Model Theoretic Stability

References:

(1) A. Pillay - Introduction to Stability Theory - Oxford Logic Guides.

(2) B. Poizat - Cours de Théorie des Modéles.

(3) S. Shelah - Classification Theory

1 STABILITY THEORY

23/11/87 (1.1) Let T be a complete $\mathcal{L}$-theory.

Mod(T) denotes the category of models of T with elementary maps as morphisms.

$\overline{\mathrm{Mod}}(T)$ is the category with:

(a) objects: $(\mathcal{M}, A)$, $\mathcal{M} \in \mathrm{Mod}(T)$, A a subset of $\mathcal{M}$;

(b) morphisms: $(\mathcal{M}, A) \to (\mathcal{N}, B)$ a map $g: A \to B$ such that for all $\mathcal{L}$-formulas $\varphi(v_1, \dots, v_n)$, and all $a_1, \dots, a_n \in A$, $\mathcal{M} \models \varphi(\vec{a}) \Rightarrow \mathcal{N} \models \varphi(g(\vec{a}))$. Notice that g must be 1-1.

TYPES OVER A:

Let I be an ordered set, and put v_i as variables.

DEF: An I-type over $(\mathcal{M}, A)$ is a set p of $\mathcal{L}_A$-formulas in the variables v_i which is maximal consistent with $Th_{\mathcal{L}_A}(\mathcal{M})$.

By compactness, p is the set of all $\mathcal{L}_A$-formulas satisfied by some I-tuple $\vec{t}$ in some $\mathcal{L}$-elementary extension of $\mathcal{M}$.

Let $S_I(\mathcal{M},A)$ be the set of I-types over $(\mathcal{M},A)$. Usually $I = n = \{0,\dots,n-1\}$. One usually writes $S_I(A)$ instead.

LEMMA 1: If there exists an $f:(\mathcal{M},A)\to(\mathcal{N},B)$, then $S_I(\mathcal{M},A) = S_I(\mathcal{N},f(A))$, where one identify $\mathcal{L}_A$ with $\mathcal{L}_{f(A)}$.

(1.2) Let $p \in S_n(\mathcal{M},A)$ $(= S_n(A))$. For any $\mathcal{L}$-formula $\Psi(\bar{x},\bar{y})$, $\bar{x}=(x_1,\dots,x_n)$, $\bar{y}=(y_1,\dots,y_r)$, define $(d_p\Psi) = \{\bar{a}\in A^r : \Psi(\bar{v},\bar{a})\in p\}$ and $d_p = ((d_p\Psi))_\Psi$.

DEF: One says p is definable if each $(d_p\Psi)$ is A-definable in the ambient model $\mathcal{M}$.

What is the dependence on $\mathcal{M}$? $\bar{a}\in(d_p\Psi) \Leftrightarrow \mathcal{M}\models\theta(\bar{a})$, θ an $\mathcal{L}_A$-formula $\Leftrightarrow \theta(\bar{a})\in p$. So,

LEMMA 2: It does not depend on $\mathcal{M}$ (in the same sense as lemma 1).

(1.3) FIRST EXAMPLE: Let $\mathcal{L}$ be language of ring theory $(\{0,1,+,-,\cdot\})$. Fix some characteristic (p or 0), and let T be the theory of algebraically closed fields of that characteristic. It is known (Tarski) that T is complete and has Q.E. (= quantifier elimination).

To show that a type in $S_n(A)$ is definable, it suffices to show (because of Q.E.) that $(d_p\Psi)$ is definable for Ψ a Boolean combination of formulas $f=0$, for $f\in\mathbb{Z}[\bar{x}]$.

Fix $(\mathcal{M},A)$, and let $p\in S_n(\mathcal{M},A)$; p is uniquely determined by knowing which $q(\bar{x},\bar{a})=0$, $q\in\mathbb{Z}[\bar{x},\bar{y}]$, are in p, and which $q(\bar{x},\bar{a})\neq 0$ are in p.

One can assume w.log. that A is a subring. To p one attaches p^+, the set of all $q\in A[\bar{x}]$ such that $q=0$

is in p. p^+ is an ideal and p^+ is prime.

To show that p is definable, one shows: for any $r(\bar{x},\bar{y}) \in \mathbb{Z}[\bar{x},\bar{y}]$, the set $\{\bar{b} \in A^r : (r(\bar{x},\bar{b}) = 0) \in p\}$ (= the same as $r(\bar{x},\bar{b}) \in p^+$) is definable in the sense that there is an A-definable relation R in $\mathcal{M}^r$ such that $R \cap A^r$ gives the above set. p^+ is finitely generated by Hilbert Basis Theorem (assume here that A is a subfield). Select a basis $q_1, \ldots, q_j$. One can apply Hilbert's Nullstellensatz, $r(\bar{x},\bar{b}) \in \langle q_1, \ldots, q_j \rangle$ iff for all $\bar{t} \in \mathcal{M}^n$, $q_1(\bar{t},\bar{b}) = 0 \wedge \ldots \wedge q_j(\bar{t},\bar{b}) = 0 \Rightarrow r(\bar{t},\bar{b}) = 0$. So $(r(\bar{x},\bar{b}) = 0) \in p \Leftrightarrow \mathcal{M} \models \forall \bar{t}\, (\bigwedge_i q_i(\bar{t}) = 0 \to r = 0)$. So, every type over A is definable. $\square$

(1.4) EXAMPLE: Let $\mathcal{L}$ be the language of abelian groups ($=\{0,+,-\}$), and T be the theory of non-trivial divisible torsion-free abelian groups. T is known to be complete and to have QE.

Fix $(\mathcal{M}, A)$, and assume A is a subgroup (not necessarily divisible). Then $p \in S_n(\mathcal{M}, A)$ is determined by conditions $\Lambda(\bar{x}) = a$ and $\Delta(\bar{x}) \neq b$, where Λ and Δ are $\mathbb{Z}$-linear combinations of $\bar{x}$, and $a, b \in A$. It suffices to show that for a fixed linear $\Gamma(\bar{x})$, the set $\{a : (\Gamma(\bar{x}) = a) \in p\}$ is definable in $\mathcal{M}$.

One has that p already forces:

(a) a certain subset of the $\bar{x}$ is a $\mathbb{Q}$-basis for the rest, over the $\mathbb{Q}$-space generated by A;

(b) the exact linear dependence of the other elements on the basis.

Write $\bar{x}'$ for a basis. Then $\Gamma(\bar{x}) = a$ is forced equal to some $\Gamma'(\bar{x}') = \Delta$ (a, other elements), Δ linear ($\bar{x}'$ is free over subspace generated by A). If $\bar{x}'$ occurs non-trivially in Γ', it is impossible for $\Gamma(\bar{x}) = a$ to be in p. Otherwise, one has $\Delta(a, \ldots) = 0$ is a necessary and sufficient condition for $\Gamma(\bar{x}) = a$ to be in p. $\square$

29/1/87: (1.5) EXAMPLE: Let $\mathcal{L}$ be the same as in (1.4), and $T = Th(\mathbb{Z})$. One has the Presburger's elimination: every formula $\varphi(v_1, \dots, v_m)$ is equivalent mod(T) to a Boolean combination of formulas $\Lambda(\bar{x}) = 0$ or $\Delta(\bar{x}) \equiv 0 \bmod(n)$, where Λ and Δ are linear over $\mathbb{Z}$.

A type $p(v_1 \dots v_m)$ over a set A is determined by various $\Lambda_i(\bar{x}) = a_i$, $\Delta_j(\bar{x}) = b_j$, where Λ_i, Δ_j are $\mathbb{Z}$-linear and $a_i, b_j \in A$.

EXERC.: Modify argument for $\mathbb{Q}$ in (1.4), to show the type is definable.

REMARK: Examples (1.4) & (1.5) are instances of a general result about modules.

Fix a ring R, let $\mathcal{L}$ be the language of abelian groups plus 1-ary functions $\bar{r}$, for all $r \in R$. So, any R-module is naturally an $\mathcal{L}$-structure. Facts:

(1) There is an analogue of Presburger elimination.

(2) All types are definable. $\square$

(1.6) A NON-EXAMPLE: (suggesting that types fail to be definable in the presence of an order.) Let $\mathcal{L}$ be the language of ordered rings $(\{0, 1, +, -, \cdot, <\})$, and $T = Th(\mathbb{R})$. One has quantifiers elimination (Tarski): every formula $\varphi(\bar{x})$ is equivalent to a Boolean combination of either $f(\bar{x}) = 0$ or $g(\bar{x}) > 0$, $f, g \in \mathbb{Z}[\bar{x}]$.

The types over A are determined by various $P_i(\bar{x}, \bar{a}) > 0$ or $Q_j(\bar{x}, \bar{a}) = 0$. Special case: $A = \mathbb{R}$, $n = 1$. The types are as follows:

(i) $x < -\infty$, i.e., $x < r$, all r; (clearly definable);

(ii) $x > +\infty$;

(iii) $x = r$; (definable)

(iv) $x > r$, $x < r^+$, i.e., $x < s$, all $s \in \mathbb{R}$, $s > r$;

(v) $x > r$, or $x < r$.

All types are definable. That all 1-types are definable is a special property of $\mathbb{R}$ among all models of T.

For example, take $A < \mathbb{R}$ countable (e.g., A = field of real algebraic numbers). Take $t \in \mathbb{R}$, $t \notin A$.

CLAIM: $\text{Type}_A(t)$ is not definable.

If so, $\{a \in A : a < t\}$ is definable in A. By Tarski, any set definable in A is a finite union of intervals whose endpoints are in A, contradiction. $\square$

EXERC.: Show that $\mathbb{R}$ is the only model of T over which all 1-types are definable.

~~(1.7) DEF: T is stable iff for every $\mathcal{M} \prec \mathcal{N}$ in Mod(T) and all k-ary $\mathcal{N}$-definable relation R on $\mathcal{N}$, $R \cap \mathcal{M}^k$ is definable.~~

~~LEMMA 3: T is stable~~

(1.7) DEF: T is stable if every type over a model of T is definable.

LEMMA 3: T is stable iff for every $\mathcal{M} \prec \mathcal{N}$ in Mod(T), and all k-ary $\mathcal{M}$-definable relation R on $\mathcal{N}$, $R \cap \mathcal{M}^k$ is definable.

PROOF:

Suppose the second assertion. Then every type p over $\mathcal{M}$ is definable, for use $\vec{t}$ in some $\mathcal{N} \succ \mathcal{M}$ realizing p. One can define $d_p\psi$ in terms of $\vec{t}$, so one can define it on $\mathcal{M}$.

Conversely, suppose $\mathcal{M} \prec \mathcal{N}$ and R is a k-ary $\mathcal{N}$-definable relation. Say R is defined by $\psi(v_1, \dots, v_k, \vec{t})$, $\vec{t}$ in $\mathcal{M}$. Let p be the type of $\vec{t}$ over $\mathcal{M}$. Select $\theta(v_1, \dots, v_k)$ defining $d_p\psi$. Then θ defines $R \cap \mathcal{M}^k$. $\square$

(2) RANK (DIMENSION NOTIONS)

(2.1) Rank (relative to T): One will define a local rank, depending on an $\mathcal{L}$-formula $\varphi(\vec{x}, \vec{y})$, length$(\vec{x}) = n$. It is a measure of complexity of splitting of definable sets using various $\varphi(\vec{x}, \vec{b})$ or $\neg\varphi(\vec{x}, \vec{c})$.

Let φ be as above, and $\psi(\vec{x})$ be a formula over A. One defines recursively the notion $R^{\varphi}_{m}(\psi) = R^{\varphi}(\psi) \geq \alpha$, for $\alpha \in On$. One has a fictitious object ∞ greater than any ordinal. So:

(i) $R^{\varphi}(\psi) \geq 0$ iff ψ is consistent;

(ii) if α is limit, $R^{\varphi}(\psi) \geq \alpha$ iff $R^{\varphi}(\psi) \geq \beta$, for all $\beta < \alpha$;

(iii) $R^{\varphi}(\psi) \geq \alpha+1$ iff there is a $\vec{b}$ (in some elementary extension of the ambient model) such that $R^{\varphi}(\psi \wedge \varphi(\vec{x},\vec{b})) \geq \alpha$ and $R^{\varphi}(\psi \wedge \neg\varphi(\vec{x},\vec{b})) \geq \alpha$.

This induces the definition of:

(1) $R^{\varphi}(\psi) = \alpha$ iff $R^{\varphi}(\psi) \geq \alpha$ & not $R^{\varphi}(\psi) \geq \alpha+1$;

(2) $R^{\varphi}(\psi) = \infty$ iff $R^{\varphi}(\psi) \geq \alpha$, for all $\alpha \in On$.

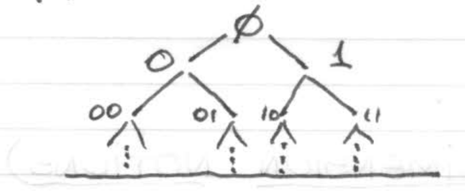
One will show that T is stable iff for all φ, $R^{\varphi}(x=x) < \omega$.

(2.2) <u>DEF</u>: Let p be a type. Then one defines $R^{\varphi}(p)$ to be the least α such that for some $\psi \in p$, $R^{\varphi}(\psi) = \alpha$.

<u>EXAMPLE</u>: (1) $R^{\varphi}(\psi) \geq 0$ iff ψ (or rather, the set defined by ψ) is non-empty.

(2) $R^{\varphi}(\psi) \geq 1$ iff there is a $\vec{b}$ such that ~~R^{φ}~~ $\psi \wedge \varphi(\vec{x},\vec{b})$ and $\psi \wedge \neg\varphi(x,\vec{b})$ are both non-empty.

30/1/87 Picture of how one witnesses that $R^{\varphi}(\psi) \geq \alpha$. Look at binary trees:



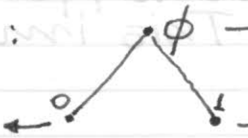
stop at α.

The branches correspond to α-sequences of 0's, 1's, i.e., they are elements of 2^{α}.

Every statement $R^{\varphi}(\psi) \geq \alpha$ has associated a tree as above, with nodes tagged by various tuples from an elementary extension. So:

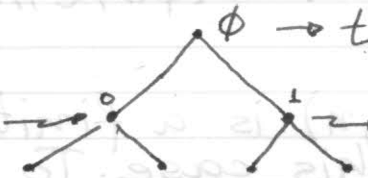
(1) $R^{\varphi}(\psi)\geq 0$: $\cdot\phi$ is tagged by some $\vec{x}$ satisfying ψ.

(2) $R^{\varphi}(\psi)\geq 1$: $\phi \to$ tag by some $\vec{b}$ such that $R^{\varphi}(\psi\wedge\varphi(\bar{x},\vec{b})), R^{\varphi}(\psi\wedge\neg\varphi)\geq 0$.



tag by an element of $\psi\wedge\varphi(\bar{x},\vec{b})$; tag by an element of $\psi\wedge\neg\varphi(\vec{x},\vec{b})$.

(3) $R^{\varphi}(\psi)\geq 2$: $\phi \to$ tag by $\vec{b}_\phi$



tag by $\vec{b}_0$; tag by $\vec{b}_1$

tag the terminal nodes by elements of $\psi\wedge\varphi(\vec{x},\vec{b}_\phi)\wedge\varphi(\vec{x},\vec{b}_0)$, $\psi\wedge\varphi(\bar{x},\vec{b}_\phi)\wedge\neg\varphi(\bar{x},\vec{b}_0)$, $\psi\wedge\neg\varphi(\vec{x},\vec{b}_\phi)\wedge\varphi(\bar{x},\vec{b}_1)$, $\psi\wedge\neg\varphi(\bar{x},\vec{b}_\phi)\wedge\neg\varphi(\bar{x},\vec{b}_1)$.

(2.3) LEMMA 4: (Properties of rank):
(i) if $T_A\vdash \psi_1\to\psi_2$, then $R^{\varphi}(\psi_1)\leq R^{\varphi}(\psi_2)$;
(ii) if $p\subseteq q$ are types, $R^{\varphi}(p)\geq R^{\varphi}(q)$.

PROOF:
Exercise. $\square$

(2.4) Fix $\varphi(\vec{x},\vec{y})$ and $\psi(\bar{x},\bar{a})$. Let $\alpha<\infty$. Consider the following $\Gamma(\varphi,\psi,\alpha)$ (maybe involving new variables). One has variables $\vec{x}_\eta$, $\eta\in 2^{\alpha}$, $\vec{y}_\tau$, $\tau\in 2^{\beta}$, for some $\beta<\alpha$. ($\vec{x}_\eta$ corresponds to terminal nodes, and $\vec{y}_\tau$ to non-terminal nodes.) $\Gamma(\varphi,\psi,\alpha)$ consists of all $\psi(\vec{x}_\eta)$, for all η, and of all $\varphi(\vec{x}_\eta,\vec{y}_{\eta\restriction\beta})$, for all $\beta<\alpha$ & such that $\eta(\beta)=0$, and $\neg\varphi(\vec{x}_\eta,\vec{y}_{\eta\restriction\beta})$ if $\eta(\beta)=1$. (For $\alpha=2$, it corresponds to the picture given if one identifies ~~[illegible]~~ a node withe the branch – i.e., function – terminating there.

LEMMA 5: (i) For $n\in\omega$, $R^{\varphi}(\psi)\geq n$ iff $\Gamma(\varphi,\psi,n)$ is consistent (equivalent to satisfied, since for $n\in\omega$, this is a finite set); (ii) $R^{\varphi}(\psi)\geq\omega$ iff $\Gamma(\varphi,\psi,\omega)$ is consistent, iff $\Gamma(\varphi,\psi,\alpha)$ is consistent, for some $\alpha\geq\omega$, iff $R^{\varphi}(\psi)=\infty$.

PROOF:
(i) Obvious, from picture.

(ii) Suppose $R^\varphi(\psi) \geq n$, for each $n \in \omega$. Then, each $\Gamma(\varphi,\psi,n)$ is consistent. So $\Gamma(\varphi,\psi,\alpha)$ is consistent, for any $\alpha \in On$ (by compactness). This implies $R^\varphi(\psi) \geq \alpha$. $\Box$

LEMMA 6: For fixed $n < \omega$, the condition on $\vec{z}$ that $R^{\varphi(\vec{x},\vec{y})}(\psi(\vec{x},\vec{z})) \geq n$ is definable.

PROOF:
$\Gamma(\varphi, \psi(\vec{x},\vec{z}), n)$ is a finite set of formulas in this case. To get definition, existentially quantify out the $\vec{x}_\eta$ and $\vec{y}_\sigma$. $\Box$

LEMMA 7: Suppose $R^\varphi(x=x) < \infty$. Then, for any A, and $p \in S_m(A)$, $d_p(\varphi)$ is definable.

PROOF:
$R^\varphi(x=x) < \infty$ implies $R^\varphi(\psi) < \infty$ for all ψ. Choose $\psi(\vec{x}, ..) \in p$ such that $R^\varphi(p) = R^\varphi(\psi) = n < \omega$. For any $\vec{b}$, either $R^\varphi(\psi \wedge \varphi(\bar{x},\vec{b})) < n$ or $R^\varphi(\psi \wedge \neg\varphi(\bar{x},\vec{b})) < n$, for, otherwise, $R^\varphi(\psi) \geq n+1$.

Claim: for $\vec{b} \in A^n$, $\varphi(\bar{x},\bar{b}) \in p$ iff $R^\varphi(\psi(\bar{x}) \wedge \varphi(\bar{x},\bar{b})) \geq n$. (Thereby defining $d_p\varphi$.)

If $\varphi(\bar{x},\bar{b}) \in p$, then $\psi(\bar{x}) \wedge \varphi(\bar{x},\bar{b}) \in p$, so $R^\varphi(\psi(x) \wedge \varphi(\bar{x},\bar{b})) \geq n$.

Now suppose $R^\varphi(\psi(\bar{x}) \wedge \varphi(\bar{x},\bar{b})) \geq n$. By the above, $R^\varphi(\psi(x) \wedge \neg\varphi(\bar{x},\bar{b})) < n$. But if $\varphi(\bar{x},\bar{b}) \notin p$, $\neg\varphi(\bar{x},\bar{b}) \in p$, and so $R^\varphi(\psi \wedge \neg\varphi(\bar{x},\bar{b})) \geq n$; $(\psi \wedge \neg\varphi(\bar{x},\bar{b}) \in p)$. $\Box$

5/2/87 One will prove that if T is stable, then all $R^\varphi(x=x)$ are finite. This requires counting types.

DEF: Let p, q be types over A. Define $p \equiv_\varphi q$ by $\varphi(\bar{x},\bar{a}) \in p \leftrightarrow \varphi(\bar{x},\bar{a}) \in q$. Say φ-type for an equivalence class for $\equiv_\varphi$.

LEMMA 8: If $d_p\varphi$ is definable for all p, then there are at most $\max(|A|, |\mathcal{L}|)$

φ-types over A.

PROOF:

Clear, since $\varphi(\bar{x},\bar{a}) \in p \leftrightarrow (d_p\varphi)(\bar{a}) \in p$, one has at most $\max(|A|,|\mathcal{L}|)$ possibilities for the formula (defining) $d_p\varphi$. □

REMARK: Let $(DS)_n(A)$ be the set of all definable n-types over A. Then $(DS)_n(A) \subseteq S_n(A)$; and: $|S_n(A)| \leq 2^{\max(|A|,|\mathcal{L}|)}$ (this is optimal).

One has $|(DS)_n(A)| \leq \prod_\varphi \#$ of φ-types. For a definable type, # of φ-types is bounded by $\max(|A|,|\mathcal{L}|)$, so one has $|(DS)_n(A)| \leq [\max(|A|,|\mathcal{L}|)]^{|\mathcal{L}|}$ ($< 2^{\max(|A|,|\mathcal{L}|)}$ for suitable choices of $|A|$).

LEMMA 9: Suppose $R^\varphi(x=x)=\infty$. (This is a property of T.) Then there exist A and p such that $d_p\varphi$ is not A-definable.

PROOF:

One uses (2.4). Fix an ordinal α and get consistency of: $\varphi(\bar{x}_\eta, \bar{y}_{\eta\restriction\beta})$, if $\eta(\beta)=1$, and $\neg\varphi(\bar{x}_\eta, \bar{y}_{\eta\restriction\beta})$, if $\eta(\beta)=0$. Take a model of this, and A the set of all coordinates of the interpretation of the $\bar{y}_\sigma$'s. There are $\gamma = \sum_{\beta<\alpha} 2^\beta$ such $\bar{y}_\sigma$, so $\mathrm{card}(A) \leq \gamma$.

Suppose η_1 and η_2 distinct. Then the interpretations of $\bar{x}_{\eta_1}$ and $\bar{x}_{\eta_2}$ have different φ-types over A. Then there are at least 2^α φ-types over A, and $|A| \leq \gamma$. If $d_p\varphi$ is definable, the # of φ-types over A is at most $\max(|A|, |\mathcal{L}|)$, and is at least 2^α. Choose $\alpha = 2^{|\mathcal{L}|} + 2^{2^{|\mathcal{L}|}} + \ldots$ (ω times). In this case, $\gamma \leq \alpha < 2^\alpha$. □

So, one has:

LEMMA 10: Suppose that for any A, p, $d_p\varphi$ is A-definable. Then $R^\varphi(x=x)<\infty$.

PROOF:

Clear. □

COROL: The following are equivalent:
(1) T is stable;
(2) all $R^{\varphi}(x=x) < \infty$;
(3) all types over any A are A-definable.

PROOF:
(2) $\Rightarrow$ (1): Done.
(1) $\Rightarrow$ (2): Modify the above proof, replacing A by a model $\mathcal{M}$ of ~~same~~ same cardinal (using Upward Löwenheim-Skolem).
The rest is trivial. $\square$

DEF: Let $\kappa \geq |\mathcal{L}|$ be a cardinal. T is κ-stable if $|S_n(A)| \leq \kappa$ for A of cardinality κ.

LEMMA 12: Suppose T stable. Then T is κ-stable for all κ such that $\kappa^{|\mathcal{L}|} = \kappa$; (e.g., if $|\mathcal{L}| = \aleph_0$, then $\kappa = 2^{\aleph_0}$ works).

PROOF:
One has $|S_n(A)| = |(DS)_n(A)| \leq \max(|A|, |\mathcal{L}|)^{|\mathcal{L}|}$, so, if $|A| = \kappa \geq |\mathcal{L}|$, and $\kappa^{|\mathcal{L}|} = \kappa$, one has the result. $\square$

The converse will be: if $\kappa \geq |\mathcal{L}|$, and T is κ-stable, then T is stable.

Strategy: assume T not stable, and try to get at least κ^+ types over some model of cardinality κ, using the rank criterion and some splitting.

(2.5) Fundamental order, heirs, etc.

DEF: (i) Let $p(\vec{x}) \in S_n(A)$, $\varphi(\vec{x}, \vec{y})$ be an $\mathcal{L}$-formula. One says that φ is represented in p if $\exists \vec{a}$, $\varphi(\vec{x}, \vec{a}) \in p$.
(ii) $cl(p)$ = class of p = $\{\varphi : \varphi$ is represented in $p\}$.
(iii) The fundamental order, $o_n(T)$, is the set of classes, under $\leq$, of n-types over models. (Write $p \leq q$, if

$cl(p) \subseteq cl(q)$, and $p \sim q$, if $cl(p) = cl(q)$.)

LEMMA 13: $p \subseteq q \Rightarrow p \leqslant q$.

PROOF:

Trivial. $\square$

DEF: Let p be a type over a model $\mathcal{M}$, and $p \subseteq q$. One says that q is an heir of p if $cl(p) = cl(q)$ in $\mathcal{L}(\mathcal{M})$.

LEMMA 14: Let p be a type over a model $\mathcal{M}$, and let $\mathcal{N} \succ \mathcal{M}$, $\mathcal{M} \subseteq A \subseteq \mathcal{N}$. Then there is a $q \supseteq p$, q over A, q an heir of p.

PROOF:

Consider $Th(\mathcal{M})$ plus all $\neg\varphi(\bar{x}, \bar{a})$, for $\vec{a}$ in A, φ not represented in p. The type of $\vec{x}$ will be an heir. The consistency is trivial. $\square$

Motivation: q is a free extension of p, e.g., a sort of "tensor product" $p \otimes A$. But, in general q is not unique.

6/2/87 LEMMA 15: Same hypothesis of lemma 14. If p is definable, q is unique.

PROOF:

One wants to decide if $\varphi(\bar{x}, \bar{a}) \in q$. Consider $\varphi(\bar{x}, \bar{y}) \wedge (d_p\psi)(\bar{y})$. This is not represented in p; (here $d_p\psi$ is definable). So, it is not represented in q. So, if $\varphi(\bar{x}, \bar{a}) \in q$, then $\mathcal{N} \models (d_p\varphi)(\bar{a})$. Apply the same argument to $\neg\varphi$ and get $\varphi(\bar{x}, \bar{a}) \in q$ iff $\mathcal{N} \models (d_p\varphi)(\bar{a})$. $\square$

3. DEFINABILITY AND AUTOMORPHISMS

(3.1) DEF: $G_{\mathcal{L}}(\mathcal{M}/A)$ denotes the group of $\mathcal{L}$-automorphisms of $\mathcal{M}$, fixing A pointwise. This group acts on relations and on types, thus: (i) if Y is a k-ary relation, and $\sigma \in G_{\mathcal{L}}(\mathcal{M})$, then

$Y^\sigma = \{\vec{x} : \sigma^{-1}\vec{x} \in Y\}$; (ii) if φ is a formula, $\varphi(\bar{x}, \bar{b})^\sigma = \varphi(\bar{x}, \bar{b}^\sigma)$; (iii) if p is a type, $p^\sigma = \{\varphi(\bar{x}, \bar{b}) : \varphi(\bar{x}, \bar{b}^{\sigma^{-1}}) \in p\}$.

(3.2) THEOREM: Let $X \subseteq \mathcal{M}^k$. Then X is definable in $\mathcal{M}$ iff for all $(\mathcal{M}, X) \prec (\mathcal{N}, Y)$ (in the language $\mathcal{L}'$ which has a k-ary relation symbol corresponding to X), and all $\sigma \in G_{\mathcal{L}}(\mathcal{N}/\mathcal{M})$ (note that it is $G_{\mathcal{L}}$ and not $G_{\mathcal{L}'}$) one has $Y^\sigma = Y$.

PROOF:
See lecuture notes on Model Theory. $\square$ → see lecture of 26/11/86.

This readily implies Beth's theorem.

Consider the following problem: let $(\mathcal{M}, X)$ be as above. If X is not definable, can one find $(\mathcal{N}, Y) \succ (\mathcal{M}, X)$ such that the orbit of Y under $G_{\mathcal{L}}(\mathcal{N}/\mathcal{M})$ is large, while $\mathcal{M}$ is small?

(3.3) One needs a little more generality. One has the following variant of theorem (3.2).

THEOREM: Let $X \subseteq \mathcal{M}^k$, $A \subseteq \mathcal{M}$. Then X is definable using only constants from A iff for all $(\mathcal{M}, X) \prec (\mathcal{N}, Y)$ and all $\sigma \in G_{\mathcal{L}}(\mathcal{N}/A)$, $Y^\sigma = Y$. $\square$

Let $\mathrm{Fix}_A(Y) = \{\sigma \in G_{\mathcal{L}}(\mathcal{M}/A) : Y^\sigma = Y\}$. The size of orbit is the index of $\mathrm{Fix}_A(Y)$.

$\mathrm{Fix}_A(Y)$ may have a bounded finite index in $G_{\mathcal{L}}(\mathcal{N}/A)$, uniformly in $\mathcal{N}$.

EXAMPLE: $\mathcal{M} = \mathbb{C}$ as a field, $R = \{i\}$, $A = \emptyset$.
Orbit (R) has exactly two elements for any $(\mathcal{N}, R') \succ (\mathcal{M}, R)$.

DEF: X is almost A-definable in $\mathcal{M}$ if Orbit (Y) is finite for all $(\mathcal{N}, Y) \succ (\mathcal{M}, X)$ (under $G_{\mathcal{L}}(\mathcal{N}/A)$).

LEMMA 16: If X is almost A-definable in $\mathcal{M}$, there is an integer m such that $|\mathrm{Orbit}(Y)| \leq m$ (under $G_{\mathcal{L}}(\mathcal{N}/A)$) for all $(\mathcal{N}, Y) \succ (\mathcal{M}, X)$.

PROOF:

Compactness. □

What is the "sintactic" equivalent of almost-definability?

Suppose X is almost A-definable in $\mathcal{M}$. Let m be the l.u.b. for the $|\mathrm{Orbit}(Y)|$, as $(\mathcal{N}, Y)$ varies. Select $(\mathcal{N}, Y) \succ (\mathcal{M}, X)$ such that $\{Y_0 = Y, Y_1, \ldots, Y_{m-1}\}$ is exactly the orbit of Y. Define an equivalence relation S on k-tuples ($k =$ arity of X) by: $S(\bar{x}, \bar{y}) \leftrightarrow \bigwedge_j (\bar{x} \in Y_j \leftrightarrow \bar{y} \in Y_j)$. Then S is $G_{\mathcal{L}}(\mathcal{N}/A)$-invariant.

By a homogeneity argument as in proof of Beth's theorem, one can conclude that S is uniformly $\mathcal{L}(A)$-definable, for all models having m distinct Y's. S has at most 2^m equivalence classes — the classes may not be A-definable. However, each is definable using parameters from any ambient model with m Y's.

Take an A-definition of S, say by $\Theta(\bar{x}, \bar{y}, \ldots)$. Then $\mathcal{N} \models \exists$ exactly j classes for Θ. So $\mathcal{M}$ satisfies it also.

One can define any equivalence class by choosing a tuple $\bar{\xi}$ from it. But Y is a finite union of equivalence classes, so is $\mathcal{M}$-definable. So, one has:

THEOREM: (Kueker): If X is almost A-definable in $\mathcal{M}$, there is an A-definable equivalence relation with finitely many classes, such that X is a union of finitely many classes.

PROOF:

See Chang–Keisler, p. 252. □

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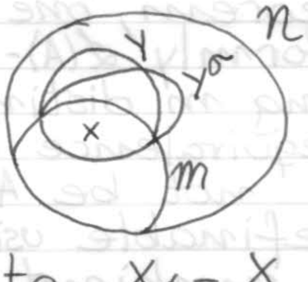
<u>Notation</u>: $FE^n(A)$ = set of equivalence relations on n-tuples, definable using parameters from A, and having finitely many classes (all relative to an ambient model).

(3.4) One goes back to Beth's theorem, i.e., take X not definable in $\mathcal{M}$ using parameters from $\mathcal{M}$. Now try to get a large orbit for X in a small model. This relates to Chang-Makkai (see Chang-Keisler, p. 255).

An easy compactness argument shows that X is not even $\mathcal{M}$-definable. So, by Beth, one gets $(\mathcal{M}, X) \prec (\mathcal{N}, Y)$ and $\sigma \in G(\mathcal{N}/\mathcal{M})$ such that $Y^\sigma \neq Y$. Clearly $\mathcal{N}$ can be chosen of cardinal less than, or equal to, card$(\mathcal{M})$, provided $|\mathcal{M}| \geq |\mathcal{L}|$.

One has:



$\sigma \in G(\mathcal{N}/\mathcal{M})$

Write $X_\emptyset = X$, $\mathcal{M}_0 = \mathcal{M}$, $\mathcal{M}_1 = \mathcal{N}$, $Y = X_0$, and $Y^\sigma = X_1$. For each pair of distinct elements, $\langle \alpha, \beta \rangle$, of $\{0,1\}$, one defines $\sigma_{\langle \alpha,\beta \rangle} \in G_{\mathcal{L}}(\mathcal{M}/\mathcal{N})$ such that $X_\alpha^{\sigma_{\langle \alpha,\beta\rangle}} = X_\beta$. So, e.g., put $\sigma_{\langle 0,1 \rangle} = \sigma$ and $\sigma_{\langle 1,0\rangle} = \sigma^{-1}$. The plan now is to construct on 2^μ (for suitable μ) a system consisting of:

(i) $\mathcal{L}$-structures $\mathcal{M}_\gamma$, $\gamma < \mu$;

(ii) for each $\delta \in 2^\gamma$, a subset X_δ of $\mathcal{M}_\gamma$;

(iii) for each pair $\langle \alpha, \beta \rangle$ of distinct elements of 2^γ, an automorphism $\sigma_{\langle \alpha,\beta\rangle}$ of $\mathcal{M}_\gamma$, so that:

(iv) $\mathcal{M}_0 = \mathcal{M}$, $X_\emptyset = X$;

(v) if $\gamma_1 < \gamma_2$, and $\delta_i \in 2^{\gamma_i}$ with $\delta_1 \subseteq \delta_2$, then $(\mathcal{M}_{\gamma_1}, X_{\delta_1}) \prec (\mathcal{M}_{\gamma_2}, X_{\delta_2})$;

(vi) if $\delta_1 \neq \delta_2$, $\delta_1, \delta_2 \in 2^\gamma$, then $X_{\delta_1} \neq X_{\delta_2}$;

(vii) if $\delta_1 \neq \delta_2$ as above, $X_{\delta_1}^{\sigma_{\langle \delta_1,\delta_2\rangle}} = X_{\delta_2}$;

(viii) if $\delta_1, \delta_2 \in 2^\gamma$, and $\delta'_1, \delta'_2 \in 2^{\gamma'}$ and $\delta_i \subseteq \delta'_i$ then $\sigma_{\langle \delta'_1, \delta'_2\rangle}$ is an extension of $\sigma_{\langle \delta_1,\delta_2\rangle}$;

(ix) $\sigma_{\langle \alpha,\beta\rangle}$ fixes $\mathcal{M}$ pointwise.

Suppose this can be done for the least μ such that $2^\mu > \mathrm{card}(\mathcal{M})\ (\geq |\mathcal{L}|)$ (Note that $\mu \leq |\mathcal{M}|$.) So, w.l.o.g., the $\mathcal{M}_\gamma$ can be chosen of cardinal $|\mathcal{M}|$. Let $\mathcal{M}_\infty = \lim\limits_{\gamma<\mu} \mathcal{M}_\gamma$. So $|\mathcal{M}_\infty| = |\mathcal{M}|$. For each $f \in 2^\mu$, X_f has a clear meaning, as $\lim\limits_{\gamma<\mu} X_{f\restriction\gamma}$, and for $f \neq g$, $X_f \neq X_g$, by (iv). By Tarski, $(\mathcal{M}_\infty, X_f) \succ (\mathcal{M}_\gamma, X_{f\restriction\gamma}) \succ (\mathcal{M}, X)$, (by (v)). Let $\sigma_{\langle f,g\rangle} = \lim \sigma_{\langle f\restriction\gamma, g\restriction\gamma\rangle}$ (use (viii)). Then $\sigma_{\langle f,g\rangle} \in G(\mathcal{M}_\infty|\mathcal{M})$, and $X_f^{\sigma_{\langle f,g\rangle}} = X_g$. So, the orbit of any X_f has cardinal $\kappa \geq 2^\mu > |\mathcal{M}_\infty| = |\mathcal{M}|$.

13/2/87 Now, one gives the idea of the proof of the consistency of (i) – (ix). As for the proof of Robinson's or Beth's theorem, construct a chain of ω-homogeneous models, at each stage doing one of three things:
(a) same as in Robinson;
(b) same as in Beth;
(c) extension of an extension of a ~~frag~~ finite fragment of the original σ.

THEOREM 18: (A version of Chang-Makkai):
Suppose X is not definable in $\mathcal{M}$, where $|\mathcal{M}| \geq |\mathcal{L}|$. Then, for all $\kappa \geq |\mathcal{M}|$, there is a model $(\mathcal{N}, Y) \succ (\mathcal{M}, X)$, $|\mathcal{N}| = \kappa$, so that Y has orbit (relative to $G_{\mathcal{L}}(\mathcal{N}/\mathcal{M})$) of cardinal $\geq \kappa^+$.

PROOF:
Done above. $\square$

THEOREM 19: Suppose T is unstable. Then T is not κ-stable, any $\kappa \geq |\mathcal{L}|$.

PROOF:
Pick $\mathcal{M}$, $p \in S_n(\mathcal{M})$, p not definable.
Pick φ such that $d_p\varphi$ is not definable. By Downward Löwenheim-Skolem, one can assume $|\mathcal{M}| = |\mathcal{L}|$. Choose $\kappa \geq |\mathcal{L}|$ and use theorem (18) to get $(\mathcal{N}, Y) \succ (\mathcal{M}, d_p\varphi)$, $|\mathcal{N}| = \kappa$, so that orbit of Y under $G_{\mathcal{L}}(\mathcal{N}/\mathcal{M})$ has cardinal $\geq \kappa^+$.
Claim: p has an heir q over $\mathcal{N}$ with $d_q\varphi = Y$. (Look at proof of existence of heirs.)
For $\sigma \in G_{\mathcal{L}}(\mathcal{M}/\mathcal{N})$, $d_{q^\sigma}\varphi = Y^\sigma \neq Y$. So $q^\sigma \neq q$.

Thus, since q^σ is also an heir, p has $\geq \kappa^+$ heirs, and there are $\geq \kappa^+$ types over $\mathcal{N}$, contradicting κ-stability. $\square$

4. FORKING

(4.1) Let $A \subseteq \mathfrak{M}$ and $p \in S_n(A)$. One wants to identify a notion of "free extension" of p to $\mathfrak{M}$. If A is itself a model, one has this via the notion of heir.

Since $A = \emptyset$ is possible, and on the other hand models have to satisfy axioms (e.g., of the form $\forall x \exists y \psi(x,y)$) which force to contain formulas to be represented, one cannot use the exact analogue of definition of heir.

It turns out that for stable T, that given an n-type p, there exists a privileged element $\gamma \in On(T)$, written $\beta(p)$ and called the bound of p, such that the types $q \supseteq p$, with $cl(q) = \beta(p)$, deserve to be called "free extensions of p".

These q are not unique, but subject to severe "orbital" constraints.

DEF: Let $A \subseteq B \subseteq \mathfrak{M}$. Suppose $p \in S_n(B)$. Then p is definable almost over A if every $(d_p\varphi) \cap B^{?}$ is of the form $X \cap B^{?}$, where X is definable almost over A.

EXAMPLE: Let $\mathcal{L} = \{\equiv\}$ and T = theory of an equivalence relation with n classes, all infinite. T is a complete theory.

There is only one 1-type over $\emptyset$.

Let $\mathfrak{M}$ be a model. There are exactly n 1-types over $\mathfrak{M}$, one for each equivalence class. Each such type is definable almost over $\emptyset$.

EXERCISE: By considering $\mathcal{L}$ with infinitely many equivalence relations, all with infinite classes, and related in a certain way (having something to do with binary trees), find an example where there is only one 1-type over ϕ, but now there are $2^{\aleph_0}$ 1-types over a model $\mathcal{M}$, each one almost over ϕ.

NON-EXAMPLE: $\mathcal{L}$ as 1st example, but with infinitely many classes, $A=\phi$, $\mathcal{M}$ a model. This time there is only one 1-type over $\mathcal{M}$ almost over ϕ, namely $p(x)$ saying "$x \not\equiv m$ all $m \in \mathcal{M}$".

DEF: (Non-forking): Let $A \subseteq B$, q be a type over B. Then q does not fork over A if there is a model $\mathcal{M}$, $A \subseteq B \subseteq \mathcal{M}$, and a type $q' \geq q$ over $\mathcal{M}$, q' definable almost over A.

19/2/87 (4.2) Completeness property of $on(T)$:

LEMMA 20: Any chain in $on(T)$ has a l.u.b. and a g.l.b.

PROOF:

Let $C = \{c_\lambda : \lambda \in (\Lambda, <)\}$ be a chain, and $c_\lambda = cl(p_\lambda)$, p_λ a type over some model.

Let $\Gamma_+ = \{\varphi : \varphi$ is represented in some $p_\lambda\}$, and $\Gamma_- = \{\varphi : \varphi$ is represented in all p_λ's$\}$.

One wants p_+ such that $\Gamma_+ = cl(p_+)$ and p_- such that $\Gamma_- = cl(p_-)$. This will prove the lemma.

More generally, given Γ, how to get p such that $cl(p) = \Gamma$?

Add to $\mathcal{L}$ constants $c_1, \dots, c_n$ and an 1-ary relation symbol N (intended to denote an elementary substructure of a model of T).

Consider the following: (i) T; (ii) N forms an elementary submodel (i.e., for each formula $\psi(t_1, \dots, t_k)$, one has axioms $\forall \bar{t}\, [Nt_1 \wedge \dots \wedge Nt_k \wedge \psi(\bar{t}) \rightarrow \psi_N(\bar{t})]$); (iii) for each $\varphi(\vec{x},\vec{y}) \in \Gamma$, one has the axiom $(\exists \vec{n} \in N)\varphi(\vec{c},\vec{n})$ (i.e., $\varphi \in \mathrm{cl}$(type of $\vec{c}$ over N)); (iv) for $\varphi \notin \Gamma$, an axiom $(\forall \vec{n} \in N)\neg\varphi(\vec{c},\vec{n})$ (i.e., $\varphi \notin \mathrm{cl}$(type of $\vec{c}$)).

If this is consistent, one gets a type p ($= \mathrm{Type}_N(\vec{c})$) such that $\Gamma = \mathrm{cl}(p)$.

When is it consistent?

Certainly for Γ_+ or Γ_-, by compactness. □

(See: V. Harnik & L. Harrington: Fundamentals of Forking – An. Pure and Applied Logic, 1984.)

(4·3) Basic problem: Let p be a type over A. What can one say about ~~[illegible]~~ various $\mathrm{cl}(q)$, $q \supseteq p$, q over a model?

DEF: A type p (over A) needs $\varphi(\bar{x},\bar{y})$ if φ is represented in all $q \supseteq p$, q over a model.

26/2/87 LEMMA-21: Suppose $p(\bar{x})$ needs $\varphi(\bar{x},\bar{y})$. Then there is a formula $\delta(\bar{w})$ (over A, with length $\bar{w} \geq$ length $\bar{y}$) and a formula $p_0(\bar{x}) \in p(\bar{x})$ such that $T_A \vdash \exists \bar{w}\, \delta(\bar{w})$, and $T_A \vdash (p_0(\bar{x}) \wedge \delta(\bar{y})) \rightarrow \bigvee_i \varphi(\bar{x},\bar{y}_i)$, where $\bar{y}_i$ are subtuples of $\bar{y}$.

PROOF:

Suppose $p(\bar{x})$ needs $\varphi(\bar{x},\bar{y})$.

Fix $\mathfrak{M} \supseteq A$. One has that φ is represented in any extension of p in $\mathfrak{M}$. So, the following is inconsistent: $\mathrm{Diag}(\mathfrak{M})$, $p(\bar{x})$, $\neg\varphi(\bar{x},\bar{m})$, all $\bar{m}$ from $\mathfrak{M}$. So, there is a finite $p_0(\bar{x}) \subseteq p(x)$, and tuples $\bar{m}_1,\dots,\bar{m}_h$, and a fragment $\delta(\bar{m})$ of $\mathrm{Th}_{\mathfrak{M}}(\mathfrak{M})$ such that $T_A \vdash p_0(\bar{x}) \rightarrow \bigvee_i \varphi(\bar{x},\bar{m}_i)$ and $T_A \vdash \delta(\bar{m})$. (W.l.o.g. the $\bar{m}_i$ are subtuples of $\bar{m}$.) So, in fact $T_A \vdash \exists \bar{w}\, \delta(\bar{w})$, and $T_A \vdash (p_0(\bar{x}) \wedge \delta(\bar{y})) \rightarrow \bigvee_i \varphi(\bar{x},\bar{y}_i)$, where $\bar{y}_i$ are subtuples of $\bar{y}$. □

The converse is clear.

<u>LEMMA 22</u> (<u>Ziegler</u>): Let T be stable. Suppose T needs $\varphi \vee \psi$. Then either p needs φ or p needs ψ.

<u>PROOF</u>:

Suppose not.

First get models $\mathcal{M}_1, \mathcal{M}_2$ over A, and elements $\bar{b}_i$ in $\mathcal{M}_i' \supseteq \mathcal{M}_i$, $i=1,2$, such that $\mathrm{Type}(\bar{b}_i|\mathcal{M}_i) \supseteq p$, and $\mathrm{Type}(\bar{b}_1|\mathcal{M}_1)$ does not represent φ, and $\mathrm{Type}(\bar{b}_2|\mathcal{M}_2)$ does not represent ψ. By Robinson's theorem, one can get $\mathcal{N} \succ \mathcal{M} \supseteq A$, $\bar{b}_1, \bar{b}_2$ in $\mathcal{N}$, $\mathrm{Type}(\bar{b}_i|\mathcal{M}) \supseteq p$, $\mathrm{Type}(\bar{b}_1|\mathcal{M})$ does not represent φ, and $\mathrm{Type}(\bar{b}_2|\mathcal{M})$ does not represent ψ. One can assume that all models are countable, and that there is a $\sigma \in \mathrm{Aut}(\mathcal{N}|A)$ such that $\vec{b}_1 = \vec{b}_2^{\,\sigma}$.

Note that $\bar{b}_2$ over $\mathcal{M}^\sigma$ behaves like $\bar{b}_1$ over $\mathcal{M}$ (and does not represent φ). And $\vec{b}_2$ over $\mathcal{M}$ does not represent ψ.

One can also assume $\mathcal{N} \cong_A \mathcal{M}$.

Since p needs $\varphi \vee \psi$, by lemma 21, there is a $p_0(\bar{x}) \in p$ and $\Theta(\bar{w})$ over A such that $T_A \vdash \exists \bar{w}\, \Theta(\bar{w})$, and $T_A \vdash$ $\vdash (p_0(\bar{x}) \wedge \Theta(\bar{y})) \rightarrow \bigvee_i (\varphi \vee \psi)(\bar{x}, \bar{y}_i)$. The right hand side disjunction may be rearranged, by replacing φ and ψ if necessary by various $\bigvee_i \varphi(\bar{x}, \bar{y}_i)$ and $\bigvee_i \psi(\bar{x}, \bar{y}_i)$. If one of these disjunctions is needed then the respective formula (φ or ψ) is needed. One can assume there is only one disjunction, with $\bar{y}_i = \bar{y}$.

Consider the elements $\bar{w}$ satisfying (in $\mathcal{N}$) $\Theta(\bar{w})$. One has $T_A \vdash p_0(\bar{x}) \wedge \Theta(\bar{x}, \bar{w}) \rightarrow$ $\rightarrow (\varphi \vee \psi)(\bar{x}, \bar{w})$. If $\bar{w}$ is in $\mathcal{M}$, one has $T_A \vdash p_0(\bar{b}_2) \wedge \Theta(\bar{b}_2, \bar{w})$, so $\mathcal{M} \models (\varphi \vee \psi)(\bar{b}_2, \bar{w})$, but $\mathcal{M} \models \neg\psi(\bar{b}_2, \bar{w})$, so $\mathcal{M} \models \varphi(\bar{b}_2, \bar{w})$. If $\bar{w}$ is in $\mathcal{M}^\sigma$, and $\Theta(\bar{w})$, then $\mathcal{M}^\sigma \models \neg\varphi(\bar{b}_2, \bar{w})$.

Use an A-isomorphism of $\mathcal{N}$ and, e.g., $\mathcal{M}$, to get inside $\mathcal{M}$ a new $\bar{b}_2$ and submodels like $\mathcal{M}$ and $\mathcal{M}^\sigma$. Do the same inside $\mathcal{M}^\sigma$, and so on.

So, one concludes that $R^\varphi(\Theta) = \infty$, contradiction. $\square$

THEOREM 23: There is a least element in the set of classes $cl(q)$, q a type over a model, $q \supseteq p$.

PROOF:

Claim: $cl(q) = \{\varphi(\bar{x},\bar{y}) : p \text{ needs } \varphi\}$.

To show this is a class of a type over a model, use usual compactness argument, and lemma (22) to show consistency. □

This is called the theorem of the bound.

One writes $\beta(p)$ (bound of p) for the class above.

New definition of dnf (= does not fork): let $p \subseteq q$; q is a nonforking extension of p if $\beta(p) = \beta(q)$.

27/2/87 (4.4) Multiplicity:

THEOREM 24: Suppose p is a type over a model, and $q \supseteq p$. Suppose $cl(q) = cl(p)$. Then q is an heir of p (and so the heir of p?). ?

PROOF:

Let $\varphi(\bar{x},\bar{y},\bar{z})$ be an $\mathcal{L}$-formula, and $\bar{a} \in \mathcal{M}$, so that $\varphi(\bar{x},\bar{y},\bar{a})$ is represented in q. One has to show $\varphi(\bar{x},\bar{y},\bar{a})$ is represented in p.

Compute ranks w.r.t. φ ($\bar{x}$ as the privileged variables). Choose an $\mathcal{L}$-formula $\psi(\bar{x},\bar{t})$ such that for some $\bar{b}$ in B (q over B), $R^{\varphi}(\psi(\bar{x},\bar{b})) = R^{\varphi}(q) = k$. (Look back at lemma 7, whose proof is being now generalized.) As in lemma 7, for any $\bar{b}'$ such that $\psi(\bar{x},\bar{b}') \in q$, & $R^{\varphi}(\psi(\bar{x},\bar{b}')) = k$, one can define (in the ambient model) $(d_p\varphi)(\bar{y},\bar{z})$ by ~~thexxxx~~ (*): "$R^{\varphi}(\psi(\bar{x},\bar{b}') \wedge \varphi(\bar{x},\bar{y},\bar{z})) \geq k$". One wants to get such a $\bar{b}'$ in $\mathcal{M}$. (*) is a definable condition on $\bar{b}'$, say $S(\bar{b}')$. One wants $\bar{b}''$ in $\mathcal{M}$ such that $\psi(\bar{x},\bar{b}'') \wedge S(\bar{b}'') \in q$ (in fact in p). Since $cl(p) = cl(q)$, this is immediate.

Use such to define $(d_q\varphi)(\bar{y},\bar{z})$ over $\mathcal{M}$.

One wants $\bar{r}$ in $\mathfrak{M}$ such that $\varphi(\bar{x},\bar{r},\bar{a}) \in$ $\in p$. Or, equivalently, ambient model $\models$ $\models (d_q\varphi)(\bar{r},\bar{a})$. One has ambient model $\models$ $\models (\exists \bar{t})(d_q\varphi)(\bar{t},\bar{a})$, and $\mathfrak{M} \prec$ ambient model. $\square$

COROL: If p is a type over a model, and q does not fork (d.n.f., as in the second definition, using β) over p, then q is an heir of p.

PROOF:

$$cl(q) \supseteq cl(p) = \beta(p) = \beta(q) \supseteq cl(q). \ \square$$

COROL (to corollary above): If p is over a model, q d.n.f. over p iff q is an heir of p.

PROOF:

Suppose q is an heir of p. Then $cl(q) = cl(p) = \beta(p)$. $\square$

LEMMA 25: Let $p \in S_n(A)$. Then there is an $A_0 \subseteq A$, $|A_0| \leq |\mathcal{L}|$, such that p dnf. over $p\restriction A_0$.

PROOF:

One uses a Löwenheim-Skolem-like argument.

For any $A' \subseteq A$, if p forks over $p\restriction A'$, then $\beta(p\restriction A') \supsetneq \beta(p)$. This will be witnessed by means of an $\mathcal{L}$-formula $\varphi(\bar{x},\bar{y})$, and there are at most $|\mathcal{L}|$ possibilities for φ. These φ will be needed by p, but not by $p\restriction A'$. By lemma 21, if p needs $\varphi(\bar{x},\bar{y})$, it is because there is a finite $p_0(\bar{x}) \subseteq p(\bar{x})$, defined over a finite set, & $\Theta(\bar{w})$, also over a finite set in A, such that ambient model $\models \exists \bar{w}\, \Theta(\bar{w})$, & $p_0(\bar{x}) \wedge \Theta(\bar{w}) \to \bigvee_i \varphi(\bar{x},\bar{w}_i)$, $\bar{w}_i$ subtuples of $\bar{w}$. Add to A' the finite set involved in the above manifestation of p needing φ. Make $|\mathcal{L}|$ steps of this kind. $\square$

LEMMA 26: There is a bound $m(p) \leq 2^{|\mathcal{L}|}$ (this inequality is optimal), p over A, such that for no $\mathfrak{M} \supseteq A$, are there no more than $m(p)$ nonforking extensions of p.

PROOF:

Let q d.n.f. over p.

Use lemma (25) to get $A_0 \subseteq A$, $|A_0| \leq |\mathcal{L}|$, p dnf over $p \restriction A_0$. Use Upward Löwenheim-Skolem to get $A_0 \subseteq \mathcal{M}_0 \prec \mathcal{M}$, $|\mathcal{M}_0| = |\mathcal{L}|$. Then q dnf over $p \restriction A_0$.

Claim: q is uniquely determined by $q \restriction \mathcal{M}_0$.

One has $\beta(q) = \beta(p) = \beta(p \restriction A_0) \subseteq cl(q \restriction \mathcal{M}_0) \subseteq \beta(q)$, or $cl(q) = cl(q \restriction \mathcal{M}_0)$, so q is heir of $q \restriction \mathcal{M}_0$, so unique.

And one has at most $2^{|\mathcal{L}|}$ possibilities. □

COROL.: Bound is independent of model. □

DEF: The minimal bound $m(p)$ is called multiplicity of p.

(4.5) "d.n.f. & almost over":

LEMMA 27: Let $p \in S_n(A)$, $p \subseteq q \in S_n(\mathcal{M})$. Suppose p d.n.f. over q. Then q is almost over A.

PROOF:

If not, then as in Kueker's thm. one gets large orbit for q over A. But this contradicts multiplicity. □

COROL.: Let $p \in S_n(A)$, $p \subseteq q \in S_n(B)$, $B \subseteq \mathcal{M}$. If q d.n.f. over p, then there is $q' \in S_n(\mathcal{M})$, $q' \supseteq q$, q' almost over A.

PROOF:

Choose q' a nonforking extension of p, and apply the above. □

5/3/87 LEMMA 28: Let $p \in S_n(A)$, $p \subseteq q \in S_n(\mathcal{M})$, q almost over A. Then q dnf over p.

PROOF:

See Pillay. □ See p as well.

COROL: The two definitions of d.n.f. agree

PROOF:

Suppose $p \in S_n(A)$, $p \subseteq q$, q d.n.f. over p. Go to $r \supseteq q$, r d.n.f. over q, $r \in S_n(\mathfrak{M})$. By lemma 27, r is almost over A. Conversely, if $p \subseteq q \subseteq r$, $r \in S_n(\mathfrak{M})$, and r almost over A, then, by lemma 28, r d.n.f. over p, & so q dnf. over p. $\square$

(4.6) The Finite Equivalence Relation Theorem:

THEOREM 29: Let $A \subseteq \mathfrak{M}$, p over A, and q_1, q_2 be distinct nonforking extensions of p to $\mathfrak{M}$. Then there is a finite equivalence relation E over A, such that $q_1(\bar{x}) \cup q_2(\bar{y}) \vdash \neg E(\bar{x}, \bar{y})$.

PROOF:

Immediate from the almost over characterization. $\square$

⑤ ORDER — FORKING SYMMETRY.

(5.1) Coheir: Let $\mathfrak{M} \subseteq A$, $p(\bar{x}) \in S_n(\mathfrak{M})$, and $q(\bar{x}) \in S_n(A)$. Then q is a coheir of p if every $\mathcal{L}(A)$-formula in q is satisfiable in $\mathfrak{M}$ (i.e., q is finitely satisfiable in $\mathfrak{M}$).

LEMMA 30: (i) Type$(\bar{c}/A)$ is an heir of Type$(\bar{c}/\mathfrak{M})$ iff Type$(\bar{a}/\mathfrak{M} \cup \{\bar{c}\})$ is a coheir of Type$(\bar{a}/\mathfrak{M})$, for all $\bar{a}$ in A;

(ii) The same, interchanging "heir" and "coheir".

PROOF:

Exercise. $\square$

(5.2) Order: $\mathfrak{M}$ has an order if for some $(2n)$-ary $\alpha(\bar{x}, \bar{y})$ over $\mathfrak{M}$ $\exists (\bar{a}_i)_{i<\omega}$ in $\mathfrak{M}^n$ such that for $i, j < \omega$, $\mathfrak{M} \models \alpha(\bar{a}_i, \bar{a}_j)$ iff $i < j$.

REMARK: if this happens, w.l.o.g. α is an $\mathcal{L}$-formula, for if $\vec{m}$ occurs, replace $\vec{a}_i$ by $\vec{a}_i{}^\frown\vec{m}$ (i.e., concatenation of $\vec{a}_i$ & $\vec{m}$).

DEF: T has an <u>order</u> if some model of T has an order.

LEMMA 31: A complete theory T has an order iff there is an $\mathcal{L}$-formula α such that for each $k<\omega$, $T \vdash$ "α can define an order of length $\geq k$".

PROOF:

Compactness. $\square$

6/3/87 (<u>Note</u>: The current discussion of forking symmetry does not depend on the equivalence between the two definitions of d.n.f.. One uses the definition in terms of <u>bounds</u>.)

THEOREM 32: (T is not assumed to be stable) Suppose $\mathcal{M}$ has no order. Let $\bar{a}, \bar{b}$ in an ambient model $\mathcal{M}^*$ be such that Type$(\bar{b}/\mathcal{M}\cup\bar{a})$ is an heir of Type$(\bar{b}/\mathcal{M})$. Then Type$(\bar{b}/\mathcal{M}\cup\bar{a})$ is a coheir of Type$(\bar{b}/\mathcal{M})$. (And so, Type$(\bar{a}/\mathcal{M}\cup\bar{b})$ is an heir of Type$(\bar{a}/\mathcal{M})$.)

PROOF:

Suppose not. Since Type$(\bar{b}/\mathcal{M}\cup\bar{a})$ is not finitely satisfiable in $\mathcal{M}$, one can choose $\alpha(\bar{x},\bar{y})$ such that $(*)$: "$\mathcal{M}^* \models \alpha(\bar{a},\bar{b})$ but $\mathcal{M}^* \models \neg\alpha(\bar{a},\bar{b}')$ for all $\bar{b}'$ in $\mathcal{M}$".

Construct by recursion $\bar{a}_i, \bar{b}_i$ in $\mathcal{M}$, $i<\omega$, such that: (i) $\mathcal{M}^* \models \alpha(\bar{a}_i,\bar{b})$, all $i<\omega$; and (ii) $\mathcal{M}^* \models \alpha(\bar{a}_i,\bar{b}_j)$ all $j\leq i$, $i<\omega$. (<u>Note</u>: (i) is needed only for inductive purposes.)

Suppose this is done. Let $\beta(\bar{x}_1,\bar{y}_1,\bar{x}_2,\bar{y}_2)$ be $\alpha(\bar{x}_1,\bar{y}_2)$. Then $\mathcal{M} \models \beta(\bar{a}_i,\bar{b}_i,\bar{a}_j,\bar{b}_j)$ iff $i\leq j$, giving order.

For (i) & (ii), proceed by induction. Suppose one has $\bar{a}_1,\ldots,\bar{a}_n,\bar{b}_1,\ldots,\bar{b}_n$. Then $\mathcal{M}^* \models \bigwedge_1^n \alpha(\bar{a}_i,\bar{b}) \wedge \bigwedge_1^n \neg\alpha(\bar{a},\bar{b}_i) \wedge \alpha(\bar{a},\bar{b})$, by (i) & $(*)$. Now, Type$(\bar{b}/\mathcal{M}\cup\bar{a})$ is an heir of Type$(\bar{b}/\mathcal{M})$, one can replace $\bar{a}$ in the above by some $\bar{a}'$ in $\mathcal{M}$. "Quantify out" $\bar{b}$ in the result, to justify replacing $\bar{b}$ by $\bar{b}'$ in $\mathcal{M}$ such that $\mathcal{M} \models \bigwedge_1^n \alpha(\bar{a}_i, \bar{b}') \wedge \bigwedge_1^n \neg\alpha(\bar{a}', \bar{b}_i) \wedge \alpha(\bar{a}', \bar{b}')$. Evidently, one can choose $\bar{a}_{n+1}$ as $\bar{a}'$ & $\bar{b}_{n+1}$ as $\bar{b}'$. $\square$

<u>THEOREM 33</u>: Suppose T stable. Then T has no order.

<u>PROOF</u>:

Let $\alpha(\bar{x}, \bar{y})$ be an $\mathcal{L}$-formula defining an order. Then $R^{\alpha}(x=x) = \infty$, since one can (by compactness) get order dense, and use splitting to boost rank. $\square$

<u>THEOREM 34</u>: Suppose T stable. $\text{Type}(\bar{a}/\mathcal{M} \cup \bar{b})$ is the heir of $\text{Type}(\bar{a}/\mathcal{M})$ iff $\text{Type}(\bar{b}/\mathcal{M} \cup \bar{a})$ is the heir of $\text{Type}(\bar{b}/\mathcal{M})$.

<u>PROOF</u>:

Theorems 32 & 33. $\square$

<u>THEOREM 35</u>: (<u>Forking Symmetry</u>): Let T be stable. Then $\text{Type}(\bar{a}/\bar{b} \cup C)$ forks over $\text{Type}(\bar{a}/C)$ iff $\text{Type}(\bar{b}/\bar{a} \cup C)$ forks over $\text{Type}(\bar{b}/C)$.

<u>PROOF</u>:

Assume the following four statements:

(i) $\text{Type}(\bar{a}/\bar{b} \cup C)$ d.n.f. over $\text{Type}(\bar{b}/C)$;

(ii) one has model $\mathcal{M}_1 \supseteq \bar{a} \cup C$ such that $\text{Type}(b/\mathcal{M}_1)$ dnf. over $\text{Type}(\bar{b}/C)$;

(iii) one has $\mathcal{M}_2 \supseteq C$ such that $\text{Type}(\bar{a}, \mathcal{M}_2)$ dnf. over $\text{Type}(\bar{a}/C)$;

(iv) the same $\mathcal{M}_2$, $\text{Type}(\bar{b}/\mathcal{M}_1 \cup \mathcal{M}_2)$ is the heir of $\text{Type}(\bar{b}/\mathcal{M}_1)$.

(Assume $\mathcal{M}_1, \mathcal{M}_2, \bar{a}, \bar{b}$ are in some ambient model.)

By (ii) & (iv), $\text{Type}(\bar{b}/\mathcal{M}_1 \cup \mathcal{M}_2)$ dnf. over $\text{Type}(\bar{b}/C)$. And, so $\text{Type}(\bar{b}/\bar{a} \cup \mathcal{M}_2)$ dnf. over $\text{Type}(\bar{b}/\mathcal{M}_2)$. By symmetry of forking over models, $\text{Type}(\bar{a}/\bar{b} \cup \mathcal{M}_2)$ dnf. over $\text{Type}(\bar{a}/\mathcal{M}_2)$, which dnf. over $\text{Type}(\bar{a}/C)$. Since $C \subseteq \mathcal{M}_2$, one has that $\text{Type}(\bar{a}/\bar{b} \cup C)$ dnf. over $\text{Type}(\bar{a}/C)$, a contradiction. $\square$

NOTE: How to justify assumptions (ii)–(iv)?

(ii) Let $p(\bar{x}) = \text{Type}(\bar{b}/\bar{a} \cup C)$. Choose $\mathcal{M}_1' \supseteq \bar{a} \cup C$, and q a non-forking extension of p to $\mathcal{M}_1'$. (Note: no suggestion that $\bar{b}$ satisfies q.) Choose some $\bar{b}'$ in the ambient model, realising q; $\bar{b}'$ realises p (p over $\bar{a} \cup C$). So, at cost of blowing up ambient model, get σ fixing $\bar{a} \cup C$, & $\sigma(\bar{b}') = \bar{b}$. Let $\mathcal{M}_1 = (\mathcal{M}_1')^{\sigma}$, giving (ii).

For (iii) & (iv), first choose $\mathcal{M}_2' \supseteq C$ in ambient model. Replace $\bar{a}$ by variables $\bar{v}$, and $\bar{b}$ by variables $\bar{w}$. Write down conditions on $\bar{v}, \bar{w}$ guaranteeing $\text{Type}(\bar{v}/C) = \text{Type}(\bar{a}/C)$, and $\text{Type}(\bar{v}/\mathcal{M}_2')$ dnf. over $\text{Type}(\bar{v}/C)$; $\text{Type}(\bar{w}/\mathcal{M}_1) = \text{Type}(\bar{b}/\mathcal{M}_1)$, and $\text{Type}(\bar{w}/\mathcal{M}_1 \cup \mathcal{M}_2')$ is the heir of $\text{Type}(\bar{b}/\mathcal{M}_1)$.

(See also hint in Pillay.)

See pp. 27–28, as well.