th(\mathbb{Z})$, $|\mathcal{M}| = \lambda \geq 2^{\aleph_0}$. $S_1(\mathcal{M})$: $p(x)$ is determined by all eq<u>ns</u> $kx = a \in \mathcal{M}$, $k \in \mathbb{Z}$; and $x \equiv b \mod(\ell)$, $b \in \mathcal{M}$, $\ell \in \mathbb{Z}$ (Presburger elimination). Essentially all you say is $x = a$, or $x \notin \mathcal{M}$ & $x \equiv r_j \mod(j)$, $j \in \mathbb{N}$, & $0 \leq r_j < j$ ($2^{\aleph_0}$ possibilities). So $|S_1(\mathcal{M})| \leq$ $\leq \lambda$.

<u>Converses of the above for modules</u> (over countable R) See Garavaglia, 1978.
<u>Ref</u>: M. Prest - Model Theory of Modules (1988).
Look at superstability. Let T be a complete theory of R-modules, R countable. Suppose T not superstable. Then (by (6.8), pp. 38-41) there is an n s.t. some $O_n(T)$ has an increasing ω-chain $\beta_0 \subsetneq \beta_1 \subsetneq \ldots$ ($\beta_j = cl(p_j)$, p_j over $\mathcal{M}_j$)

(48)

Look at $\beta_0 \subsetneq \beta_1$. Some fla. represented in β_1 is not represented in β_0. Any fla. is a Boolean combination of positive primitive flae., & such flae. $\varphi(\vec{x},\vec{y})$ assign $\vec{x}$ to a coset (defined from $\vec{y}$) of the group $\{\vec{t} : \varphi(\vec{t},\vec{0})\}$. Some hand waving implies that some coset of the definable group $\{\vec{t} : \varphi(\vec{t},\vec{0})\}$ is available in $\mathcal{M}_1$, not in $\mathcal{M}_0$, & p_0 saying that $\vec{x}$ is not in any coset of any element of $\mathcal{M}_0$ mod $\{\vec{t} : \varphi(\vec{t},\vec{0})\}$. So the index, in any model, of $H_0 = \{\vec{t} : \varphi(\vec{t},\vec{0})\}$ is infinite. Now consider $\beta_1 \subsetneq \beta_2$ to get H_1, with $[H_0 : H_1]$ infinite, etc. This gives:

<u>LEMMA 64</u>: ($\mathcal{M}$ is an R-module) $Th(\mathcal{M})$ is superstable iff no model of $Th(\mathcal{M})$ has an infinite descending chain of definable subgroups, each of infinite index on its predecessor. □

8/6/87 Alternative proof (of ⇐ of lemma 64): Let $p \in S_1(A)$. To every positive primitive fla. $\varphi(\vec{x},\vec{a})$ there corresponds a group $\varphi(\vec{x},\vec{0})$. (Except for trivial case.) Select $\varphi(\vec{x},\vec{a})$ in p st. $\varphi(\vec{x},\vec{0})$ is minimal for the infinite index ordering above. Note that there are $\leq |A|$ possibilities for $\varphi(\vec{x},\vec{a})$. For every $\psi(\vec{x},\vec{b})$ in p, $[\varphi(\vec{x},\vec{a}) : \varphi(\vec{x},\vec{a}) \wedge \psi(\vec{x},\vec{b})] < \omega$; o.w. one would have $[\varphi(\vec{x},\vec{0}) : \varphi(\vec{x},\vec{0}) \wedge \psi(\vec{x},\vec{0})] \geq \omega$, contradicting minimality of φ. So, there are only finitely many cosets of $\varphi \wedge \psi$ in $\varphi(\vec{x},\vec{a})$. Therefore, $< \omega$ choices for representatives of $\psi(\vec{x},\vec{y})$ in p. Any choice of representative determines all other $\psi(\vec{x},\vec{b})$ (by additivity). So # of possibilities for $p \leq$ # of possibilities for $\varphi(\vec{x},\vec{a}) \cdot \aleph_0^{\#\psi} = |A| \cdot 2^{\aleph_0}$. Hence, superstability. (<u>Note</u>: $\#\psi$ only consider those with finitely many cosets of $\varphi \wedge \psi$ in $\varphi(\vec{x},\vec{a})$.) □

<u>Exercise</u>: Prove the analogous result for ω-stable & the usual DCC on definable

subgroups.

Ways to get new stable structures from old:

(1) Interpretations: Let T be a complete $\mathcal{L}$-theory, $\mathcal{M} \models T$. Take a definable equiv. reln. E on $\mathcal{M}^n$ and form the set of equiv classes $\mathcal{M}^n/E$. One can interpret this. One may also have various definable relns. & fns. for which E is a consequence. This gives interpretations of these on $\mathcal{M}^n/E$. (E.g., definable normal subgroups)

METATHEOREM: If you interpret some $\mathcal{N}$ inside $\mathcal{M}$ in the above style, & $Th(\mathcal{M})$ is κ-stable, then so is $Th(\mathcal{N})$. (See discussion on G. Ahlbrandt & M. Ziegler, An. Pure & Appl. Logic, (1986?).)

(2) Finite products: $\mathcal{M}_1$ & $\mathcal{M}_2$ $\mathcal{L}$-structures; $\mathcal{M}_1 \times \mathcal{M}_2$ is an $\mathcal{L}$-structure in obvious way.

FACT: If $Th(\mathcal{M}_i)$ are κ-stable, so is $Th(\mathcal{M}_1 \times \mathcal{M}_2)$.

(Easily deduce from work of Feferman-Vaught, Fund. Math. c1959)

COUNTEREXAMPLE: $\mathcal{M} = S_3$, $\mathcal{N} = S_3^{\aleph_0}$; $Th(\mathcal{M})$ is κ-stable, since $\mathcal{M}$ is finite, but $Th(\mathcal{N})$ is not stable. Define order via inclusion of centralizers of elements. (But Feferman-Vaught theory applies to $\mathcal{N}$.)

CLASSIFICATION OF ω-STABLE ABELIAN GROUPS:

(1) Suppose G is an ω-stable abelian gp. By DCC on definable subgroups, $G \supseteq 2G \supseteq 6G \supseteq \ldots$ $\ldots \supseteq n!G \supseteq \ldots$, becomes stationary. Same is true for any elementary extensions of G. Let D be the subgroup of divisible elements of G. Put $G = D \oplus B = (n!G) \oplus B$, for some n, $n!B = 0$. So G ω-stable $\Rightarrow$ $G = D \oplus B$, D divisible, B of bounded exponent.

(2) At cost of going to elementary

extension, $D = \oplus$ (copies of $\mathbb{Q}$) $\oplus$ (copies of $\mathbb{Q}(p^\infty)$ = gp. of p^n roots of 1, all n). It is straightforward to show that D is ω-stable. Let B be of bounded exponent. By Prüfer, B is the direct sum of finitely many finite cyclic gps, or $B = \oplus_{i=1}^{n}$ (copies of finite cyclic gp H_i). To show B is ω-stable, show that $H_i^{(\omega)}$ is ω-stable. So, one has:

THEOREM: (A.J. Macintyre, Fund. Math, 1971): G is ω-stable $\iff G = D \oplus B$, D divisible, and B of bounded exponent.

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(8) RANKS (FOR T STABLE)

The Lascar U-rank:

DEF: (U_n, a rank on $S_n(A)$, $A \subseteq \mathcal{M} \models T$) One actually defines (by recursion on $\alpha \in$ Ord) the notion of $U_n(p) \geq \alpha$, and then defines $U_n(p) = \alpha$ to be $U_n(p) \geq \alpha$ and not $U_n(p) \geq \alpha + 1$. Clauses:

(i) $U_n(p) \geq 0$;

(ii) if α is limit, $U_n(p) \geq \alpha \iff \forall \beta < \alpha \; U_n(p) \geq \beta$;

(iii) $U_n(p) \geq \alpha + 1$ if p has a forking extension q with $U_n(q) \geq \alpha$.

DEF: $U_n(p) = \infty$ if $U_n(p) \geq \alpha$, $\forall \alpha \in$ Ord.

LEMMA 65: Suppose $p \subseteq q$, $U(q) < \infty$. Then $U(p) = U(q)$ iff q dnf over p.

PROOF:

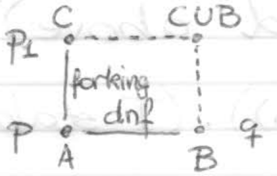
(i) Suppose q forks over p. Let $U(q) \geq \gamma$. Then $U(p) \geq \gamma + 1$, by definition, so if $U(q) < \infty$, then $U(p) > U(q)$.

(ii) One proves by induction on α that if $U(p) \geq \alpha$ & q dnf over p, then $U(q) \geq \alpha$.

This is evident for $\alpha = 0$, or α = limit.

Suppose that $U(p) \geq \alpha = \beta + 1$. So $\exists \, p_1 \supseteq p$, p_1 forks over p, and $U(p_1) \geq \beta$. One has to construct a forking extension of q, with $U \geq \beta$.

<u>CLAIM</u>: Wlog. on p_1, $\exists \bar{d}$ st $\bar{d}$ realizes q & $tp(\bar{d}/B \cup C)$ is a non-forking extension of p_1.



Assume this. So, by above, $U(tp(\bar{d}/B \cup C) \geq \beta$. Also $tp(\bar{d}/B \cup C)$ is a forking extension of q. Therefore $U(q) \geq \beta + 1$.

Now one proves the claim. Choose any C and p_1 as above, and $\bar{d}$ realizing q. Use compactness to get C'' (automorphic to C over A, via $c \mapsto c''$) st. $tp(\bar{d}/C'') = p_1(C/C'')$ (i.e., for every $\varphi(\bar{x}, c_1, \ldots, c_k) \in p_1$, write $\varphi(\bar{x}, c_1'', \ldots, c_k'')$). Let:

(1) C' realize non-forking extension of $tp(C''/A \cup \bar{d})$ to $B \cup \bar{d}$.

Recall:

(2) $tp(\bar{d}/B)$ dnf over A.

One has $(2) \Rightarrow (3)$, where:

(3) $tp(B/A \cup \bar{d})$ dnf over A.

And $(1) \Rightarrow (4)$, where:

(4) $tp(B/A \cup C' \cup \bar{d})$ dnf over $A \cup \bar{d}$.

So, by (3) & (4):

(5) $tp(B/A \cup C' \cup \bar{d})$ dnf over A.

So:

(6) $tp(C' \cup \bar{d}/B)$ dnf over A.

So, by lemma 43:

(7) ~~tp~~ $tp(\bar{d}/B \cup C')$ dnf over $A \cup C'$.

Final claim: $A \cup C' = C'$, since $A \subseteq C$, and $C'' \sim_A C$ (so $A \subseteq C''$), so $tp(C''/A \cup \bar{d})$ contains equality statements for $A \subseteq C''$, & so this transfers to equality statements giving $A \subseteq C'$. $\square$

The <u>foundation</u> <u>rank</u>:

Recall that one showed that T is superstable iff $\exists$ no increasing ω-chains in $O_n(T)$, (iff $O_n(T)$ with reverse order is well-founded).

DEF: (V_n, the foundation rank on $O_n(T)$.)
Essential clause: $V_n(\xi) \geq \alpha + 1 \Leftrightarrow \exists \xi' \gneq \xi$, $V_n(\xi') \geq \alpha$. (For $\alpha = 0$ or limit, as usual.) (Note: ξ' is below ξ in reverse order mentioned above.)

LEMMA 66: $U_n(p) = V_n(\beta(p))$. (T stable).

PROOF:

Easy exercise. $\square$

THEOREM 67: (T stable): T is superstable iff $\forall p$, $U(p) < \infty$.

PROOF:

Lemma 66 translate this to condition that $O_n(T)$ is well-founded. $\square$

EXERCISE: Connect Lascar rank for superstable modules to a rank defined via DCC. for definable subgps. under the infinite index inclusion.

23/6/87 THEOREM 68: Suppose T is $\aleph_0$-categorical and superstable. Then T is ω-stable. (Lachlan, c. 1972).

PROOF:

Let $\mathcal{M} \models T$, $\mathcal{M}$ countable. Let $p \in S_1(\mathcal{M})$; p dnf over A, for some finite A. Then $FE'(A)$ is finite by $\aleph_0$-categoricity, so $Stp_1(A)$ is finite, so mult$(p \restriction_A)$ is finite. So, #types $\leq$ #(finite subsets)$\cdot \aleph_0 = \aleph_0$. $\square$

OPEN PROBLEM: Can one weaken the hypothesis (of thm 68) to only $\aleph_0$-categorical and stable? No. (By E. Hrushovski.)

(9) INDISCERNIBLES & INDEPENDENCE:

Recall: Order indiscernibles (existence of these are not tied up to stability). Let $\mathcal{M} \models T$, $X \subseteq \mathcal{M}$, and $<$ on X. $(X, <)$ is a set of order indiscernibles on $\mathcal{M}$ iff for all $x_1 < \dots < x_n$, $y_1 < \dots < y_n$ from X, and all φ, $\mathfrak{M} \models \varphi(\bar{x}) \leftrightarrow \varphi(\bar{y})$.

<u>Pure indiscernibility</u> of X: as long as $x_1, \dots, x_n$ are distinct, & $y_1, \dots, y_n$ are distinct, then $\mathfrak{M} \models \varphi(\bar{x}) \leftrightarrow \varphi(\bar{y})$

<u>FACT</u>: (Ehrenfeucht–Mostowski). If T has an infinite model & $(X, <)$ is any ordered set, $\exists \mathfrak{M} \models T$ st $(X,<)$ is order indiscernible in $\mathfrak{M}$.

<u>Note</u>: One cannot usually get pure indiscernible, e.g., if $|X| \geq 2$ & $\mathfrak{M}$ ordered.

<u>LEMMA 69</u>: T stable iff every infinite order indiscernible set of n-tuples is pure indiscernible.

<u>PROOF</u>:

Go via order property. $\square$

One connects (in stable T) independence & indiscernibility.

<u>LEMMA 70</u>: (About Morley sequences). Let $\{\bar{c}_\alpha : \alpha < \gamma\}$ be a set of n-tuples st $tp(\bar{c}_\alpha / \{\bar{c}_\beta : \beta < \alpha\} \cup A)$ dnf over A. Then $\{\bar{c}_\alpha : \alpha < \gamma\}$ is independent over A.

<u>PROOF</u>:

Because of finitary of forking (it does not happen at limits) it suffices to show that for $\beta \geq \alpha$, $tp(\bar{c}_\alpha / \{\bar{c}_\delta : \underset{\delta \neq \alpha}{\delta \leq \beta}\} \cup A)$ dnf over A.

Do it by induction on β.

If $\beta = 0$, $\alpha = 0$. This is given.

For limit β, is trivial.

Suppose known for β. Want to get for $\beta+1$. If $\alpha \leq \beta+1$, either $\alpha \leq \beta$ or $\alpha = \beta+1$. $tp(\bar{c}_{\beta+1} / \{\bar{c}_\lambda : \lambda \leq \beta\} \cup A)$ dnf over A is given. If $\alpha \leq \beta$, use forking symmetry to get $tp(\bar{c}_\alpha / \{\bar{c}_\lambda : \lambda \leq \beta+1, \lambda \neq \alpha\} \cup A)$ dnf over $A \cup \{\bar{c}_\lambda : \alpha \neq \lambda \leq \beta\}$. And by induction hypothesis, $tp(\bar{c}_\alpha / \{\bar{c}_\lambda : \alpha \neq \lambda \leq \beta\} \cup A)$ dnf over A. $\square$

<u>LEMMA 71</u>: Let I be an infinite set indiscernible over A. Then all similar tuples from I have same strong type over A.

PROOF:
Exercise. (I is infinite, & consider equiv. classes with finitely many classes). $\square$

LEMMA 72: Suppose $C = \{c_i : i \in I\}$ is A-indep., and all elements of C have the same strong type over A. Then C is indiscernible.

PROOF:
Pick $c \in C$. Since $tp(c/(C \setminus \{c\}) \cup A)$ dnf over A, c over $(C \setminus \{c\}) \cup A$ realizes the unique non-forking strong type ext$^{\underline{n}}$ of $stp(c/A)$. Impose a well order $< K$ on I. One shows $(I, <)$ is order indiscernible, so, by stability, indiscernible.

One shows that if φ is an $\mathcal{L}(A)$-fmla, and $\models \varphi(c_0, \ldots, c_n, c_\alpha)$, then $\models \varphi(c_{\beta_0}, \ldots, c_{\beta_n}, c_\alpha)$, if $n < \omega$, $n < \alpha$, $\beta_0 < \cdots < \beta_n < \alpha$.

Apply lemma (40) to get d_1 defining "membership" in non-forking extension of $p = tp(c/A)$. Suppose p stationary. Suppose $\models \varphi(c_0, \ldots, c_n, c_\alpha)$. So $\models (d_1\varphi)(c_0, \ldots, c_n)$. (Note $tp(c_\alpha / c_0, \ldots, c_n, A)$ dnf over A.) Use the obvious induction on # of putative indiscernibles to get $\models (d_1\varphi)(c_{\beta_0}, \ldots, c_{\beta_n})$. So p has a non-forking extension over $c_{\beta_0}, \ldots, c_{\beta_n}$, containing $\varphi(c_{\beta_0}, \ldots, c_{\beta_n}, x)$. So, by stationarity, $\models \varphi(c_{\beta_0}, \ldots, c_{\beta_n}, c_\alpha)$.

24/6/87 Reduction of general case to above case (p stationary).

Go to $\mathcal{M} \supseteq A$ independent from $C \cup A$ over A. It suffices (by above) to get:

$\begin{cases} (1)\ C \text{ independent over } \mathcal{M}; \\ (2)\ \text{all elements of } C \text{ have the same type over } \mathcal{M}. \end{cases}$

(2): Take $c_1, c_2 \in C$, $tp(c_i/\mathcal{M})$ dnf over A. If distinct, they have different strong types over A (by Finite Equivalence Relation thm.), contradiction.

(1): Let $c \in C$, and C' a finite subset of $C \setminus \{c\}$. One has $tp(c/\mathcal{M} \cup C')$ dnf over A (by choice of $\mathcal{M}$). So, by lemma 43, tp($c/\mathfrak{M} \cup C'$) dnf over $A \cup C'$. One has tp($c/A \cup C'$) dnf over A (by independence of C over A). So tp($c/\mathfrak{M} \cup C'$) dnf over A. So tp($c/\mathfrak{M} \cup C'$) dnf over $\mathfrak{M}$. □

<u>LEMMA 73</u>: (T stable): Let I be an infinite indiscernibles set of n-tuples, and let $\varphi(\bar{x},\bar{y})$ have length$(\bar{x})=n$. Then $\exists N<\omega$ st. for any $\bar{c}$, either $|\{\bar{a}\in I : \models \varphi(\bar{a},\bar{c})\}| < N$, or $|\{\bar{a}\in I : \models \neg\varphi(\bar{a},\bar{c})\}| < N$.

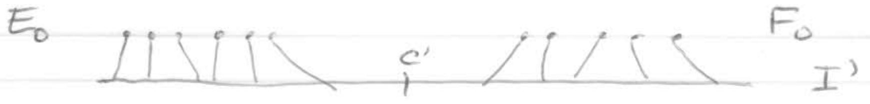
<u>PROOF</u>:

(Note: N depends only on φ.)

If not, by compactness, one gets an infinite I', and $\bar{c}$ st. $|\{\bar{a}\in I' : \models \varphi(\bar{a},\bar{c})\}| \geq \aleph_0$, and $|\{\bar{a}\in I' : \models \neg\varphi(\bar{a},\bar{c})\}| \geq \aleph_0$. (One can get I' of power κ, any κ, and the above cardinalities equal κ.)

Choose I' of power κ. Let $E \subseteq I'$, E infinite & coinfinite. Consider (the partial type) $p_E(\bar{t})$ saying $\varphi(\bar{x},\bar{t})$ iff $\bar{x}\in E$ (or $\neg\varphi(\bar{x},\bar{t})$ o.w.). It suffices to show p_E consistent, all E. If not, it is inconsistent because of finite $E_0 \subseteq E$ and finite $F_0 \subseteq I' \setminus E$, i.e. it is inconsistent to have $\varphi(\bar{x},\bar{t})$, $\bar{x}\in E_0$, & $\neg\varphi(\bar{x},\bar{t})$, $\bar{x}\in F_0$.

Picture:



Use indiscernibility. □

<u>DEF</u>: (T stable) Let I be an infinite set of indiscernibles, & B any set. The <u>average</u> of I over B, $Av(I/B) = \{\varphi(\bar{x},\bar{b}) : \bar{b}\in B$ st. $\models \varphi(\bar{a},\bar{b})$ for almost all $\bar{a}\in A\}$.

<u>Exercise</u>: $Av(I/B)$ is a type.

<u>LEMMA 74</u>: $Av(I/B)$ dnf over I, if $B \supseteq I$.

<u>PROOF</u>:

Suppose it forks. Then, by the open mapping thm., $\exists$ a fmla $\varphi(\bar{x},\bar{b}) \in Av(I/B)$, $\bar{b}\in B$, st. any type including this fmla forks over I. One wants to get some q over B, $\varphi(\bar{x},\bar{b})\in q$, & q

dnf over I. Choose $\bar{i} \in I$ st $\models \varphi(\bar{i}, \bar{b})$.
Let $q = tp(\bar{i}/B)$. $\square$

COROL.: $Av(I/I)$ is stationary.
PROOF: (Idea)
Go to $B \supseteq I$, $q = Av(I/B)$. Show it is definable over I. $\square$

DEF: Let p, q be stationary types over A, B resp. Then $p \parallel q$ (p parallel to q) if $\exists C \supseteq A \cup B$, with a common non-forking extension of p & q.

Exercise: (Easy from above). Start with p stationary. Let I be a Morley sequence. Let $q = Av(I/I)$. So $p \parallel q$.

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